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Universal observable detecting all two-qubit entanglement and determinant-based separability tests

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We construct a single observable, measurement of whose mean value on four copies of an unknown two-qubit state is sufficient to determine unambiguously whether the state is separable or entangled. In other words, there exists a universal collective entanglement witness detecting all two-qubit entanglement. The test is directly linked to a function that characterizes the entanglement quantitatively to some extent. This function is an entanglement monotone under so-called local pure operations and classical communication which preserve local dimensions. Moreover, it provides tight upper and lower bounds for negativity and concurrence. An elementary quantum computing device estimating unknown two-qubit entanglement is designed.

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One of the main challenges of both theoretical and experimental quantum-information theory (QIT) is the determination of entanglement properties of a given state. There is an extensive literature covering the problem of determining the entanglement of a state [1–7]. As one knows from the seminal paper of Peres and Wootters [8], collective measurement on several copies of a system in a given quantum state may provide better results than measurements performed on each copy separately. This fact was reflected in the method of entanglement detection with collective measurements. The method, initiated for pure states [9,10], and then developed for mixed states with the help of quantum networks [11–17] and the concept of collective entanglement witnesses [18], found its first experimental demonstration in coalescence-anticoalescence coincidence experiments [19]. In particular, somewhat surprisingly, it was shown how to estimate and/or even measure the amount of entanglement (concurrence) without prior state reconstruction [11–13]. Recently the method got a new twist thanks to application of collective measurements [20–22] that are directly related to quantum concurrence (see [23]), including the photon polarization-momentum experiments on pure states in distant laboratories paradigm [20]. Recently, collective entanglement witnesses were also shown to lead to easily measurable lower bounds on entanglement [21].

We show that a single observable, if measured on four copies of an unknown two-qubit state, is sufficient for discrimination between entanglement and separability of it. Moreover it can serve limited quantitative purposes. With this aim we explore the two-qubit separability test equivalent to the positive partial transpose (PPT) one [2,24] stating that a state is separable if and only if the determinant of its partially transposed density matrix is non-negative [25,26]. The result, known for a few years, has barely been mentioned in the literature in that form (see, e.g., Ref. [27]) and to our knowledge this is the first time an operative physical meaning has been assigned to it. That is, we introduce a state function, straightforwardly connected to the test, which is a monotone under pure local operations and classical communication (PLOCC) with fixed dimensions (see Refs. [28,30]) and a single collective observable is enough to measure it experimentally. Moreover, it provides tight upper and lower bounds for the two-qubit negativity and concurrence.

Further, we discuss how the result allows us to build a small quantum device implementing a kind of elementary algorithm, namely, detecting entanglement in an unknown two-qubit state. Our method has a significant advantage over prior methods [12,13] as we require only one collective measurement. In comparison to the result of Ref. [21], where a single observable provides a concurrence lower bound which sometimes is not conclusive, we achieved a sharp test that is to some extent quantitative.

We also discuss higher-dimensional and multiparty generalizations. In particular, we find that the reduction criterion [31,32] on composite $2\otimes d$ systems with the map applied to the second subsystem is equivalent to a single determinant condition, and as such can be checked via measurement of a single observable.

Here we discuss the necessary and sufficient condition for two-qubit separability in terms of a determinant of a partially transposed density matrix. The observation follows from the facts given by Sanpera *et al.* [25] and Verstraete *et al.* [26]. Here we prove a more general statement about the reduction criterion, exploiting its equivalence to the PPT test on two qubits. Let us consider the reduction map defined as $\Lambda_r(A)=\mathrm{Tr}(A)\mathbb{1}_d-A$ on any $d\times d$ matrix A with $\mathbb{1}_d$ standing for an identity acting on $\mathbb{C}^d$. The following proposition holds.

Proposition 1. For any $2\otimes d$ state ϱ the reduction criterion with respect to the system B is satisfied if and only if

$$\det\{[I\otimes\Lambda_r](\varrho)\}\geq 0. \tag{1}$$

In particular any two-qubit state is separable if and only if

$$\det\varrho^{\Gamma}\geq 0. \tag{2}$$

Proof. The necessity of the condition is obvious. Let us prove sufficiency. With this aim, we may assume that our $2\otimes d$ state ϱ has nonsingular reduced density matrix $\varrho_A=\mathrm{Tr}_B\varrho$, as otherwise it would be a product state. Applying a local filter $\mathcal{V}_A=(\varrho_A^{-1}/2)^{1/2}$ and utilizing previous observations, one obtains $\det[I\otimes\Lambda_r](\varrho)=[\det(\varrho_A/2)]^2\det\{[I$

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$\otimes \Lambda_r](\tilde{\varrho})\}$, where the state $\tilde{\varrho}$ is a result of local filtering. Now there is an immediate observation that for any positive Λ positivity of $[I\otimes\Lambda](\varrho)$ is *equivalent* to the positivity of the new state being the result of local filtering on system A with arbitrary nonsingular filter $\mathcal{V}_A$. Since we deal with the nonsingular $\mathcal{V}_A$, the original state ϱ violates the reduction criterion iff the state $\tilde{\varrho}$ does. Suppose this is the case. Since the first subsystem of the latter is in a maximally mixed state, i.e., $\tilde{\varrho}_A=(1/2)\mathbb{1}_2$, one easily infers (cf. [15]) that in order to violate the criterion $\tilde{\varrho}$ *must* have one eigenvalue that is greater than one-half. Then the operator $[I\otimes\Lambda_r](\tilde{\varrho})=(1/2)\mathbb{1}_{2d}-\tilde{\varrho}$ clearly has a spectrum with all nonzero values in which only one is negative. This finally gives $\det\{[I\otimes\Lambda_r](\tilde{\varrho})\}<0$ which (as we already mentioned) is equivalent to $\det\{[I\otimes\Lambda_r](\varrho)\}<0$. Thus violation of the reduction criterion by a $2\otimes d$ state on the second subsystem is equivalent to violation of (1).

To prove the second part, we only need to observe that $\det \varrho^{\Gamma}=\det(\mathbb{1}_2\otimes\sigma_y\varrho^{\Gamma}\mathbb{1}_2\otimes\sigma_y)=\det\{[I\otimes\Lambda_r](\varrho)\}$ and recall that the reduction criterion is equivalent to the PPT test on two-qubit states. This concludes the proof.

A question important from an experimental point of view is whether a function of ϱ^{Γ} can serve for *quantitative* purposes. We obtain a partially positive answer.

First we introduce a function defined on $d_1\otimes d_2$ states,

$$\pi_{d_1,d_2}(\varrho)=\begin{cases}0, & \det\varrho^{\Gamma}\geq 0,\\ \sqrt{d_1d_2}\sqrt[d_1d_2]{|\det\varrho^{\Gamma}|}, & \det\varrho^{\Gamma}<0.\end{cases} \tag{3}$$

Let us observe that for $d_1=d_2=d$, we have $\pi_{d,d}(|\psi\rangle)\equiv\pi_d(|\psi\rangle)=d|\det A^{\psi}|^{2/d}$ for any $|\psi\rangle=\sum_{i,j}A^{\psi}_{i,j}|i\rangle|j\rangle$. This leads to $\pi_d(|\psi\rangle)=G_d(|\psi\rangle)$, where G_d is called the G concurrence and is defined as a geometric mean value of Schmidt numbers, scaled by the dimension factor (see [29,30]). It is known to be a monotone under LOCC, not changing the dimensions of the state, and as such is considered as an entanglement measure [28–30]. Below we prove that π_{d_1,d_2} satisfies the monotonicity property under some restricted class of LOCC (invariance under local unitary operations is obvious), namely, the ones for which local operations consist only of a single Kraus operator. We call them pure LOCC. Assuming nonseparability of ϱ we have the following.

Proposition 2. For any PLOCC preserving the dimensions of a state, which transform the initial state ϱ to $\varrho^{(i)}$ with probability p_i the following holds:

$$\sum_i p_i\pi_{d_1,d_2}(\varrho^{(i)})\leq\pi_{d_1,d_2}(\varrho). \tag{4}$$

Proof. Reasoning from [33] (the measure is symmetric under a change of particles) allows us to restrict ourselves to a single measurement on Bob's side, described by the family of completely positive operators $\mathcal{M}_i$ with single Kraus operators (we consider only PLOCC), i.e., acting as $\mathcal{M}_i(\varrho)=\mathbb{1}_{d_1}\otimes M_i\varrho\mathbb{1}_{d_1}\otimes M_i^{\dagger}$. We take the square M_i ($\sum_i M_i^{\dagger}M_i\leq\mathbb{1}_{d_2}$) to satisfy dimensionality preservation. Let $D=d_1d_2$; then since $[\mathcal{M}_i(\varrho)]^{\Gamma_A}=\mathcal{M}_i(\varrho^{\Gamma_A})$, we have

$$\begin{aligned}\sum_i p_i\pi_{d_1,d_2}(\varrho^{(i)}) &= \sqrt{D}\sum_i p_i|\det(1/p_i)[\mathcal{M}_i(\varrho)]^{\Gamma_A}|^{1/D}\\ &= \sqrt{D}\sum_i|\det\mathcal{M}_i(\varrho^{\Gamma_A})|^{1/D}\\ &= \sum_i[\det(\mathbb{1}_{d_1}\otimes M_i^{\dagger}M_i)]^{1/D}\,\pi_{d_1,d_2}(\varrho)\\ &\leq\left(\det\sum_i\mathbb{1}_{d_1}\otimes M_i^{\dagger}M_i\right)^{1/D}\pi_{d_1,d_2}(\varrho)\\ &\leq\pi_{d_1,d_2}(\varrho),\end{aligned}$$

where the inequalities follow from the Minkowski determinant theorem and the normalization condition for M_i.

It may be easily checked that π_{d_1,d_2} is not a monotone in the weak sense in general. It suffices to apply the twirling on entangled Bell-diagonal states, which increases the value of π_{d_1,d_2}.

Let us now focus on two-qubit states. Below we will establish a connection of π_2 with the concurrence C and negativity N [34]. As shown in Ref. [35], the concurrence of a density matrix transformed with a filter $A\otimes B$ changes by the factor $|\det AB|/\mathrm{Tr}(AA^{\dagger}\otimes BB^{\dagger}\varrho)$. As it turns out, π_2 of the state transformed in this way changes identically. Moreover, the filters are known to be sufficient for transformation of any nonsingular two-qubit state to a Bell-diagonal one [36]. It is then enough to check the relation between C and π_2 for these states. Taking the entangled state ϱ to be a mixture of Bell states with probabilities $\{p_i\}_{i=1}^4$, we obtain $\pi_2(\varrho)=\Pi_i\sqrt[4]{|1-2p_i|}$, which with the assumption $p_1\geq p_i$ gives $\pi_2(\varrho)\geq 2p_1-1$. This, however, means that π_2 is bounded from below by C as for Bell-diagonal states it is just equal to the right-hand side of the above inequality. Obviously π_2 provides also an upper bound for negativity as the latter is always less than or equal to C [26]:

$$N(\varrho)\leq C(\varrho)\leq\pi_2(\varrho). \tag{5}$$

One may also provide tight lower bounds on $N(\varrho)$ and $C(\varrho)$ in terms of $\pi_2(\varrho)$. With this aim, notice that $\pi_2(\varrho)=2\sqrt[4]{(1/2)N(\varrho)\lambda_1\lambda_2\lambda_3}$, where the λ_i's are the positive eigenvalues of ϱ^{Γ}. Their product is maximal when they are equally distributed. This observation with the aid of the fact that $\sum_{i=1}^3\lambda_i-N/2=1$ leads us to

$$\pi_2(\varrho)\leq\sqrt[4]{N(\varrho)\left(\frac{N(\varrho)+2}{3}\right)^3}\leq\sqrt[4]{C(\varrho)\left(\frac{C(\varrho)+2}{3}\right)^3}. \tag{6}$$

In conclusion, π_2, although not a full entanglement monotone, quantifies all the two-qubit entanglement in a nontrivial way, providing tight lower and upper bounds for other entanglement measures (see Fig. 1).

Now we address a natural question arising in the context of the results from the previous section: Is a measurement of a determinant of ϱ^{Γ} possible by means of a single observable? Following Ref. [18] we define the collective witness to be a Hermitian operator $W^{(n)}$, whose mean value on n copies of separable ϱ is non-negative, i.e., $\langle\langle W^{(n)}\rangle\rangle_{\varrho^{\otimes n}}:=\mathrm{Tr}(W^{(n)}\varrho^{\otimes n})\geq 0$, and negative on some entangled state. Reformulating this question in terms of the above, we ask if

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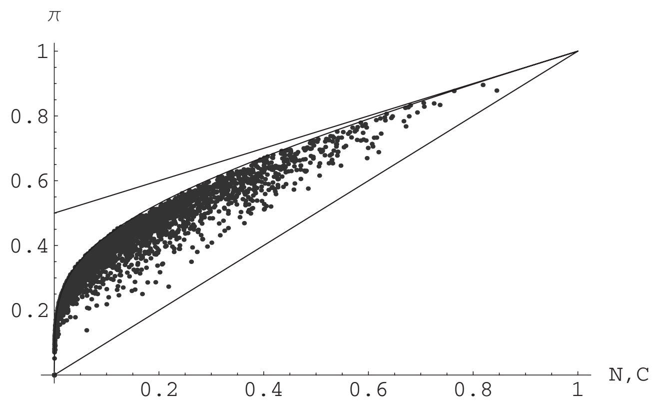

FIG. 1. Plot of π_2 versus N (or C) for randomly generated density matrices with bounds obtained in (5) and (6). The bound $\pi_2 \leq (1/2)(N+1)$ obtained from the geometric-arithmetic inequality applied to the absolute values of the eigenvalues of ϱ^Γ is added. The reader is encouraged to consult [37].

there exists such an observable that $\langle\langle W_{\text{univ}}^{(4)}\rangle\rangle_{\varrho^{\otimes n}} = \det \varrho^\Gamma$. It was shown [38] that any mth-degree polynomial of the elements of ϱ (in particular, its determinant) may be found by determining an expectation value of two observables each on m copies of a state corresponding to the real and imaginary parts of the value of the polynomial. With the guarantee (*a priori* knowledge) that a polynomial is real valued, we need only a single observable (cf. [18]). Obviously the determinant (2) is such a polynomial. Its degree is four so the necessary number of copies is also four. This positively resolves the problem of the existence of a single observable $W_{\text{univ}}^{(4)}$. To find its explicit form, we first introduce the polynomials $\Pi_k(\vec{x}) = \sum_{i=1}^m x_i^k$, which for $\vec{x} = \vec{\lambda}$, a vector consisting of the eigenvalues of a given matrix, are its kth moments. We know that each $\Pi_k(\vec{\lambda})$ is a mean value of an observable $\mathcal{O}^{(k)} = (1/2)(V^{(k)} + V^{(k)\dagger})$ on k copies of ϱ with permutations $V^{(k)}$ defined as $V^{(k)}|\Phi_1\rangle \cdots |\Phi_{k-1}\rangle|\Phi_k\rangle = |\Phi_k\rangle|\Phi_1\rangle \cdots |\Phi_{k-1}\rangle$ $(k=1,\ldots,m)$, with $|\Phi_i\rangle \in \mathcal{H}$.

Now the crucial step is to connect the determinant of a matrix with its easily measurable moments. The Newton-Girard formulas [39] provide us with $\det \varrho^\Gamma = (1/24)[1 - 6\Pi_4(\vec{\lambda}) + 8\Pi_3(\vec{\lambda}) + 3\Pi_2^2(\vec{\lambda}) - 6\Pi_2(\vec{\lambda})]$. Noting that $V^{(k)}$ can be written as $\widetilde{V}^{(k)} \otimes \widetilde{V}^{(k)}$ where $\widetilde{V}^{(k)}$ are permutations on the same subsystems of $\varrho^{\otimes k}$, and using the approach from Ref. [14] we arrive at

$$W_{\text{univ}}^{(4)} = (1/24)\mathbb{1}_{256} - (1/8)(\widetilde{V}^{(4)} \otimes \widetilde{V}^{(4)T} + \widetilde{V}^{(4)T} \otimes \widetilde{V}^{(4)}) + (1/6)\mathbb{1}_4 \otimes (\widetilde{V}^{(3)} \otimes \widetilde{V}^{(3)T} + \widetilde{V}^{(3)T} \otimes \widetilde{V}^{(3)}) + (1/8)V^{(2)} \otimes V^{(2)} - (1/4)\mathbb{1}_{16} \otimes V^{(2)},$$

whose mean value on four copies of ϱ gives $\det \varrho^\Gamma$.

Next we consider the problem of the designation of a network measuring $W_{\text{univ}}^{(4)}$. The issue of avoiding unimportant data (frequency probabilities corresponding to all eigenvalues of the observable) while measuring the observable was considered in Refs. [15,40]. The question about the dimension of ancillas involved in the measurement was answered

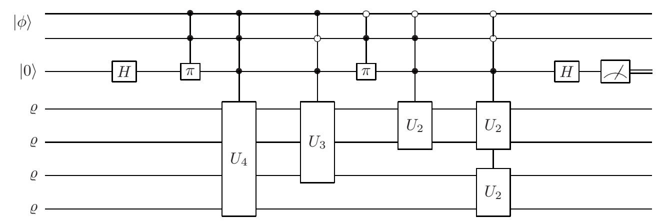

FIG. 2. Network determining entanglement properties of a two-qubit state by a single measurement of $\langle\sigma_z\rangle$ on a control qubit. Here $|\phi\rangle = (1/\sqrt{23})(\sqrt{3}|00\rangle + \sqrt{6}|01\rangle + \sqrt{8}|10\rangle + \sqrt{6}|11\rangle)$; unitary U_i's are combinations of SWAP operations such that $\text{Tr}(U_i \varrho^{\otimes i}) = \Pi_i$. The state would be declared entangled if and only if the measurement yielded a result less than $-1/23$.

in Ref. [41], where it was shown that, via unitary interaction with a single qubit and final measurement of σ_z on it, one can get the mean value of an arbitrary observable with a bounded spectrum. Finally, in Ref. [38] it was shown that interaction between the systems in question and the ancilla can be conducted as a controlled unitary operation. Note that the above single-qubit universality in a mean value estimation is compatible with the further proof that single qubits are in a sense universal quantum interfaces [42].

The most efficient network in number of systems involved involves nine qubits interacting via unitary operations which can be constructed in the way described in [14]. We present here (Fig. 2) an alternative network that requires two more ancillary qubits. However, with these additional systems we achieve simplicity of the structure of the controlled unitary operations, which are just SWAPs. This device shows how one can easily combine mean values of many observables. We do not go into detail concerning the optimality of both networks in number of gates.

Now we discuss the above approach in the context of the entanglement of an arbitrary bipartite state ϱ. Let Λ be a positive, but not completely positive, map. According to Ref. [2], Λ constitutes a necessary separability criterion for bipartite states, which now can be easily reformulated in terms of a determinant.

Fact. Let Λ be a positive map. If $[I \otimes \Lambda](\varrho) \geq 0$ holds, then $\det\{[I \otimes \Lambda](\varrho)\} \geq 0$.

In general, the converse fails, which can be shown by embedding an entangled $2 \otimes 2$ state in a $3 \otimes 3$ space. Note that, as shown in Proposition 1, it is true for reduction applied to the second subsystem of a $2 \otimes d$ system, which is useful in the context of entanglement distillability (see [31]).

Construction of the observable along the lines of Ref. [14] gives the one whose mean value reproduces the desired determinant, i.e., $\langle\langle \widetilde{W}_\Lambda^{(n)}\rangle\rangle_{\varrho^{\otimes n}} = \det\{[I \otimes \Lambda](\varrho)\}$. The idea generalizes immediately to the multiparty case, where maps positive on product states [43] are involved.

In conclusion, we have constructed a single-observable test that detects entanglement of an unknown two-qubit state. In addition, the function corresponding to it provides bounds for the negativity and concurrence. We have also designed a quantum network that can also be interpreted as a quantum computing method that solves quantitatively a problem with quantum data structure (cf. [11]).

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Some research toward higher-dimensional generalizations has been initialized; however, the results suffer from a lack of sufficient character. Nevertheless, a very natural question arises: is there a way to generalize the main result, i.e., to find a single collective observable detecting entanglement of other $d_1 \otimes d_2$ quantum systems without ambiguity (see, e.g., Ref. [44] for some partial results). One would first need a counterpart of the analytical criterion (2), the existence of which is a long-standing open problem in QIT. The first question could be whether there exists a positive map which, applied to one subsystem of any bipartite density matrix, produces a full-rank matrix with an odd number of negative eigenvalues so that the criterion based on the determinant remains true.

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