Postprint of: Mikielewicz J., Mikielewicz D., A simple dissipation model of circular hydraulic jump, International Journal of Heat and Mass Transfer, Vol. 52, Iss. 1–2 (2009), pp. 17-21, DOI: 10.1016/j.ijheatmasstransfer.2008.06.007

Manuscript submitted for consideration to publish in
International Journal of Heat and Mass Transfer

A simple dissipation model of circular hydraulic jump

Jarosław Mikielewicz[1], Dariusz Mikielewicz[2]

[1]*The Szewalski Institute of Fluid-Flow Machinery PAS, ul. Fiszera 14, Gdańsk, Poland,* *jarekm@imp.gda.pl*

[2]*Gdańsk University of Technology, Faculty of Mechanical Engineering, ul. Narutowicza 11/12, 80-952 Gdańsk, Poland,* Dariusz.Mikielewicz@pg.gda.pl

Abstract - Considered has been the phenomenon of hydraulic jump formed as a result of circular jet impingement on the horizontal surface .The horizontal spreading of liquid is induced mainly by action of inertia forces. In the case of supercritical conditions of film flow the phenomenon of hydraulic jump may appear. Postulated model includes in the analysis dissipation effects present in the flow which lead to formation of the hydraulic jump. The model is derived from the analysis of Bernoulli equation. Some preliminary analysis on the formation of the type I and type II hydraulic jump, i.e. featuring formation of one or two eddies, has also been given. Presented model was compared with available data base of experimental results. Satisfactory consistency has been achieved.

1. Introduction

Jets impinging on horizontal surfaces can be used to cleanse metal surfaces, induce atomisation and effect high heat and mass transfer in the industry. Spreading of a film formed in such a way may experience a sudden change of the film thickness, i.e. the hydraulic jump, accompanied by a significant loss of energy and production of turbulence. During such change the flow changes from supercritical condition to subcritical condition with a sudden of the liquid height and a decrease in velocity of liquid. In some cases also entrainment of air is present as a result of effects of turbulent mixing and buoyancy, Waniewski et al. (2001). Precise knowledge of liquid film depth before and after the jump enables determination of accurate rates of heat which can be removed from surfaces. It is apparent that the heat transfer is much more efficient in the case of thin films or highly turbulent thick films and our efforts should aim at such a configuration of nozzles cooling the surface which would eliminate the issue of increased film thickness for laminar or low turbulence cases of film.

The formation of the thin layer and the circular jump was first noticed and described by Lord Rayleigh (1914) who considered inviscid flow along a channel of constant breadth. The speed ahead of the hydraulic jump was assumed uniform. Wilson (1964) extended that theory slightly by assumption of the flow in the thin layer being radial and strongly influenced by viscosity, whereas the principles of momentum and continuity at the jump location were similar as in the theory due to Rayleigh. Such theory for a long time served as a benchmark for testing other models of hydraulic jump. A very interesting research into the understanding of the hydraulic jump was carried out by Ishigai et al. (1977) who studied the problem both theoretically and experimentally. He classified the hydraulic jump in relation to the Froude number before the jump as smooth ($Fr<2$), S-shaped ($2<Fr<7$), round and narrow ($7<Fr<15$) and unstable entraining air ($Fr>15$). It has also been noticed that the mean location of hydraulic jump could be calculated from the principle of momentum conservation. Craik et al. (1981) experimentally detected the regions of "reversed flow" just after the occurrence of the hydraulic jump. Such reversed flow region was also detected by Nakoryakov et al. (1978). The eddy changed its dimensions with regard to the flow conditions, namely the upstream Froude and Reynolds numbers. The dimension of the eddy shortened as the outer depth of the film increased. In the study by Craik et al. (1981) the equations of motion were integrated across the thin liquid layer to solve for the velocity and height of the liquid film. Bowles and Smith (1992) carried out numerical study in which the conclusion was made that hydraulic jumps are governed by a free interaction predominantly between surface tension and viscosity upstream and then further downstream between gravitational pressure gradient and viscosity. The latter was claimed vital in controlling these jumps. Liu and Lienhard V (1993) concluded that the radial position of the jump, which was elaborated by various researchers have not been confirmed to a satisfactory extent with experimental data available to date. On the basis of momentum balance the classical theory by Raleigh, reproduced by Massey (1989), arrived at a relation for the change of the film depth before and after the jump in the form:

$$h_2 / h_1 = 0.5(-1 + \sqrt{1 + 8Fr_1^2}) \tag{1}$$

That study was one of the first to track theoretically the occurrence of one and two eddies formed after the hydraulic jump. Therefore two types of eddies were distinguished and the "traditional" one was named "type I" and "the roller", the another eddy which appeared just beneath the surface - "type II" respectively, see fig. 1. Up to date none of the studies were able to provide explanation for the formation of the roller eddy. Numerical studies of Yokoi and Xiao (1999) showed that transition from type I to type II is associated with an increase in pressure beneath the surface in the region after the hydraulic jump. They found that there is an immediate increase of pressure after the jump due to interaction between the surface tension and the main flow. This increase in pressure needs to balance the surface tension and the driving flow to produce steady jump. They concluded also that the type II jump tends to occur when the depth of the film after the jump is greater than 0.5mm, whereas the type I develops in cases of smaller depths. The results of their numerical simulations also show that the radius of the jump decreases with increasing kinematic viscosity of liquid, which is consistent with experimental evidence, see for example Bohr et al. (1996). Modelling of dissipation effects was, amongst the others, introduced by Hewakanamby and Zimmerman (2001). The energy dissipation was attributed to small eddies. Yokoi and Xiao (2002) in their numerical examination of the hydraulic jump managed to reproduce both types of jump. The transition between the two was due to pressure increase beneath the jump. The roller is formed as a result of a proper pressure pattern.

The phenomenon of hydraulic jump is a very important issue in the case when heat transfer on a body, where the liquid jet impinges, is considered. Beyond the hydraulic jump the intensity of heat transfer abruptly deteriorates. Therefore knowledge on such phenomenon and ability to predict its location is of paramount importance in the design of surface cooling processes. The hydraulic jump is formed at a location where the balance between forces resulting from momentum change in thinner and thicker liquid layers and forces stemming from hydrostatic thrust of thicker liquid layer and surface tension force is obeyed. Presented below is a simple model enabling determination of liquid film thickness before and after hydraulic jump as well as its radial position. The model is derived from the analysis of the Bernoulli equation rather then of the momentum equation. The results obtained using the model are based on the existence of a single or two eddies formed at the location of hydraulic jump.

2. Model of hydraulic jump

Let's consider a liquid jet impinging on a horizontal plane. The spreading of liquid takes place by means of action of inertia forces. As mentioned earlier the hydraulic jump occurs in places where the flow suddenly changes from supercritical one (Froude number, $Fr > 1$) into subcritical one ($Fr<1$).

Phenomenon of hydraulic jump features a sudden change of liquid film thickness. According to existing theories, the critical flow conditions, in the case of thin films, correspond to a value of Froude number of one, i.e. when the mean velocity of liquid and propagation velocity of disturbances on a flat liquid surface are equal. The hydraulic jump seems to be analogical to the phenomenon of a shock wave in gas flow, when the flow conditions change from supercritical (Mach number, Ma>1) into subcritical conditions (Ma<1). In the subsequent considerations a case will be considered where the value of Froude number at inlet will exceed unity, so that favourable conditions for the occurrence of hydraulic jump will be present.

The postulated model of hydraulic jump is based on the analysis of Bernoulli energy equation for a viscous liquid, which includes energy losses present due to a sudden flow expansion beyond the jump as well as the presence of one or two eddies following the change of film thickness. The Bernoulli equation for an averaged streamline, as in Fig. 1, yields:

$$\frac{p_1}{\rho g} + H_1 + \frac{u_1^2}{2g} = \frac{p_2}{\rho g} + H_2 + \frac{u_2^2}{2g} + \Delta h_{loss} \tag{2}$$

Symbols appearing in equation (2) have been explained in Fig. 1. In the considered case we are dealing with a free surface flow and therefore $p_1=p_2$, $H_1=h_1/2$ and $H_2=h_2/2$. Incorporating the latter we obtain from (2):

$$\frac{h_1}{2} + \frac{u_1^2}{2g} = \frac{h_2}{2} + \frac{u_2^2}{2g} + \Delta h_{loss} \tag{3}$$

On the other hand the continuity equation for radially spreading outwards film yields:

$$Q = \frac{\pi d^2}{4} u_d = 2\pi r_1 u_1 h_1 = 2\pi r_2 u_2 h_2 \tag{4}$$

Additionally examined were losses due to friction with the wall at the location of hydraulic jump, but they proved to be an order of magnitude lower than the ones modelled in the study. Equations (3) and (4) describe the phenomenon of hydraulic jump. Losses of mechanical energy, Δh_{exp}, present during the hydraulic jump, are modelled in a similar way as pressure losses in a channel where sudden change of cross-section of the channel takes place and therefore are named the expansion losses. Such losses can be estimated from formulas presented in numerous textbooks on fluid mechanics, where such topics are considered, for example Douglas et al. (1998):

$$\Delta h_{\exp} = (u_1 - u_2)^2 / 2g \tag{5}$$

Moreover, as stems from examination of experimental data as well as numerical calculations, beyond the location where the sudden change of the film thickness there is formed a strong one eddy or two eddies, schematically presented in Fig. 1 and 2, which contribute to further energy losses. The sketch showing the geometry of a model for the case of one eddy is presented in Fig. 2a, whereas the case of two eddies is considered in Fig. 2b. Head losses caused by rotation of liquid between two locations within the eddy, namely r_1 and r_2 are:

$$\Delta h_{ed} = \frac{p_2}{\rho g} - \frac{p_1}{\rho g} = \frac{\omega^2 r_2^2}{2g} - \frac{\omega^2 r_1^2}{2g} \tag{6}$$

In relation (6) ω denotes the eddy vorticity. Assuming r_1=0 and averaging the pressure change in the limits of the mean radius of the eddy curvature we obtain:

$$\Delta h_{ed} = \frac{\omega^2}{2gh_2} \int_0^{R/2} r^2 dr \tag{7}$$

In relation (8) R stands for the mean eddy radius. That is also related to the curvature of the free surface at the location of the jump, as seen from Fig. 1. In the model the eddy occupies half of the film depth after the jump and divides it into four imaginary layers, ones without the eddy on top and bottom of the film after the jump and the other two layers occupied by the eddy with diameter R. In order to complete our considerations of head losses due to presence of eddies we need to determine the vorticity resulting from the presence of the single eddy. The rotation of the eddy with respect to its centroid can be approximated using the velocity difference acting on two sides of the eddy. In such a case the vorticity reads:

$$\omega = \left(u_1 - u_2\right) / R \tag{8}$$

Generally the radius of eddy is a function of the film depth and parameter: p which models the unknown eddy size, and can be expressed as $R \approx h_2/p$. Parameter p requires more detailed experimental evidence to be examined. Surveyed literature did not show any data related to that topic. Therefore in the present study it has been studied as a parameter. Substitution of (8) into (7) returns the head loss due to the presence of the eddy:

$$\Delta h_{ed} = (u_1 - u_2)^2 / (2g\, 8p) \tag{9}$$

Finally, the total losses due to the flow expansion and the presence of one eddy yield:

$$\Delta h_{loss} = \Delta h_{\exp} + \Delta h_{ed} = \left(1 + \frac{1}{8p}\right)\frac{(u_1 - u_2)^2}{2g} = k\,\frac{(u_1 - u_2)^2}{2g} \tag{10}$$

In the paper were assumed two values of parameter p, namely $p_1=4$ and $p_2=8$, corresponding in such a way to the division of the film after the jump into respective layers, see Fig. 2. Such a division corresponds also to one or two eddies, respectively. In case of two eddies it is easy to show, on the basis of condition of same head losses for a single and two eddies that $p_2=2p_1$. In relation (11) $k=(1+1/(8p))$. Introducing (10) and (4) into (3) we can obtain a relation in which the film thickness before and after the hydraulic jump are related through the definition of the Froude number:

$$Fr_1 = \frac{u_1}{\sqrt{gh_1}} = \frac{h_2/h_1}{\sqrt{(h_2/h_1+1) - (h_2/h_1-1)k}} \tag{11}$$

Relation (11), when losses due to eddy are not considered, i.e. k=1 reduces to:

$$h_2/h_1 = \sqrt{2}u_1/\sqrt{gh_1} = \sqrt{2}Fr_1 \tag{12}$$

Such relation should be compared with the traditional relation for the change of film depth (1). Relations (1) and (12) give similar results for large values of Froude number (discrepancy less then 2%.. At values of $Fr<10$ discrepancies are more pronounced and if necessary (1) can be used. In order to find a specific relation for localisation of the hydraulic jump we require additional information about relation of film depth before and after the jump. To do that let's compare difference in hydrostatic pressure due to hydraulic jump with the pressure difference resulting from surface tension, which sustains that pressure difference:

$$(h_2 - h_1)\rho g = \sigma / R \tag{13}$$

In (13) R denotes the jump curvature radius. Such radius can be estimated and examination of literature shows that it amounts to more or less of half of the film thickness beyond the jump. Hence:

$$(h_2 - h_1) = \sqrt{2\sigma/(\rho g)} \tag{14}$$

In the case of laminar film flow the film thickness before the jump can be estimated from the distance from stagnation point, Mikielewicz and Mikielewicz (2004). In the case of developing flow, assuming inviscid flow in films and (4), as well as that nozzle velocity is equal to undisturbed velocity in film $u_d=u_1$ we obtain:

$$h_1 = d^2/(8r) \tag{15}$$

A good approximation of film thickness is r=d/2 in (15), Mikielewicz (2004), which leads to:

$$h_1 = d/4 = const \tag{16}$$

A more general relation for film thickness before the jump stems from account of viscous and inertia forces in momentum equation, Mikielewicz and Mikielewicz (2004):

$$h_1 = \left(\frac{10Qr^2\nu}{3\pi u_d^2}\right)^{0.2} = \left(\frac{5d^3r^2}{6\,\mathrm{Re}_d}\right)^{0.2} \tag{17}$$

where $\mathrm{Re}_d = u_d d/\nu$. Utilising (14) we can derive the relation for Froude number:

$$Fr_1 = \frac{u_1}{\sqrt{gh_1}} = \frac{1 + 8\dfrac{\sqrt{2}}{We}\dfrac{r_h}{d}}{2\sqrt{2}\left(\dfrac{r_h}{d}\right)^{0.5}\sqrt{2 + \dfrac{8\sqrt{2}}{We}\dfrac{r_h}{d}(1-k)}} \tag{18}$$

where $Fr_d = u_d/\sqrt{gd}$, $We = d\sqrt{\rho g/\sigma}$.

Making use of (16) leads to relation describing the location of hydraulic jump in the form:

$$\frac{r_h}{d} = \frac{Fr_d \sqrt{2 + 4\sqrt{2}(1-k)/We}}{4\sqrt{2}/We + 1} \tag{19}$$

On the other hand implementation of (18) leads to the following relation for hydraulic jump:

$$\frac{1}{8} Fr_d \left(\frac{5}{6}\right)^{-0.3} \left(\frac{r_h}{d}\right)^{-1.6} \mathrm{Re}^{0.3} \sqrt{2 + a\frac{(1-k)}{(r_h/d)^{0.4}}} - \left[1 + \left(\frac{r_h}{d}\right)^{-0.4}\right] = 0 \tag{20}$$

where: $a = \sqrt{2}/We(6\,\mathrm{Re}/5)^{0.2}$.

3. Calculations

Considered has been supercritical flow of liquid film (Fr>1). In calculations a value of the parameter p, describing the eddy size, was chosen to be p=4 and 8, influencing in such a way value of h_2/h_1. Such a choice of values of parameter p were corresponding to the presence of 1 or 2 eddies. In Fig. 3 presented are experimental data from literature due to Craik et al. (1981), Ishigai et al (1977), Liu et al. (1993), Bykuć (2004), Gumkowski (2004) compared with the model predictions. Good consistency has been achieved. Most of experimentral data collected up to date was gathered for the inlet Froude number less then 20. Only some of data due to Liu and Lienhard V (1993) have been established for higher values of Froude number. Relation (1) proves to be a good model for small values of Froude number, however that relation fails to reflect the experimental data for higher values of the Froude number. The proposed in the present work model (11) proves to be of the same quality as relation (1) in the case of small values of Froude number. In case of large Froude numbers the model is capable of predicting a non-linear behaviour of experimental data, contrary to equation (1). In Fig. 3 the results of calculations performed using equation (11) have been presented where values of parameter p were selected to be respectively p_1=4, one eddy, and p_2=8 (two eddies). It must be admitted that the calculations are very fragile to a value of parameter p. Anyway, the way in which parameter p is considered in the present work has some physical meaning, as it is the number of regions into which the film depth is divided after the jump. In case of a single eddy it is four layers and in case of two eddies it is eight layers, see fig. 2 a and 2b. The type II jump, i.e. when two eddies are present, occurs mostly at higher values of Froude which correspond to thicker films. A good quality of results showing the jump location compared against experimental data can be seen in Fig. 4. In this figure there are contained envelopes corresponding to respective models of film thickness distribution for different values of parameter p. Again it is apparent that the simple models of film thickness distribution (15), (16) and (17) are sufficient to be combined with the model of hydraulic jump (11) to enable calculation of the location of hydraulic jump. Most promising results are obtained with the model (15), however even in that case some further refinement would be necessary.

4. Conclusions

In the paper presented has been a simple model of circular hydraulic jump. The model is based on the solution of Bernoulli equation for a viscous fluid flow, which incorporates the dissipation losses due to change of film thickness as well as the presence of eddies following the jump, instead of the momentum equation. Film thickness before and after the jump is a local quantity and depends merely on Reynolds and Weber numbers. Proposed model features a parameter describing the size of the eddy, the parameter p. The model is very sensitive to selection of a value of that parameter. It results from the presented analysis that value of parameter p_1=4 corresponds to the presence of a single eddy, whereas p_2=8 corresponds to two eddies. Calculations performed for two cases show an envelope where all considered experimental data can be found. Consistency between model predictions and experimental data seems satisfactory. The type II jump, occurs mostly at higher values of Froude which correspond to thicker films.

NOMENCLATURE

d	- nozzle diameter, (m)
$Fr = u_d/(gd)^{0.5}$	- Froude number
G	- acceleration due to gravity, (m/s^2)
h	- film thickness, (m)
H	- head losses, (m)
k	- parameter
p	- pressure, (Pa),
P	- parameter modelling unknown eddy size
r	- radial coordinate,
R	- mean radius of curvature of the free surface, (m)
$Re_s = u_d d/\nu$	- Reynolds number.
Q	- volumetric flow rate, (m^3/s)
U	- velocity in film, (m/s)
u_d	- nozzle velocity, (m/s)
$We=d(\rho g/\sigma)^{0.5}$	- Weber number
ν	- kinematic viscosity, (m^2/s)
ρ	- liquid density, (kg/m^3)
σ	- surface tension, (N/m)
ω	- eddy vorticity, (1/s)

Subscripts

1	- inlet,

2 - outlet

d - nozzle

h - location of hydraulic jump

ed - eddy

exp - expansion

loss - losses

r - radial

References

1. Bohr T., Ellegaard C., Hansen E.A., Haaning A., Putkaradze V., Watanabe S., *Separation and pattern formation in hydraulic jumps*, Physica A, 249, 111-117, 1998.
2. Bowles R.I., Smith F.T., The standing hydraulic jump: theory, computations and comparisons with experiments, J. Fluid Mechanics, 242, 145-168, 1992.
3. Bykuć S., Barnik D., Experimental investigation of film thickness due to jet impingement on a flat plate, internal report of IFFM, No. 4313/2004.
4. Chaudhury Z.H., Heat transfer in a radial liquid jet, J. Fluid Mechanics, 20, 501-511, 1964.
5. Craik D.D., Latham R.C., Fawkes M.J., Gribbon W.F., *The circular hydraulic jump*, J. Fluid Mech., 12, 347-362, 1981.
6. Douglas J.F., Gasiorek J. M.,Swaffield J. A., *Fluid Mechanics*, Longman, 1998.
7. Gumkowski S., Experimental investigations of heat transfer and hydrodynamics of liquid layers formed as a result of jet impingement on a plate, internal report of IFFM, No. 4282/2004.
8. Hewakandamby B.N., Zimmerman W.B., Characterisation of hydraulic jumps/falls with surface tension variations in thin film flows, Dynamics of Atmospheres and Oceans, 34, 349-364, 2001.
9. Ishigai S., Nakanishi S., Mizuno M., Imamura T., Heat transfer of the Impinging Round Water Jet in the Interference Zone of Film Flow along the Wall, Bulletin of the JSME, vol. 20, No. 139, 1977.
10. Liu X., Lienhard V J. H., *The hydraulic jump in circular jet impingement and in other thin liquid films*, Experiments in Fluids 15, 108-116, 1993.
11. Lord Rayleigh, On the theory of long waves and bores, *Proc. of the Royal Society of London. Series A*, 90, 324-328, 1914.
12. Massey B.S., Mechanics of Fluids, Chapman & Hall, London, 1989.
13. Mikielewicz J., Mikielewicz D. *Heat transfer in fully developed laminar film form due to jet impingement*, Proc. XII Symp. of Heat and Mass Transfer, 549-557, Cracow, 2004.
14. Nakoryakov V.E., Pokusaev B.G., Troyan E.N., Impingement of an Axisymmetric Liquid Jet on a Barrier, Int. J. Heat Mass Transfer, 21, 1175-1184, 1978.
15. Thomas S., Hankey W., Faghri A., Swanson T., One-dimensional Analysis of the Hydrodynamic and Thermal Characteristics of Thin Film Flows Including the Hydraulic Jump and Rotation, Trans. of ASME, 112, 728-735, 1990.
16. Waniewski T.A., Hunter C., Brennen C.E., Bubble measurements downstream of hydraulic jumps, Int. J. of Multiphase Flow, 27, 1271-1284, 2001.

17. Watson E.J., The radial spread of a liquid jet over a horizontal plane, J. Fluid Mechanics, 20, 481-499, 1964.
18. Yokoi K., Xiao F., *A numerical study of the transition in circular hydraulic jump*, Physics Letters A, 257, Elsevier, 153-157,1999.
19. Yokoi K., Xiao F., Mechanism of structure formation in circular hydraulic jumps: numerical studies of strongly deformed free-surface shallow flows, Physica D, 161, 202-219, 2002.

Figure captions:

Fig. 1. Schematic of hydraulic jump
Fig. 2. Schematic geometry of a model of a jump featuring: a) one eddy, b) two eddies
Fig. 3. Froude number before the jump in function of film thickness ratio h_2/h_1.
Fig. 4. Hydraulic jump location r_h/d in function of Froude number before the jump Fr_1.

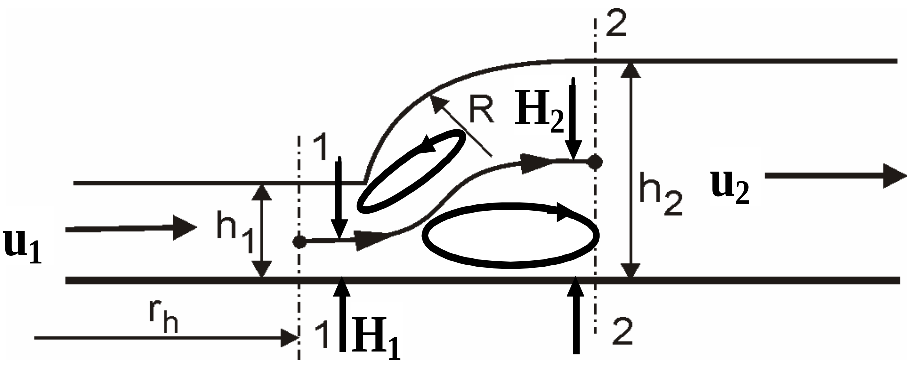

Fig. 1. Schematic of hydraulic jump

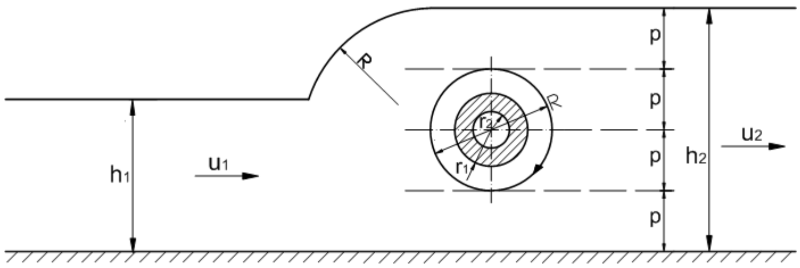

Fig. 2. Schematic geometry of a model of a jump featuring: a) one eddy

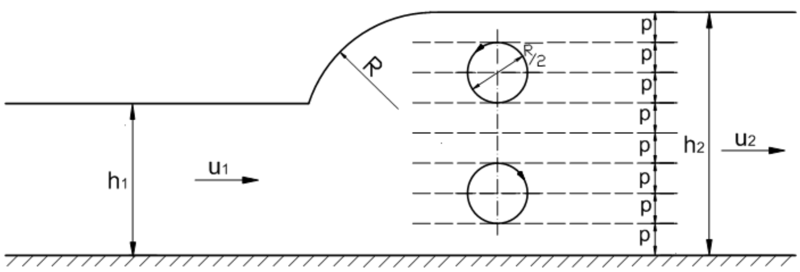

Fig. 2. Schematic geometry of a model of a jump featuring: b) two eddies

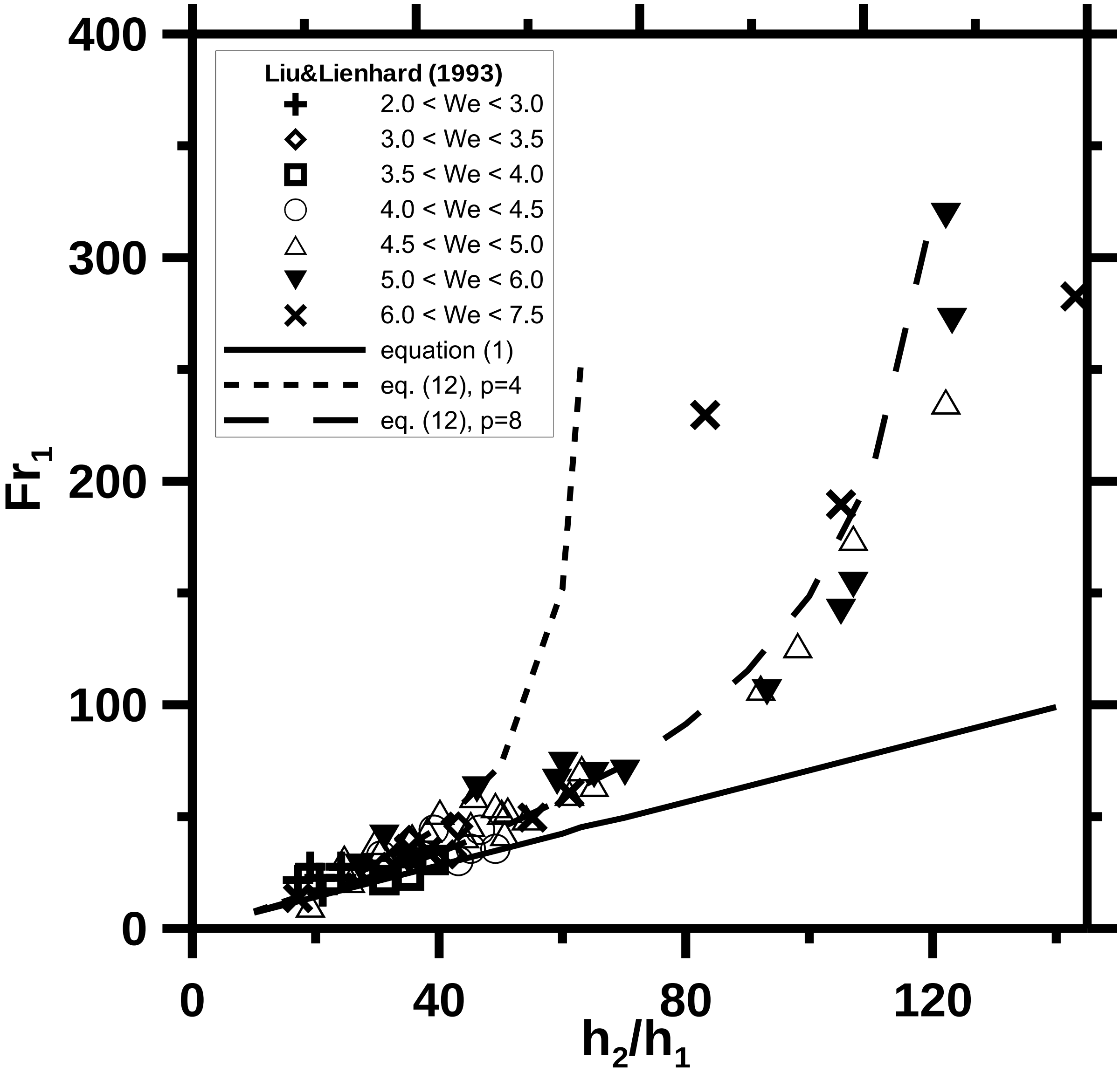

Fig. 3. Froude number before the jump in function of film thickness ratio h_2/h_1.

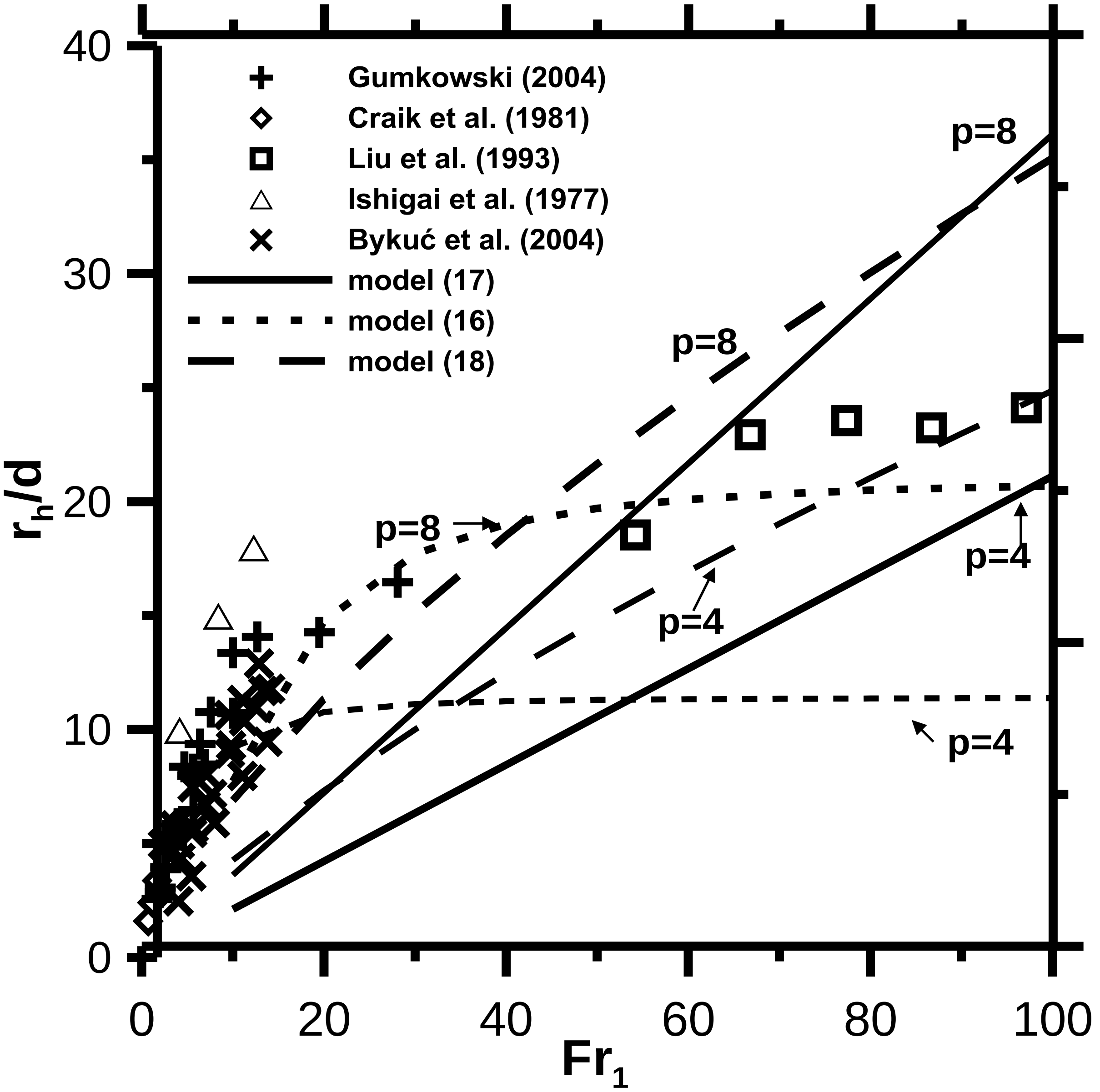

Fig. 4. Hydraulic jump location r_h/d in function of Froude number before the jump Fr_1.