Opuscula Mathematica • Vol. 28 • No. 4 • 2008

Dedicated to the memory of Professor Andrzej Lasota

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THE WORK OF PROFESSOR ANDRZEJ LASOTA ON ASYMPTOTIC STABILITY AND RECENT PROGRESS

Abstract. The paper is devoted to Professor Andrzej Lasota's contribution to the ergodic theory of stochastic operators. We have selected some of his important papers and shown their influence on the evolution of this topic. We emphasize the role A. Lasota played in promoting abstract mathematical theories by showing their applications. The article is focused exclusively on ergodic properties of discrete stochastic semigroups $\{P^n : n \geq 0\}$. Nevertheless, almost all of Lasota's results presented here have their one-parameter continuous semigroup analogs.

Keywords: invariant measure, asymptotic stability, asymptotic periodicity, compact attractor, lower function, smoothing, cell cycle, sweeping, genericity.

Mathematics Subject Classification: Primary 37A30, 47A35; Secondary 45D05, 46B42, 60J20, 92D15.

1. INTRODUCTION

An essential part of Professor Andrzej Lasota's mathematical work concerns asymptotic behaviour of iterates of linear contractions and their stability. The goal of this article is to describe it briefly from the perspective of the past 30 years. We will emphasize the role that Professor A. Lasota played in both abstract operator ergodic theory and its applications to biology, medicine and technology by selecting a few representative results. We will show how his ideas have developed into well established theories. It seems to be an easy task, as his contribution to this topic is enormous. On the other hand, this is not a survey article, neither does it aim to be complete in any sense. Definitely, it might omit some important material due to the author's ignorance, poor memory and the limited time for its preparation. The reader should not expect a comprehensive and updated review of A. Lasota's work in such a restricted volume. Frankly speaking, the choice of the material is very subjective and follows the paths which the author stepped into after the Master.

The paper is organized as follows. After this short Introduction, we proceed to Section 2, where we first provide a historical background for A. Lasota's works. Then we present classical results of A. Lasota on the existence of absolutely continuous invariant measures for piecewise monotonic functions of the unit interval $[0,1]$. In Section 3, we describe original methods introduced by A. Lasota for studying asymptotic stability and periodicity of Markov operators. The most fruitful notions which are still in use are the constrictivity and lower bound techniques. They have been borrowed by numerous pure and applied mathematicians (including the author) and generalized into different directions. We mention A. Lasota's studies on genericity of ergodicity or mixing in the class of semigroups of stochastic operators. Section 4 deals with applications to biology and medicine. We discuss A. Lasota's models of cell cycles based on asymptotic stability of stochastic operators. Final Section 5 brings information on recent progress in the noncommutative versions of the above topics. All papers discussed in that section use, more or less openly, methods introduced by A. Lasota.

2. INVARIANT MEASURES FOR PICEWISE EXPANDING MAPS

Given a measure space $(X, \sum, \mu)$, we say that a transformation $\tau : X \to X$ is *measurable* if $\tau^{-1}(\sum) \subseteq \sum$. The system $(X, \sum, \mu, \tau)$ is called a measurable dynamical (discrete time) system. In order to understand it, we need to know the (asymptotic) behaviour of its (typical) trajectories $\tau^n(x) : n \geq 0, x \in X$. We restrict our studies to the case when τ is *nonsingular* (which means that if $\mu(A) = 0$ then $\mu(\tau^{-1}(A)) = 0$). It follows from the famous Birkhoff Individual Ergodic Theorem that whenever τ is measure μ preserving (i.e., $\mu \circ \tau^{-1} = \mu$), then for each $f \in L^1(\mu)$ the Cesaro averages

$$A_N^\tau f(x) = \frac{1}{N} \sum_{j=0}^{N-1} f(\tau^j(x))$$

converge pointwise (for μ almost all $x \in X$) and in the L^1 norm to $\overline{f}(x)$, where $\overline{f} \in L^1(\mu)$ and is τ–invariant (i.e., $\overline{f} \circ \tau = \overline{f}$). τ is said to be *ergodic* if $\sum_{\text{inv}} = \{A \in \sum : \tau^{-1}(A) = A\} = \{\emptyset, X\}$. For ergodic and measure preserving τ there is $\overline{f}(x) = \frac{1}{\mu(X)} \int_X f d\mu$ for μ almost all $x \in X$ if μ is finite and $\overline{f} = 0$ if μ is infinite. Even in the very classical case of $\tau_\alpha(x) = (x + \alpha)_{\text{mod } 1}$, where $x = [0,1)$, $\sum$ is the σ–algebra of Borel (Lebesgue) subsets of $[0,1)$ and $\mu = \lambda|_{[0,1)}$ (the Lebesgue measure), studying Cesaro means $A_N^\tau f(x)$, and not using the Birkhoff theorem, is a nontrivial task in analysis. It appears that τ_α is $\lambda|_{[0,1)}$ preserving and ergodic if and only if α is irrational. In this case,

$$\lim_{N\to\infty} \frac{\sharp\{k : 0 \leq k < N, \tau_\alpha^k(x) \in [a,b]\}}{N} = b - a$$

for all $x \in [0,1)$ and $0 \le a < b < 1$. What can be said about other transformations of the unit interval like the Renyi transformation $\tau(x) = (rx)_{\mathrm{mod}1}$, where $r > 1$, $\tau(x) = (\ln(x))_{\mathrm{mod}1}$ or

$$\gamma(x) = \begin{cases} \frac{x}{1-x} & \text{for} \quad 0 \le x < \frac{1}{2}, \\ 2x-1 & \text{for} \quad \frac{1}{2} \le x \le 1? \end{cases}$$

As long as we know that a transformation τ preserves measure μ or another *equivalent* measure ν (i.e. $\mu(A) = 0$ if and only if $\nu(A) = 0$) or at least measure ν which is *absolutely continuous* with respect to μ (which means that $\nu(A) = 0$ whenever $\mu(A) = 0$), the Birkhoff Ergodic Theorem describes asymptotic average behaviour of trajectories $\tau^n(x)$ as far as the frequency of visiting measurable sets is concerned. Therefore, the first step towards understanding the system $(X, \sum, \mu, \tau)$ is to answer the question of the existence of τ invariant measures. In general, there is no technique available for studying the asymptotic behaviour of $A_N^\tau f(x)$, even if τ is given by a simple explicit formula. Lifting our transformation τ (it may be very nonlinear) to some linear operator $P_\tau : L^1(\mu) \to L^1(\mu)$ seems to be an artificial step. Luckily it will work.

In 1973, in [65], A. Lasota and J.A. Yorke solved S. Ulam's problem (see [103]): whether transformations $\tau : [0,1] \to [0,1]$ which are piecewise C^2 and $\inf |\tau'| > 1$ admit invariant measures ν which are equivalent to $\lambda|_{[0,1]}$. They openly used techniques from functional analysis. For this, let $P_\tau : L^1(\mu) \to L^1(\mu)$ be defined as a linear operator

$$\int_A P_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu.$$

P_τ is called the Frobenius–Perron operator corresponding to the transformation τ. It can easily be verified that P_τ is *positive* (i.e., $P_\tau f \ge 0$ for $f \ge 0$) and the *integral preserving* (i.e., $\int_X P_\tau f d\mu = \int_X f d\mu$ for all $f \in L^1(\mu)$). It easily follows from the last properties that $\|P_\tau f\|_1 \le \|f\|_1$ (i.e., P_τ is a *contraction*). The adjoint operator $P_\tau^* : L^\infty(\mu) \to L^\infty(\mu)$ (called the Koopman operator) is the composition operator $P_\tau^* h(x) = h(\tau(x))$. Clearly, $P_\tau^n = P_{\tau^n}$. If $f_* \in L^1(\mu)$ is nonnegative and normalized (we call such a function a *density*) and $P_\tau f_* = f_*$, then the measure $\nu_{f_*}(A) = \int_A f_* d\mu$ is τ invariant $(\nu_{f_*}(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f_* d\mu = \int_A P_\tau f_* d\mu = \int_A f_* d\mu = \nu_{f_*}(A))$. This leads us to a conclusion: finding all fixed densities of P_τ means finding all τ invariant absolutely continuous probabilities. In the case of $\tau : [0,1] \to [0,1]$, we have a representation

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f(s) ds.$$

We recall that a transformation $\tau : [0,1] \to [0,1]$ is called *piecewise* $C^{1+\delta}$ (we will consider $\delta > 0$) if there exists a partition $0 = a_0 < a_1 < \cdots < a_p = 1$ of the unit interval such that for each index i the restriction $\tau_i = \tau|_{(a_{i-1}, a_i)}$ is a $C^{1+\delta}$ function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a $C^{1+\delta}$ function, τ need not be continuous at the points a_i (we recall that a function $\tau : [a,b] \to [c,d]$ is called

$C^{1+\delta}, \delta > 0$ if τ is differentiable and $|\tau'(x) - \tau'(y)| \le K|x-y|^\delta$ for all $x, y \in [a,b]$ and some positive K). In [65], A. Lasota and J.A. Yorke proved:

Theorem 1. *Let $\tau : [0,1] \to [0,1]$ be a piecewise C^2 function such that $\inf |\tau'| > 1$. Then for any $f \in L^1(\mu)$ the sequence*

$$\frac{1}{N} \sum_{j=0}^{N-1} P_\tau^j f$$

converges in the norm $\| \cdot \|_1$ to a function $f_* \in L^1(\mu)$. The limit function has the following properties:

1. $f \ge 0 \Rightarrow f_* \ge 0$,
2. $\int_0^1 f_* d\lambda = \int_0^1 f d\lambda$,
3. $P_\tau f_* = f_*$,
4. *the function f_* is of bounded variation; moreover there exists a constant c independent of the choice of initial f such that the variation of the limiting f_* satisfies the inequality $\bigvee_0^1 f_* \le c\|f\|_1$.*

It is also noticed in [65] that the assumption $\inf |\tau'| > 1$ is essential. In particular, this assumption is violated for the transformation γ defined above and it may be proved that it has no invariant absolutely continuous finite measures.

The proof of the above theorem is based on the K. Yoshida and S. Kakutani (see [106]) mean ergodic theorem and the C.T. Ionescu-Tulcea and G. Marinescu (see [37]) spectral decomposition theorem. The observation that iterates $P_\tau^k f$ for f from the unit ball of $L^1(\mu)$ are attracted to a norm $\| \cdot \|_1$ compact subset $\mathcal{F} \subset L^1(\mu)$ is crucial. This motif has further been exploited in a generalized form by A. Lasota, his coauthors, students, collaborators and numerous others (see [20, 22, 24, 25, 32, 35, 68–70, 76–78, 85]). I will specially emphasize the works of Z. Kowalski [50, 51] and [52] on ergodic properties of piecewise expanding transformations of the unit interval, because it was him who brought (in the mid-1970s) this topic from A. Lasota (at that time lecturing in Kraków) to Wrocław. Attending in 1978 Z. Kowalski's lectures on Ergodic Theory, I first heard about the pioneering results of A. Lasota and realized that there was still room for more results in this direction. Extensions to expanding transformations of n–dimensional cubes were intensively studied by M. Jabłoński (see [41]). On the other hand, ergodic properties of transformations of the unit interval with piecewise expanding transformations and with countably many components were studied by M.R. Rychlik (see [90]). His beautiful compact result states that for every transformation $\tau : [0,1] \to [0,1]$ which has the property that on partition intervals $J_i = (a_i, b_i)$ the transformation is expanding ($J_i \cap J_k = \emptyset$ for $i \ne k$, $\overline{\bigcup_{i=1}^\infty J_i} = [0,1]$ and $\inf |\tau'|_{J_i}| \ge a > 1$) and the total variation $\bigvee_U (\frac{1}{|\tau'(x)|}) < \infty$, where $U = \bigcup_{i=1} J_i$, there exists a τ invariant absolutely continuous measure. In this section we have given a very brief account of ergodic theory of piecewise monotonic and expanding transformations of the unit interval. For more detailed accounts, we refer the reader to the books [56] and [19].

The above topic, even though it looks very abstract, provides theoretical background for technological applications. For instance, it was used (see [62] and more recently [23]) for modeling the dynamics of a rotary drill and finally to design its improved version. This was formally acknowledged by technicians who granted A. Lasota a patent. Other applications are mentioned in [19]. Recently, piecewise expanding transformations, and their invariant measures have been used in mathematical modeling of option pricing (see [6]).

3. CONSTRICTORS, ASYMPTOTIC PERIODICITY AND GENERICITY

A measure preserving transformation τ defined on a measure space $(X, \sum, \mu)$ is called *exact* if $\sum_\infty = \bigcap_{n=1}^{\infty}\{\tau^{-n}(A) : A \in \sum\} = \{\emptyset, X\}$. M. Lin proved in [71] that a (doubly measurable and measure preserving) τ is exact if and only if for every $f \in L^1(\mu)$ there is $\lim_{n\to\infty} \|P_\tau^n f - (\int_X f d\mu)\mathbf{1}\|_1 = 0$ (i.e., if P_τ is asymptotically stable). A. Lasota and J.A. Yorke's paper [66] originated a series of publications on exactness and equivalent conditions. They introduced the concept of lower and upper bounds. These ideas have been adapted to a more general setting; therefore, we introduce a modern version here. First of all, instead of the Frobenius-Perron operator, we may consider what is known as *stochastic operators*. Namely, for a σ–finite measure space $(X, \sum, \mu)$, a linear operator $P : L^1(\mu) \to L^1(\mu)$ is called *stochastic* (*Markov*) if it is positive and preserves the integral ($f \in L^1_+(\mu) \Rightarrow Pf \in L^1_+(\mu)$ and $\int_X Pf d\mu = \int_X f d\mu$). The set of all stochastic operators on $L^1(\mu)$ is denoted by $\mathcal{S}$ (clearly $\mathcal{S}$ is a convex unital semigroup of operators, closed in the weak operator topology in $\mathcal{L}(L^1(\mu))$). A stochastic operator P is called *asymptotically stable* if there exists a density f_* such that for all $f \in L^1(\mu)$ there holds $\lim_{n\to\infty} \|Pf - (\int_X f d\mu) f_*\|_1 = 0$ (in other words, P^n converge in the strong operator topology to the one dimensional projection $\mathbf{1} \otimes f_*$). Clearly, if P is asymptotically stable, then the limit density f_* is unique and it is a fixed point (hence ν_{f_*} is τ invariant). A nonzero function $h \in L^1_+(\mu)$ is called *a lower bound* if for each density $f \in L^1(\mu)$ there is $\lim_{n\to\infty} \|(h - P^n f)^+\|_1 = 0$. On the other hand, a nonzero function $h \in L^1_+(\mu)$ such that $\|h\|_1 < 2$ is called *an upper bound* if for each density $f \in L^1(\mu)$ there is $\lim_{n\to\infty} \|(P^n f - h)^+\|_1 = 0$. In [66] the authors proved

Theorem 2. *Let τ be a doubly measurable nonsingular transformation of a σ-finite measure space $(X, \sum, \mu)$. Then τ is exact (the corresponding Frobenius-Perron operator P_τ is asymptotically stable) if and only if P_τ has a lower bound and only if P_τ has an upper bound.*

The notion of lower and upper bounds appeared to be very fruitful, relatively easy to verify and therefore was generalized in many directions (see [33, 34, 53, 63, 84, 86, 93, 97, 98, 109]). We focus on some of them only. The following concept emerges directly. We will say that a stochastic operator P *overlaps supports* if for every two nonzero and positive $f_1, f_2 \in L^1(\mu)$ there exists n such that $P^n f_1 \wedge P^n f_2 \neq 0$. For quite a long time it was an intriguing question (posed by A. Lasota) whether any stochastic operator which overlaps supports and posses an (necessarily unique) invariant density is asymptotically stable. It was disapproved by R. Rudnicki in [89] (compare also [17]). The notion of overlapping in its different versions was discussed in [10]. If there exists $\varepsilon > 0$ such that for every two densities f_1, f_2 there exists n_0 such that $\|P^{n_0} f_1 \wedge P^{n_0} f_2\|_1 \geq \varepsilon$, then for every two densities f_1, f_2 there holds $\lim_{n\to\infty} \|P^n f_1 - P^n f_2\|_1 = 0$ (such operators are called *mixing* $_o$ $\equiv$ the iterates P^n converge to zero in the strong operator topology on $L^1_0(\mu) = \{f \in L^1(\mu) : \int_X f d\mu = 0\}$). ε-overlapping is equivalent to asymptotic stability as long as there exists an invariant density. *Uniform ε-overlapping* means that there exist n_0 and $\varepsilon > 0$ such that for every two densities f_1, f_2 there is $\|P^{n_0} f_1 \wedge P^{n_0} f_2\|_1 \geq \varepsilon$. It can be proved that uniform ε-overlapping implies that a (unique) invariant density f_* does exist and in this case $\lim_{n\to\infty} |||P^n - \mathbf{1} \otimes f_*||| = 0$ (i.e., the iterates P^n converge in the operator norm on $\mathcal{L}(L^1(\mu))$ to the one dimensional projection $\mathbf{1} \otimes f_*$). Moreover, the rate of convergence is exponential.

A version of this notion, the so-called mean lower bound, was recently discussed in [29].

The next paper [54] of A. Lasota, T.Y. Li and J.A. Yorke may be recognized as another milestone in the stability theory of Markov operators. The following important notion has been introduced: given a stochastic operator $P : L^1(\mu) \to L^1(\mu)$, where $(X, \sum, \mu)$ is a fixed σ-finite measure space, we say that P is *strongly constrictive* if there exists a norm compact subset $\mathcal{F} \subseteq \mathcal{D}$ such that for all densities $f \in \mathcal{D}$ there holds

$$\lim_{n\to\infty} \operatorname{dist}(P^n f, \mathcal{F}) = 0.$$

The set $\mathcal{F}$ is called a *norm constrictor* (this concept appeared even earlier in the proof of Theorem 1 in [65]). The following theorem is the main result of [54]

Theorem 3. *Let P be a strongly constrictive stochastic operator on $L^1(\mu)$. Then there exist finite sequences of densities $g_1, g_2, \ldots, g_r$ and nonnegative functions $0 \leq h_1, h_2, \ldots, h_r \leq 1$ such that for any function $f \in L^1(\mu)$ there is*

$$\lim_{n\to\infty} \|P^n(f - \sum_{j=1}^{r} (\int_X f h_j d\mu) g_j)\|_1 = 0.$$

Moreover, the densities $g_1, g_2, \ldots, g_r$ have pairwise disjoint supports and $Pg_i = g_{\alpha(i)}$ for some permutation α of the set of indices $\{1, 2, \ldots, r\}$.

Therefore, the iterates of any strongly constrictive operator P admit a decomposition $P^n f = \sum_{j=1}^{r} (\int_X f h_j d\mu) g_{\alpha^n(j)} + R_n f$, where $\|R_n f\|_1 \to 0$ (this property is called *asymptotic periodicity*).

This result served as a stepping-stone for several later papers ([36,83,93,107,108]). We will mention some of them. First of all, let us call a linear operator $P : \mathfrak{X} \to \mathfrak{X}$, where $(\mathfrak{X}, \|\cdot\|)$ is a fixed Banach space, strongly (uniformly) constrictive if, as above, there exists a norm compact subset $\mathcal{F} \subset \mathfrak{X}$ such that for every vector x from the unit ball $B_1(\mathfrak{X})$ there is $\lim_{n\to\infty} \operatorname{dist}(P^n x, \mathcal{F}) = 0$ ($\lim_{n\to\infty} \sup_{x \in B_1(\mathfrak{X})} \operatorname{dist}(P^n x, \mathcal{F}) = 0$, respectively). It turns out (see [9]) that if $(\mathfrak{X}, \|\cdot\|, |\cdot|)$ is a Banach lattice and T is a positive contraction, then uniform constrictivity is equivalent to quasi-compactness of T and to the convergence of T^{nd} (for some d) in the norm operator topology to a finite dimensional projection. Let us recall that an operator T is *quasi-compact* if it can be approximated in the operator norm by compact operators or if there exists a linear compact operator $Q : \mathfrak{X} \to \mathfrak{X}$ such that for some power n there is $|||T^n - Q||| < 1$. As far as strong constrictivity is concerned, several generalizations of [54] were obtained instantly (compare [9, 36, 75, 92, 104]). We emphasize [11] (see also [74]), where the effect of asymptotic periodicity was obtained for strongly constrictive linear operators acting on ordered F-spaces with the Riesz Decomposition Property.

An essential extension of the notion of strong constrictivity is due to J. Komornik, who in [44] defines a stochastic operator $P : L^1(\mu) \to L^1(\mu)$ to be *weakly constrictive* if there exists a weakly compact subset $\mathcal{F} \subset \mathcal{D}$ such that $\lim_{n\to\infty} \operatorname{dist}(P^n f, \mathcal{F}) = 0$ holds for all $f \in \mathcal{D}$. In [44] it is proved that weakly constrictive stochastic operators are strongly constrictive and therefore asymptotically periodic.

Another notion (see [46, 48]) useful in studying asymptotic behaviour of iterates $P^n f$ of a stochastic operator P is known as *quasi-constrictivity.* Namely, $P : L^1(\mu) \to L^1(\mu)$ is called quasi-constrictive (or *smoothing*) if there exist a measurable set C of finite measure μ and constants $\kappa < 1$, $\delta > 0$ such that for any density $f \in \mathcal{D}$ there exists an integer n_f such that

$$\int_{E\cup C^c} P^n f d\mu \le \kappa$$

whenever $n \ge n_f$ and $\mu(E) \le \delta$. A. Lasota and J. Komornik obtained, in [46]:

Theorem 4. *A stochastic operator $P : L^1(\mu) \to L^1(\mu)$ is asymptotically periodic if and only if it is smoothing.*

Let us call a stochastic operator P *uniformly smoothing* (cf [12]) if there exist a measurable set C of finite measure μ, natural number n_0 and constants $\kappa < 1$, $\delta > 0$ such that for any density $f \in \mathcal{D}$ there is $\sup\limits_{\mu(E)\le\delta} \int_{E\cup C^c} P^{n_0} f d\mu \le \kappa$. It was proved in [12] that a stochastic P is uniformly smoothing if and only if it is quasi-compact (and therefore uniformly constrictive).

Another generalization of constrictivity has been recently applied in papers [94–96] to the problem of supercyclicity. Namely, a power bounded operator T acting on a Banach space $(\mathfrak{X}, \|\cdot\|)$ is said to have an *occasionally attracting* compact set $\mathcal{F} \subset \mathfrak{X}$ (called *an occasional constrictor*) if for each vector $x \in B_1(\mathfrak{X})$ there holds $\liminf_{n\to\infty} \operatorname{dist}(T^n x, \mathcal{F}) = 0$. It follows from [96] that a stochastic operator P (generally any positive linear contraction acting on a Banach lattice) is asymptotically periodic if and only if it has an occasional constrictor.

In paper [59], A. Lasota and J. Myjak addressed the problem of genericity of stochastic operators possessing strictly positive invariant densities. We recall that a subset of a metric space is *residual* if its complement is contained in a set of the first Baire category. A property is said to be *generic* or *generically satisfied* if the elements enjoying it form a residual subset. The semigroup $\mathcal{S}$ of all stochastic operators on $L^1(\mu)$ has been studied quite comprehensively in this regard (see [8, 38–40]).

In [59] the following was proved:

Theorem 5. *The set $\mathcal{S}_*$ of all stochastic operators on $L^1(\mu)$ which are asymptotically stable and have strictly positive densities is a residual subset of $\mathcal{S}$ with the operator norm topology.*

Papers [14, 88] contain further generalizations. Using the notion of uniform ε-overlapping, W. Bartoszek proved in [14] that $\mathcal{S}_*$ is an operator norm dense and strong operator G_δ. On the other hand, in [88] R. Rudnicki provides a general technique for category-type results in subsemigroups of Banach algebras. Interesting results concerning genericity of chaos for piecewise monotone transformations of the unit interval were obtained by J. Piórek in [85], after discussions with Professor Andrzej Lasota as mentioned in the paper. Finally, we mention T. Szarek's paper [98] as it exploits two different ideas coming from A. Lasota (cf. [67]). Namely, using the technique of lower bound and the Fortet-Mourier norm it is proved that the set of asymptotically stable (in this norm) Markov operators having invariant measure with a full support and zero Hausdorff dimension is generic. We hope that the studies of A. Lasota on asymptotic behaviour of Markov operators on Polish (in general non-σcompact) spaces with applications to fractals will be addressed elsewhere, by a specialist more involved in this subject (we merely mention [42, 60, 61, 64, 82, 97, 99]).

4. MARKOV MODELS OF A CELL CYCLE

All the projects and papers of Professor Andrzej Lasota stemmed from practical needs. Many of his papers on stochastic operators include sections devoted to applications. To A. Lasota, stochastic operators (which may be formally treated as abstract linear positive operators acting on AL Banach lattices $L^1(\mu)$) appeared to be a perfect tool to build biological models of proliferating cell populations. Describing these phenomena with stochastic kernels, A. Lasota and M.C. Mackey in [55] originated a long series of interesting papers where biology and pure mathematics intertwine. We recall that a kernel stochastic operator $P : L^1(\mu) \to L^1(\mu)$ is defined as

$$Pf(x) = \int_X k(x,y)f(x)d\mu(y),$$

where $k : X \times X \to \mathbb{R}$ is a *stochastic kernel*, i.e., k is jointly measurable on $X \times X$ and satisfies $k(x,y) \geq 0$ and $\int_X k(x,y)d\mu(x) = 1$ for every $y \in X$. After some time (compare contributions by O. Arina and M. Kimmel [3], again A. Lasota and M.C. Mackey [57], J. Tyrcha [102], G.F. Webb and A. Grabosch [105]) A. Lasota and K. Baron published paper [7], where they extracted the mathematical core of above mentioned models. Namely, stochastic operators $P : L^1([0,\infty)) \to L^1([0,\infty))$ used there are defined by kernels of an abstract form

$$k(x,y) = \begin{cases} -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) & \text{if} \quad 0 \leq y \leq \lambda(x), \\ 0 & \text{otherwise,} \end{cases}$$

where functions $H, Q, \lambda : [0,\infty) \to [0,\infty)$ are assumed to be absolutely continuous. Moreover, they satisfy the following conditions: $H(0) = 1$, $\lim_{x\to+\infty} H(x) = 0$, H is nonincreasing, $Q(0) = \lambda(0) = 0$, $\lim_{x\to+\infty} Q(x) = \lim_{x\to+\infty} \lambda(x) = +\infty$, and Q, λ are nondecreasing. These Volterra like operators will be denoted by LMT.

Papers [31] and [7] are devoted to asymptotic properties of iterates of such operators. They summarize, unify and finally generalize results of J.J. Tyson and K.B. Hannsgen (cf. [100] and [101]) and J. Tyrcha [102]. A. Lasota and H. Gacki proved in [31] that

Theorem 6. *If an LMT operator $P : L^1([0,\infty)) \to L^1([0,\infty))$ satisfies*

$$\liminf_{x\to\infty} H(x) > 1$$

then P is asymptotically stable.

After three years, in A. Lasota and K. Baron's paper [7], there appeared:

Theorem 7. *If P is a LMT kernel operator and there exists an $\alpha \in (0,1]$ such that*

$$\int_0^\infty x^\alpha h(x)dx < \liminf_{x\to\infty}((Q(\lambda(x)))^\alpha - Q(x)^\alpha)$$

then the operator P has a stationary density.

Before formulating another result from this paper, we recall that a stochastic operator $P : L^1((X,\sum,\mu)) \to L^1((X,\sum,\mu))$, where (X, d) is a locally compact Polish metric space and $\sum$ is the Borel σ–algebra on X is called *sweeping* if for any density $f \in \mathcal{D}$ and any compact subset $K \subseteq X$ there is $\lim_{n\to\infty} \int_K P^n f d\mu = 0$.

Theorem 8. *If P is a LMT kernel operator satisfying*

$$\sup_{x\geq x_0}((Q(\lambda(x)))^\beta - Q(x)^\beta < \int_0^\infty x^\beta h(x)dx < \infty$$

for an $x_0 \geq 0$ and $\beta \geq 1$ and $\int\limits_{Q(\lambda(x_0))}^{\infty} h(x)dx > 0$, then P is sweeping.

Nevertheless, the question of the behaviour of iterates $P^n f$ has not been answered satisfactorily. There were several more papers devoted to this class of operators (see [49, 58, 72, 87, 107]). J. Komornik and I. Melicherčik proved (see [47, 73]) that for the class of LMT operators the so-called Foguel alternative holds (i.e., such an operator either possesses an invariant density or it is sweeping). Step by step, we have arrived (cf. [16]; see also [13] and [15]) to the final form. Namely, if P is a LMT operator then for every $f \in L^1([0,\infty))$ and compact subset $K \subset [0,\infty)$ there holds $\lim_{n\to\infty} \int_K P^n f dx = \int_K S f dx$, where $S : L^1([0,\infty)) \to L^1_{\text{fix}}([0,\infty))$ is the projection onto the sublattice of P invariant functions. In other words, LMT operators are always weak* asymptotically stable. Moreover, $P^n f$ converge strongly on $L^1(F)$, where F stands for the center of an operator P (we recall that the center of a stochastic operator is the union of all supports of invariant densities).

5. STOCHASTIC OPERATORS ON NONCOMMUTATIVE SPACES

We conclude the article by mentioning a few relatively fresh noncommutative results directly linked to some ideas from Andrzej Lasota works. Before we reach this point, we must realize that in some applications, describing stochastic/statistical evolution of the system, requires more than objects and theories like density functions, probability measures or spaces $L^1(\mu)$, $L^\infty(\mu)$. Of course, we have in mind quantum physics. Classical probability is replaced by its "noncommutative" counterpart build on C* algebras (specifically von Neumann algebras) and W^* algebras.

For the sake of completeness of this section, we add that asymptotic properties of iterates of generalized stochastic operators acting on ordered vector spaces with norm which is additive on positive elements have been studied by Sh.A. Ayupov, T.A. Sarymsakov and N.P. Zimakov ([5, 91]). For more detailed accounts concerning general quantum dynamical systems, we refer the reader to books [1, 4] and [21].

In order to show how the theory discussed above may be adapted to a new environment, we will briefly introduce necessary notions. Let us start with a background for the most classical von Neumann model of quantum mechanics based on the so-called Schatten classes (a primer for modern models of quantum dynamical systems).

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable (infinite dimensional) complex Hilbert space. As usual, the norm is denoted by $\|\cdot\|$ and the Banach algebra of linear and bounded operators on $(\mathcal{H}, \|\cdot\|)$ is denoted by $\mathcal{L}(\mathcal{H})$. The operator adjoint to $A \in \mathcal{L}(\mathcal{H})$ is denoted by A^*. An operator $A \in \mathcal{L}(\mathcal{H})$ is called Hermitian if $A = A^*$, i.e., $\langle Ax, y\rangle = \langle x, Ay\rangle$ holds for all $x, y \in \mathcal{H}$. Moreover, if $\langle Ax, x\rangle \in [0, \infty)$ holds for all $x \in \mathcal{H}$ then we say that A is positive. Clearly, positive operators on $\mathcal{H}$ form a cone in $\mathcal{L}(\mathcal{H})$, denoted by $\mathcal{L}(\mathcal{H})_+$. Each Hermitian operator A may be uniquely decomposed as $A = A_+ - A_-$ (with $A_+A_- = A_-A_+ = 0$), where A_+ and A_- are called a positive and negative part of A respectively. By $|A|$ we mean $A_+ + A_-$. Obviously, $|A| \in \mathcal{L}(\mathcal{H})_+$ and it is called a modulus of A. The modulus may be equivalently introduced as $|A| = \sqrt{A^*A}$. Having the cone, we introduce in $\mathcal{L}(\mathcal{H})$ a partial order as follows: $A \leq B$ if and only if $B - A \in \mathcal{L}(\mathcal{H})_+$. It is well known that $\mathcal{L}(\mathcal{H})$ endowed with this order is not a (vector) lattice and it does not satisfy the so-called Riesz decomposition property. A general bounded operator A may be written as $A = B + iC = (B_+ - B_-) + i(C_+ - C_-)$, where both B, C are Hermitian. Let us recall that $A \in \mathcal{L}(\mathcal{H})$ is compact if $A(x_n)$ has a (norm) convergent subsequence for each bounded sequence $x_n \in \mathcal{H}$ (or, equivalently, when A is a norm operator limit of finite dimensional operators). The ideal of compact operators on a Hilbert space plays an important role (it is denoted by $\mathcal{C}_0$). We say that an operator $X \in \mathcal{L}(\mathcal{H})$ is trace-class if for each orthonormal basis $e_1, e_2, \dots \in \mathcal{H}$ there is $\sum_{j=1} \langle |X| e_j, e_j\rangle < \infty$. The trace is defined as $\sum_{j=1} \langle X e_j, e_j\rangle$ and it is denoted by $\mathrm{tr}(X)$. Then the functional

$$X \to \mathrm{tr}(|X|) = \|X\|_1$$

defines a norm (stronger than the operator norm). The trace-operators form a two sided ideal in $\mathcal{L}(\mathcal{H})$, which is called the Schatten class 1 and is denoted by $\mathcal{C}_1$. The trace norm is complete on $\mathcal{C}_1$. It may be easily verified that whenever $\mathcal{H}$ is not finite dimensional, $\mathcal{C}_1$ is not closed in the operator norm in $\mathcal{L}(\mathcal{H})$. It is well known that by dual operation $\langle A, X\rangle = \mathrm{tr}(XA)$, where $A \in \mathcal{C}_0$ and $X \in \mathcal{C}_1$, the adjoint space to $(\mathcal{C}_0, \|\cdot\|)$ may be identified with $(\mathcal{C}_1, \|\cdot\|_1)$. Further, the dual space to $(\mathcal{C}_1, \|\cdot\|_1)$ is $(\mathcal{L}(\mathcal{H}), \|\cdot\|)$ (denoted in this context as $\mathcal{C}_\infty$) with dual operation $\langle X, B\rangle = \mathrm{tr}(BX)$, where $B \in \mathcal{C}_\infty$ and $X \in \mathcal{C}_1$. In particular, $\mathcal{C}_1$ is not reflexive. The space $\mathcal{C}_1$ is commonly seen as the noncommutative counterpart of the ℓ^1 space. Since the operators of finite rank are norm dense in $\mathcal{C}_1$, and the Hilbert space $\mathcal{H}$ is separable (by our assumption), $\mathcal{C}_1$ is separable too. The following additivity property (like in the Banach lattice $L^1(\mu)$) of norm $\|\cdot\|_1$ is preserved

$$\forall_{X_1, X_2 \in \mathcal{C}_1} \left(X_1,\ X_2 \geq 0 \ \Rightarrow \|X_1 + X_2\|_1 = \|X_1\|_1 + \|X_2\|_1\right).$$

A positive operator X from $\mathcal{C}_1$ is called a *state* if $\mathrm{tr}(X) = 1$ (they obviously play the role of classical densities). The set of all states is denoted by S. It is easy to verify that S is a convex and closed subset of $\mathcal{C}_1$, in the weak topology (hence, for the both operator and trace norms). Direct verification proves that the set is not closed in the weak* topology (if $\dim \mathcal{H} = \infty$). A bounded linear operator $P : \mathcal{C}_1 \to \mathcal{C}_1$ is said to be *positive* if $P(\mathcal{C}_{1+}) \subseteq \mathcal{C}_{1+}$. A positive operator P is called *stochastic (Markovian)* if for every $X \in \mathcal{C}_{1+}$ there is $\|P(X)\|_1 = \|X\|_1$ (equivalently, we may say that $P(\mathrm{S}) \subseteq \mathrm{S}$). The set of all stochastic operators on $\mathcal{C}_1$ is denoted by $\mathcal{S}$.

Let us give a few examples of stochastic operators. Given a unitary operator U, define $P(X) = U^*XU$ and $Q(X) = UXU^*$. Clearly, both P and Q are stochastic. Moreover, they are invertible isometries. Let V be a linear contraction (onto) of $\mathcal{H}$ such that V^* is isometric. Similarly as above, we define $R(X) = V^*XV$. It is easy to check that R is stochastic (non-invertible in general). It follows that any convex combination

$$\sum_j \alpha_j P_j + \sum_k \beta_k Q_k + \sum_l \gamma_l R_l$$

is stochastic as long as $\alpha_j, \beta_k, \gamma_l \geq 0$ and $\sum_j \alpha_j + \sum_k \beta_k + \sum_l \gamma_l = 1$. A slight modification gives $\int P(s) d\nu(s) \in \mathcal{S}$, whenever each $P(s)$ is in $\mathcal{S}$ and the integral over a probabilistic measure ν is properly defined. The following theorem (see [18], where we use the technique of lower bounds) is fundamental for genericity of mixing.

Theorem 9. *Let P be a stochastic operator on $\mathcal{C}_1$. Then the following conditions are equivalent:*

(i) *there exist a one-dimensional projection $Q_{X_*} \in \mathrm{S}$ (i.e., $Q_{X_*}(X) = tr(X)X_*$ for some $X_* \in \mathrm{S}$) and constants $C > 0$, $0 < a < 1$ such that*

$$|||P^n - Q_{X_*}||| < Ca^n \quad \textit{for} \quad n \in \mathbb{N},$$

(ii) *there exists a one-dimensional projection $Q_{X_*} \in \mathrm{S}$ such that*

$$\lim_{n\to\infty} |||P^n - Q_{X_*}||| = 0,$$

(iii) *for each $\epsilon > 0$, there exists an index n_0 such that for all X_1, $X_2 \in \mathrm{S}$ there holds*

$$||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 < \varepsilon,$$

(iv) *there exists an index n_0 such that*

$$\lambda = \sup_{X_1, X_2 \in \mathrm{S}} ||P^{n_0}(X_1) - P^{n_0}(X_2)||_1 < 2.$$

We say that a stochastic operator $P \in \mathcal{S}$ is *norm mixing* (uniformly stable) if one of the conditions of the above theorem is satisfied for some n and some $\varepsilon < 2$. The family of all norm mixing stochastic operators is denoted by $\mathcal{S}_{nm}$. A state $X \in \mathrm{S}$ *is strictly positive* if for each nonzero $x \in \mathcal{H}$ there is $\langle Xx, x \rangle > 0$ (or, equivalently, the eigenvectors of X span the whole space $\mathcal{H}$, or X is "1-1"). The set of all strictly positive states is denoted by S_+. The set of all norm mixing stochastic operators possessing a strictly positive invariant state is denoted by $\mathcal{S}_{nm+}$. It was proved in [18] by W. Bartoszek and B. Kuna that

Theorem 10. *The set $\mathcal{S}_{nm+}$ is a dense G_δ subset of $\mathcal{S}$ in the norm operator topology.*

In comparison with the norm topology, stochastic operators with iterates converging to one dimensional projections form a meager set in the strong operator topology (compare [39]). If there exists $X_* \in \mathrm{S}$ such that for all $X_1 \in \mathrm{S}$ there holds $\lim_{n\to\infty} \|P^n(X_1) - X_*\|_1 = 0$, then the operator P is called *strong operator topology (s.o.t.) mixing* (asymptotically stable). The set of all s.o.t. mixing stochastic operators is denoted by $\mathcal{S}_{sm}$. We say that a stochastic operator P on $\mathcal{S}$ *is almost mixing* in the strong operator topology if for each pair of states $X_1, X_2 \in \mathrm{S}$ there is $\lim\limits_{n\to\infty} \|P^n(X_1) - P^n(X_2)\|_1 = 0$. The set of all almost mixing operators is denoted by $\mathcal{S}_{sam}$. It is proved in [18] that

Theorem 11. *The set $\{P \in \mathcal{S}_{sam} : P \text{ has no invariant state}\} = \mathcal{S}_{sam} \setminus \mathcal{S}_{sm}$ is a strong operator topology dense G_δ subset of $\mathcal{S}$.*

We conclude the paper with a very brief discussion of recent papers which deal with lower/upper bound, smoothness techniques or asymptotic stability/periodicity of stochastic operators on preduals $\mathcal{M}_*$ to von Neumann algebras.

In [2,26] and [27], the authors extend the notion of constrictors beyond Banach lattices. In [28] E.Yu. Emel'yanov and M.P.H. Wolff adopt constrictivity and smoothness to stochastic semigroups defined on $\mathcal{M}_*$. They study the structure of attractors under a specific property (strong normality) of cones, mainly for stochastic operators defined on C* algebras and preduals of von Neumann algebras. Whenever T is a positive constrictive operator on $\mathcal{M}_*$ with a constrictor of the form $[-y, y] + \kappa B_1(\mathcal{M}_*)$, with $\kappa \in [0, 1)$ and some $y \in \mathcal{M}_{*+}$, T is weakly almost periodic (in particular, the Cesaro means $\frac{1}{N}\sum_{j=0}^{N-1} T^j$ converge in the strong operator topology). These four articles are summarized in [30]. Several results about Cesaro (mean) ergodicity, asymptotic stability, smoothness, constrictivity and asymptotic periodicity of stochastic semigroups on preduals of von Neumann algebras are obtained.

The so-called Dobrushin coefficient of ergodicity (related to Doeblin property or to the uniform ε -overlapping condition) is used in [79, 80] and [81] to obtain a criterion for the asymptotic stability of stochastic operators on $L^1(A, \tau)$ spaces associated with finite von Neumann algebras or finite Jordan algebras. We also mention a paper [43] by A. Katz, who studied asymptotic stability of iterates of automorphisms of an arbitrary von Neumann algebra.

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Received: February 15, 2008.
Accepted: May 28, 2008.