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Study of free convective boundary layer of isothermal lateral surface of axisymmetrical horizontal body

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ABSTRACT

Approximate analytical solution of simplified Navier–Stokes and Fourier–Kirchhoff equations describing free convective heat transfer from isothermal surface has been presented. It is supposed that the surface has the horizontal axis of symmetry and its axial cross-section lateral boundary is a concave function. The equation for the boundary layer thickness is derived for typical for natural convection assumptions. The most important are that the convective fluid flow is stationary and the normal to the surface component of velocity is negligibly small in comparison with the tangential one. The theoretical results are verified by two characteristic cases of the revolution surfaces namely for horizontal conic and vertical round plate. Both limits of presented solution coincide with known formulas.

1. Introduction

The applied mathematical modeling of the convective heat transfer phenomena is traditional based from the celebrated papers of Prandtl and Schlichting on a notion of boundary layer [1]. The introduction of boundary layer theory allowed to solve this complicated (especially for natural convective) problem of heat transfer in terms of the functions that describes the boundary layer form and the velocity and temperature profiles across the layer. Such approach was very useful to determine convective heat losses from apparatus, devices, pipes in industrial or energetic installations, electronic equipment, architectonic objects and so on by engineers and designers. It especially important now in microfluidics phenomena and need a study of bodies with various configurations and high precision [2].

The results of theoretical and experimental study of free convective heat transfer from different configurations of heating objects are widely published. From the analysis of literature data it is obvious that heat transfer from the objects' surfaces is described mainly by Nusselt–Rayleigh relations $Nu = CRa^n$ with constants C and exponents n individual for each cases of surface. In the review Churchill's paper [3] among about 120 theoretical and experimental results only a few positions of rotational surfaces (spheres, hemispheres, horizontal cylinders). For such surfaces may also be included a vertical round plate investigated by Lewandowski et al. [4] and horizontal conic experimentally studied by Oosthuizen [5]. At our recent papers [6,7] free convective boundary layer on isothermal horizontal cone have been studied theoretically and experimentally as well. Such cone is a specific case of a rotational body as for example the mentioned above vertical round plate, hemisphere with horizontal axis of symmetry, hemispheroid and so on.

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In this paper we are going to consider a convective heat transfer from general concave rotational isothermal surface with horizontal axis of symmetry. Such generalization is based on the same geometric idea that was used in the case of cone: we build the curves on the heated surface along which the boundary layer develops. Such curves is defined by the resulting force (the sum of gravitation, buoyancy and reaction) which moves the liquid elements in each point near the surface.

We use this same basic equations and assumptions typical for the natural convection heat transfer from isothermal surfaces in the steady condition of a convective fluid flow as in [6]. A solution of such problem is generated on the base of energy and momentum balance on control fluid volume restricted by surface built by the coordinate curves.

The main result of our paper is the derivation of the boundary layer thickness equation which is the ordinary differential equation in the variable along the curve described above. Each such curve is marked by its initial cylindrical coordinate (boundary layer starting point) that enter the resulting equation as a parameter. This resulting equation is rather easy for a numerical treatment.

2. Geometry

Assume the cylindrical variables z', ρ, φ are defined so that the Cartesian variable $z = z'$ coincides with cylindrical one and for others the relations $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$ are valid.

Due to the physical symmetry with respect to the reflection of $y \to -y$, instead of the variable $\varphi \in (0, 2\pi)$ we introduce the variable $\varepsilon \in (-\pi/2, \pi/2)$ that is $\varepsilon = -\varphi + \pi/2$ (see Fig. 1).

Let us define the surface Σ by the function (one-to-one correspondence) $\rho = \rho(z)$, $z \in [0, h]$. Assume that a circumference $z = h$ creates the edge of the heated isothermal surface Σ of temperature T_w. For example for the cylinder one has

$$\rho = R, \tag{1}$$

where R is the radius of the cylinder base; for the cone:

$$\rho = z\cot\alpha, \tag{2}$$

where α is the cone angle. The spherical segment case $z < h$ is determined by its radius R

$$\rho = \sqrt{z(2R - z)}. \tag{3}$$

The Cartesian coordinates (x,y,z) on the surface are expressed in the terms of the cylindrical ones (z,ρ,ε), the notation of the axisymmetrical coordinate z is preserved

$$x = \rho(z)\sin\varepsilon, \quad y = \rho(z)\cos\varepsilon. \tag{4}$$

We use the variables (z, ε) as ones that define a point on the surface. At arbitrary point M of the lateral surface Σ one may distinguish two tangent unit vectors τ_z and τ_ε

$$\tau_z = \frac{\partial \mathbf{r}}{\partial z}, \quad \tau_\varepsilon = \frac{\partial \mathbf{r}}{\partial \varepsilon} \tag{5a}$$

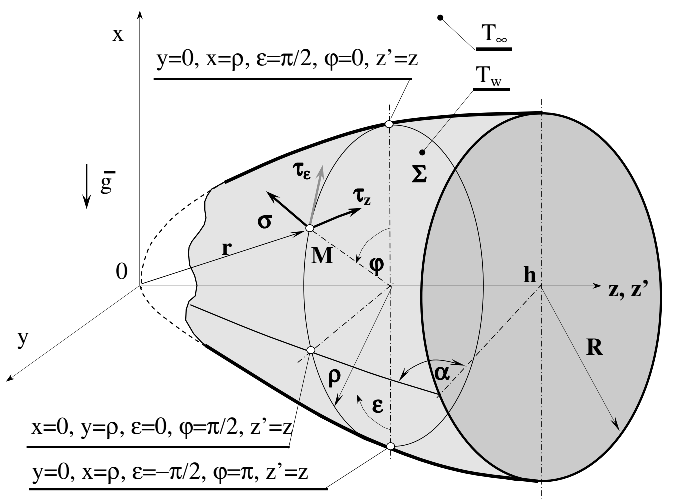

Fig. 1. Coordinate system of free convective heat transfer from isothermal rotational surface with horizontal axis of symmetry.

and one normal σ to surface

$$\sigma = \frac{\tau_z \times \tau_\varepsilon}{|\tau_z \times \tau_\varepsilon|}, \tag{6}$$

where $\mathbf{r} = (x, y, z) \in \Sigma$.

The Cartesian coordinates of the vectors τ_ε, τ_z, σ are

$$\tau_{zx} = \frac{\partial x}{\partial z} = \rho'(z)\sin\varepsilon, \quad \tau_{zy} = \frac{\partial y}{\partial z} = \rho'(z)\cos\varepsilon, \quad \tau_{zz} = \frac{\partial z}{\partial z} = 1 \tag{7}$$

$$\tau_{\varepsilon x} = \frac{\partial x}{\partial \varepsilon} = \rho(z)\cos\varepsilon, \quad \tau_{\varepsilon y} = \frac{\partial y}{\partial \varepsilon} = -\rho(z)\sin\varepsilon, \quad \tau_{\varepsilon z} = \frac{\partial z}{\partial \varepsilon} = 0 \tag{8}$$

$$\sigma_x = \frac{\sin\varepsilon}{\sqrt{1 + (\rho'(z))^2}}, \quad \sigma_y = \frac{\cos\varepsilon}{\sqrt{1 + (\rho'(z)^2}}, \tag{9}$$

$$\sigma_z = \frac{-\rho'(z)}{\sqrt{1 + (\rho'(z))^2}}, \tag{10}$$

where $\rho'(z) = \rho'$ is the derivative $d\rho/dz$.

The vector of gravity acceleration (Fig. 2) in our coordinates is

$$g_x = -g, \quad g_y = 0, \quad g_z = 0. \tag{11}$$

It is convenient to built the local coordinate system by the three orthogonal vectors, one - normal to the surface σ, next – τ, which is based on the gravitational vector $\mathbf{g}$ with the extracted projection to σ

$$\tau = \frac{\mathbf{g} - \mathbf{g}_\sigma}{|\mathbf{g} - \mathbf{g}_\sigma|} = \frac{\mathbf{g} - (\mathbf{g}, \sigma)\sigma}{|\mathbf{g} - (\mathbf{g}, \sigma)\sigma|}. \tag{12}$$

Cartesian coordinates of the τ are correspondingly

$$\begin{aligned} \tau_x &= -\frac{\sqrt{\rho'^2 + \cos^2\varepsilon}}{\sqrt{1 + \rho'^2}}, \\ \tau_y &= \frac{1}{\sqrt{\rho'^2 + \cos^2\varepsilon}\sqrt{1 + \rho'^2}}\sin\varepsilon\cos\varepsilon, \\ \tau_z &= -\frac{1}{\sqrt{\rho'^2 + \cos^2\varepsilon}\sqrt{1 + \rho'^2}}(\sin\varepsilon)\rho'. \end{aligned} \tag{13a}$$

The third vector of the local basis is chosen as $\xi = [\sigma \times \tau]$.

Let us remark that the gravitation vector $\mathbf{g}$ belongs to the plane, built by the vectors σ, τ so decomposition of the gravity according to these coordinates gives two components of gravity force along unit vectors σ and τ that acts in normal and in tangent direction to the lateral surface (see Fig. 2).

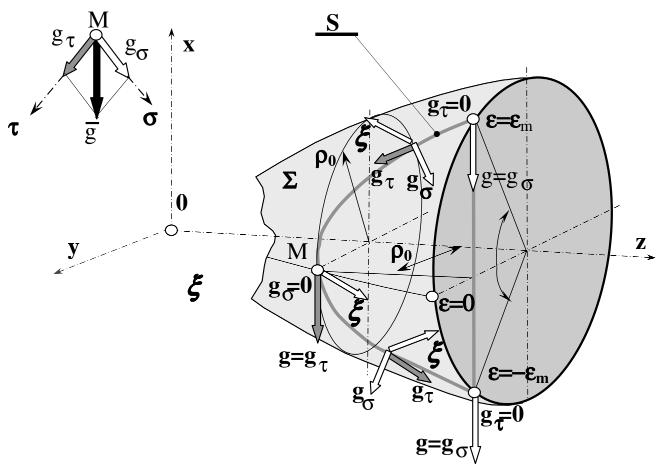

Fig. 2. Construction of curve S on considered rotational surface with horizontal axis of symmetry.

Let us define a curve S on the surface Σ, as a curve for which the unit vector τ is tangent. This curve is created by the cross-section of the surface Σ with a vertical surface of the gravity vector action. At any point M the components of gravity g_σ and g_τ are the normal and tangent ones to the curve S.

3. Physical model

We consider and solve the stationary problem for equations: Navier–Stokes, Fourier–Kirchhoff and continuity with respect to the three mentioned characteristic directions σ, τ and ξ. We do not consider the projection of the Navier–Stokes equations over the direction of ξ because the gravity force component in this direction is zero. Hence, we consider the fluid flow over the heated surface along the curve S only.

In these notations and after typical for natural convection assumptions such as [8]

- temperature of the considered surface Σ is constant and equal T_w,
- temperature of the fluid outside the disturbed region T_∞ is also constant,
- physical parameters a, ρ_f, v, β, λ of the fluid inside the boundary layer are taken as constant; in comparison with experiment it is taken at average temperature $T_{av} = (T_w + T_\infty)/2$,
- fluid is incompressible and its flow is laminar,
- inertia terms are negligible in comparison with viscosity ones,
- thicknesses of the thermal and hydraulic boundary layers are the same,
- tangent component of the velocity inside the boundary layer is significantly larger than normal one $W_\tau \gg W_\sigma$.

The last assumption is not valid for two marginal regions: first one is where the boundary layer arises at $\varepsilon = -\pi/2$ and second where the layer is transformed into the free buoyant plum at $\varepsilon = \pi/2$.

On the bases of such assumptions the Navier–Stokes equations may be written as

$$v\frac{\partial^2 W_\tau}{\partial\sigma^2} - g_\tau\beta(T - T_\infty) - \frac{1}{\rho_f}\frac{\partial p}{\partial\tau} = 0, \tag{14}$$

$$-g_\sigma\beta(T - T_\infty) - \frac{1}{\rho_f}\frac{\partial p}{\partial\sigma} = 0, \tag{15}$$

where $W_\tau = (\tau, \mathbf{W})$, $g_\tau = (\tau, \mathbf{g})$, $g_\sigma = (\sigma, \mathbf{g})$ are components of velocity and gravitation acceleration vectors in the direction of the unit vectors τ and σ. The corespondent components of pressure gradient are denoted as $\partial p/\partial\tau$, $\partial p/\partial\sigma$. From the Eqs. (6)–(12), respectively, one can found the gravity acceleration vector components:

$$g_\sigma = -\frac{g\sin\varepsilon}{\sqrt{(1+\rho'^2)}}, \quad g_\tau = \frac{g\sqrt{(\rho'^2 + \cos^2\varepsilon)}}{\sqrt{(1+\rho'^2)}}. \tag{16}$$

We assumed that relation for temperature distribution inside boundary layer can be used as solution of Fourier–Kirchhoff equation [8]:

$$\Theta = \frac{T - T_\infty}{T_w - T_\infty} = \left(1 - \frac{\sigma}{\delta}\right)^2, \tag{17}$$

or

$$T - T_\infty = \Delta T\left(1 - \frac{\sigma}{\delta}\right)^2. \tag{18}$$

Plugging, Eqs. (16) and (17) into Eqs. (14) and (15) gives

$$v\frac{\partial^2 W_\tau}{\partial\sigma^2} - g\beta\Delta T\left(1 - \frac{\sigma}{\delta}\right)^2\frac{\sqrt{(\rho'^2 + \cos^2\varepsilon)}}{\sqrt{(1+\rho'^2)}} - \frac{1}{\rho_m}\frac{\partial p}{\partial\tau} = 0, \tag{19}$$

$$-g\beta\Delta T\left(1 - \frac{\sigma}{\delta}\right)^2\frac{\sin\varepsilon}{\sqrt{(1+\rho'^2)}} - \frac{1}{\rho_m}\frac{\partial p}{\partial\sigma} = 0. \tag{20}$$

4. Transformations of basic equations

Integration of the Eq. (20) for the boundary condition $p(\tau, \sigma) = p_\infty$ at $\sigma = \delta$ gives a formula for the pressure distribution across the boundary layer

$$p = -p_\infty - g\rho_m\beta\Delta T\left(\sigma - \frac{\sigma^2}{\delta} + \frac{\sigma^3}{3\delta^2} - \frac{\delta}{3}\right)\frac{\sin\varepsilon}{\sqrt{(1+\rho'^2)}}. \tag{21}$$

Pressure p_∞ represents the excess of pressure over the hydrostatic pressure outside the boundary layer. Because our considerations are concerned with unlimited space the value of this pressure p_∞ is constant.

Differentiating of the Eq. (21) with respect to τ along the curves S gives

$$\frac{\partial p}{\partial \tau} = \frac{\partial p}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \tau}, \tag{22}$$

where

$$\frac{\partial p}{\partial \varepsilon} = -\frac{g\rho_m\beta\Delta T}{\sqrt{(1+(\rho')^2)}}\left[\left(\sigma - \frac{1}{\delta(\varepsilon)}\sigma^2 + \frac{1}{3\delta^2(\varepsilon)}\sigma^3 - \frac{1}{3}\delta(\varepsilon)\right)\cos\varepsilon + \left(\frac{1}{\delta^2(\varepsilon)}\sigma^2 - \frac{2}{3\delta^3(\varepsilon)}\sigma^3 - \frac{1}{3}\right)(\sin\varepsilon)\frac{\partial\delta(\varepsilon)}{\partial\varepsilon}\right]. \tag{23}$$

Calculation of the derivative $\partial\varepsilon/\partial\tau$ along the curve S needs some explanations. Let us remind that the vector τ is tangent in each point of the curve S from the starting point $z = h, \varepsilon = \varepsilon_{\min}$ until the final point $z = h, \varepsilon = \varepsilon_{\max}$. Introduction of Eqs. (13a) and (4) creates Cartesian coordinates of a point on the curve S in three dimensional space. Differentiating of Eq. (4) gives

$$\frac{dx}{d\varepsilon} = \frac{d\rho}{dz}\frac{dz}{d\varepsilon}\sin\varepsilon + \rho\cos\varepsilon, \tag{24}$$

$$\frac{dy}{d\varepsilon} = \frac{d\rho}{dz}\frac{dz}{d\varepsilon}\cos\varepsilon - \rho\sin\varepsilon. \tag{25}$$

The ratio of Eq. (24) and τ_x Eq. (13a) is equal to the ratio of Eq. (25) and τ_y Eq. (13a), that give the equation for $\frac{dz}{d\varepsilon}$

$$(\rho'^2 + \cos^2\varepsilon)\left(\frac{d\rho}{dz}\frac{dz}{d\varepsilon}\cos\varepsilon - \rho\sin\varepsilon\right) + \sin\varepsilon\cos\varepsilon\left(\frac{d\rho}{dz}\frac{dz}{d\varepsilon}\sin\varepsilon + \rho\cos\varepsilon\right) = 0,$$

or

$$\frac{dz}{d\varepsilon} = \rho\frac{\rho'}{\rho'^2+1}\tan\varepsilon = F(z)\tan\varepsilon, \tag{26}$$

where $F(z) = \frac{\rho\rho'}{\rho'^2+1}$ is the function that depends only on the surface shape. For example for cylinder $F(z) = 0$, for conic $F(z) = z\cos^2\alpha$ and for the spherical segment $F(z) = (-R+z)z\frac{-2R+z}{R^2}$.

The integral of the Eq. (26) is obtained by variables division:

$$\int_h^z \frac{dz'}{F(z')} = \int_{-\varepsilon_m}^{\varepsilon} \tan\varepsilon' d\varepsilon' = -\ln(\cos\varepsilon) + \ln(\cos\varepsilon_m). \tag{27}$$

The solution of the Eq. (27) is the function $z(\varepsilon)$ that defines the curve S. In the trivial case of the cylinder $z(\varepsilon) = 0$, while the cone gives

$$\int_h^z \frac{dz'}{z'\cos^2\alpha} = \frac{\ln z - \ln h}{\cos^2\alpha} = -\ln(\cos\varepsilon) + \ln(\cos\varepsilon_m), \tag{28}$$

or

$$z = h\left(\frac{\cos\varepsilon_m}{\cos\varepsilon}\right)^{\cos^2\alpha}, \tag{29}$$

and for the segment of the sphere

$$\int_h^z \frac{1}{(-R+z)z\frac{-2R+z}{R^2}}dz = \ln\frac{1}{-R+z}\sqrt{z}\sqrt{(-2R+z)}\frac{-R+h}{\sqrt{h}\sqrt{(-2R+h)}} = -\ln(\cos\varepsilon) + \ln(\cos\varepsilon_m). \tag{30}$$

After simplification

$$\frac{1}{-R+z}\sqrt{z}\sqrt{(-2R+z)}\frac{-R+h}{\sqrt{h}\sqrt{(-2R+h)}} = \frac{\cos\varepsilon_m}{\cos\varepsilon}. \tag{31}$$

The derivative of the pressure in the direction τ (12) may be evaluated by means of the connection:

$$d\tau^2 = dx^2 + dy^2 + dz^2, \tag{32}$$

where dx, dy and dz are taken from Eqs. (24)–(26). The link Eq. (24) gives

$$d\tau = d\varepsilon\sqrt{\left(\frac{d\rho}{dz}\frac{dz}{d\varepsilon}\sin\varepsilon + \rho\cos\varepsilon\right)^2 + \left(\frac{d\rho}{dz}\frac{dz}{d\varepsilon}\cos\varepsilon - \rho\sin\varepsilon\right)^2 + \left(\frac{dz}{d\varepsilon}\right)^2}.$$

Simplifying yields

$$\frac{d\varepsilon}{d\tau} = \frac{1}{\rho\sqrt{\left[\frac{(\rho')^2}{(\rho')^2+1}\tan^2\varepsilon + 1\right]}}. \tag{33}$$

In the case of a cone where $\rho = z\cot\alpha$ one has

$$\frac{d\varepsilon}{d\tau} = \frac{\sin\alpha\cos\varepsilon}{z(\cos\alpha)\sqrt{1-\sin^2\alpha\sin^2\varepsilon}}. \tag{34}$$

From Eqs. (27) and (28) the explicit expression for the variable z as a function of ε on the curve $z = h\left(\frac{\cos\varepsilon_m}{\cos\varepsilon}\right)^{\cos^2\alpha}$, finally

$$\frac{d\varepsilon}{d\tau} = \left(\frac{\cos\varepsilon}{\cos\varepsilon_m}\right)^{\cos^2\alpha}\frac{\sin\alpha\cos\varepsilon}{h(\cos\alpha)\sqrt{1-\sin^2\alpha\sin^2\varepsilon}}. \tag{35}$$

Introduction of the result (33) into the Eq. (22) gives

$$\frac{\partial p}{\partial\tau} = -\frac{g\rho_f\beta\Delta T\cos\varepsilon}{\rho\sqrt{\cos^2\varepsilon+(\rho')^2}}\left[\left(\sigma - \frac{\sigma^2}{\delta(\varepsilon)} + \frac{\sigma^3}{3\delta^2(\varepsilon)} - \frac{\delta(\varepsilon)}{3}\right)\cos\varepsilon + \left(\frac{\sigma^2}{\delta^2(\varepsilon)} - \frac{2\sigma^3}{3\delta^3(\varepsilon)} - \frac{1}{3}\right)(\sin\varepsilon)\frac{\partial\delta(\varepsilon)}{\partial\varepsilon}\right]. \tag{36}$$

5. Control volume energy balance: boundary layer thickness equation derivation

Plugging the relation Eq. (36) into Eq. (19) and double integration with boundary conditions ($W_\tau = 0$ when: $\sigma = \delta$, 0) after averaging across the boundary layer yields

$$\overline{W_\tau} = \frac{1}{\delta}\int_0^\delta W_\tau d\sigma = \frac{g\beta\Delta T\delta^2}{\nu\sqrt{((\rho')^2+\cos^2\varepsilon)}}\left(\frac{(\cos\varepsilon)\delta}{180\rho} + \frac{\partial\delta(\varepsilon)}{\partial\varepsilon}\frac{(\cos\varepsilon)(\sin\varepsilon)}{72\rho} - \frac{((\rho')^2+\cos^2\varepsilon)}{40\sqrt{(1+(\rho')^2)}}\right), \tag{37}$$

where $\rho = \rho(z(\varepsilon))$ is defined via the solution of the Eq. (27) on the curve S.

Taking into account the choice of the unit vector τ the change in mass flow intensity is

$$dm = -d(A\cdot\overline{W_\tau}\cdot\rho_m), \tag{38}$$

where (A) is the cross-section area of the boundary layer (see Fig. 3).

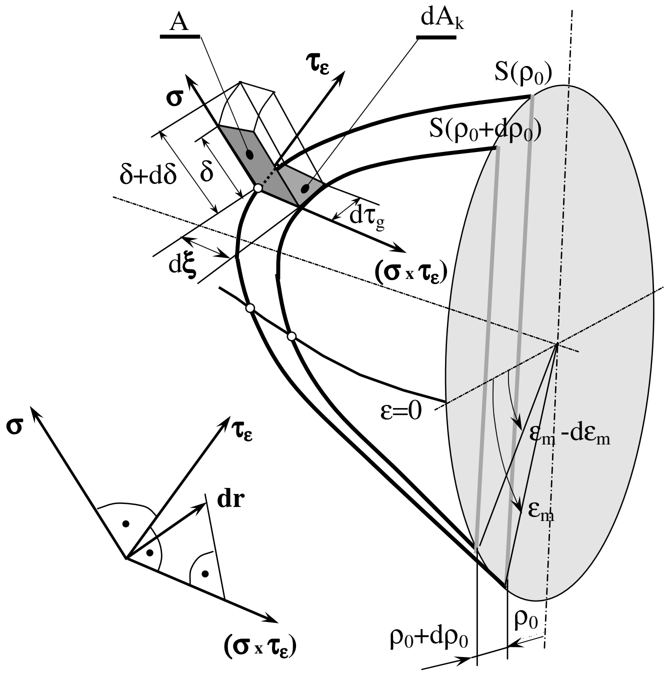

Fig. 3. Graphical explanation of estimation control surfaces A and dA_k on considered heated surface.

The amount of heat necessary to create this change in mass flux is

$$dQ = \Delta i \cdot dm = -\rho_m \cdot c_p \cdot (\overline{T - T_\infty}) d(A \cdot \overline{W_\tau}). \tag{39}$$

Substitution of the mean value of the temperature

$$(\overline{T - T_\infty}) = \frac{1}{\delta} \int_0^\delta \Delta T \cdot \left(1 - \frac{\sigma}{\delta}\right)^2 d\sigma = \frac{\Delta T}{3} \tag{40}$$

gives

$$dQ = -\frac{1}{3} \rho_m \cdot c_p \cdot \Delta T \cdot d(A \cdot \overline{W_\tau}). \tag{41}$$

The heat flux described by Eq. (41) should be equal to the heat flux determined by Newton's Eq. (42)

$$dQ = \alpha \cdot \Delta T \cdot dA_k = -\lambda \cdot \left(\frac{\partial \Theta}{\partial \sigma}\right)_{\sigma=0} \cdot \Delta T \cdot dA_k, \tag{42}$$

where dA_k is the control surface of the body boundary (see Fig. 3).

From simplifying assumption of the temperature profile inside boundary layer (17), the dimensionless temperature gradient on the heated surface may be evaluated as

$$\left(\frac{\partial \Theta}{\partial \sigma}\right)_{\sigma=0} = -\frac{2}{\delta}. \tag{43}$$

Substitution Eq. (43) into Eq. (42) and equating the result with Eq. (41), one obtains the control volume energy balance equation

$$\frac{1}{6\lambda} \cdot \rho_m \cdot c_p \cdot \delta \cdot d(A \cdot \overline{W_\tau}) = -dA_k. \tag{44}$$

Derivation of formulas for the cross-sectional area and the control surfaces A and dA_k is pictorial shown on Fig. 3 as it is presented below.

As one can see in Fig. 3 for the both control surfaces A and A_k the differential width $d\xi$ is the scalar product of

$$d\xi = |[\sigma \times \tau] \cdot d\mathbf{r}|, \tag{45}$$

where the vector product of normal (6) and tangent (12) to the curve on the surface vectors is

$$[\sigma \times \tau] = \frac{\mathbf{j}(\rho') + \mathbf{k} \cos \varepsilon}{\sqrt{\cos^2 \varepsilon + \rho'^2}}. \tag{46}$$

Differentiation of three dimensional coordinate vector ($r = r(x,y,z)$) on the surface with constant ε and account of the relations (4) and the Eq. (27) leads to the expression for the differential form of this vector

$$d\mathbf{r} = \frac{dz}{d\varepsilon_m}((\rho') \sin \varepsilon, (\rho') \cos \varepsilon, 1) d\varepsilon_m = F(z) \tan \varepsilon_m ((\rho') \sin \varepsilon, \rho' \cos \varepsilon, 1) d\varepsilon_m. \tag{47}$$

Plugging (46) and (47) into (45) gives

$$d\xi = F(z) \tan \varepsilon_m (\cos \varepsilon) \frac{1 + (\rho')^2}{\sqrt{((\rho')^2 + \cos^2 \varepsilon)}} d\varepsilon_m. \tag{48a}$$

Using the definition of F Eq. (26) the result allow to find the width of the control surfaces Eq. (48a) and next the relations for the cross-sectional area A

$$A = d\xi \cdot \delta = \frac{\rho \rho' \delta}{\sqrt{((\rho')^2 + \cos^2 \varepsilon)}} \tan \varepsilon_m \cdot \cos \varepsilon \cdot d\varepsilon_m. \tag{49}$$

Similar expression may be obtained using Eq. (33) for the control surface dA_k

$$dA_k = d\xi \cdot d\tau = \frac{\rho^2 \rho'}{\sqrt{(1 + \rho'^2)}} \tan \varepsilon_m \cdot d\varepsilon \cdot d\varepsilon_m. \tag{50}$$

Substituting the Eqs. (37), (49) and (50) into Eq. (44) leads to the nonlinear differential equation for boundary layer thickness $\delta = \delta(\varepsilon)$

$$\frac{g\beta\Delta T}{6\lambda\nu} \rho_m c_p \delta d \left(\frac{\rho \rho' \delta}{\sqrt{((\rho')^2 + \cos^2 \varepsilon)}} \cos \varepsilon \frac{\delta^2}{\sqrt{(\rho'^2 + \cos^2 \varepsilon)}} \left(\frac{(\cos \varepsilon)\delta}{180\rho} + \frac{\partial \delta(\varepsilon)}{\partial \varepsilon} \frac{(\cos \varepsilon)(\sin \varepsilon)}{72\rho} - \frac{(\rho'^2 + \cos^2 \varepsilon)}{40\sqrt{(1 + \rho'^2)}} \right) \right) = -\frac{\rho^2 \rho'}{\sqrt{(1 + \rho')}} d\varepsilon$$

or

$$K\cdot\delta\cdot d\left(\rho\rho'\delta^3\cos\varepsilon\left(\frac{(\cos\varepsilon)\delta}{((\rho')^2+\cos^2\varepsilon)180\rho}+\frac{\partial\delta(\varepsilon)}{\partial\varepsilon}\frac{(\cos\varepsilon)(\sin\varepsilon)}{((\rho')^2+\cos^2\varepsilon)72\rho}-\frac{1}{40\sqrt{1+\rho'^2}}\right)\right)=-\frac{\rho^2\rho'}{\sqrt{(1+\rho'^2)}}d\varepsilon.$$

Introducing the notations X_i for the coefficients one have

$$K\cdot\delta\cdot d\left(X_1\cdot\delta^3+X_2\cdot\delta^4+X_3\cdot\delta^3\cdot\frac{\partial\delta}{\partial\varepsilon}\right)=X_4\cdot d\varepsilon,$$

where

$$K=\frac{g\cdot\beta\cdot\Delta T\cdot\rho_m\cdot c_p}{6\cdot\lambda\cdot\nu}=\frac{Ra_R}{6\cdot R^3}, \tag{51}$$

$$X_1=-\frac{\rho\rho'\cos\varepsilon}{40\cdot\sqrt{1+(\rho')^2}}, \tag{52}$$

$$X_2=\frac{\rho'\cos^2\varepsilon}{180\cdot((\rho')^2+\cos^2\varepsilon)}, \tag{53}$$

$$X_3=\frac{\rho'\sin\varepsilon\cos^2\varepsilon}{72\cdot((\rho')^2+\cos^2\varepsilon)}, \tag{54}$$

$$X_4=-\frac{\rho^2\rho'}{\sqrt{1+\rho'^2}}. \tag{55}$$

5.1. On formulation of a problem for the boundary layer thickness equation

After introducing dimensionless variables:

$$\delta^*=\delta\cdot K^{1/3},\quad \rho^*=\rho\cdot K^{1/3}\quad \text{and}\quad z^*=z\cdot K^{1/3} \tag{56}$$

and after dropping the stars in δ, ρ and z one obtains

$$X_3\left(\delta^4\frac{d^2\delta}{d\varepsilon^2}+3\delta^3\left(\frac{d\delta}{d\varepsilon}\right)^2\right)+\left(3X_1+4X_2\delta+\frac{dX_3}{d\varepsilon}\delta\right)\delta^3\frac{d\delta}{d\varepsilon}+\frac{dX_2}{d\varepsilon}\delta^5+\frac{dX_1}{d\varepsilon}\delta^4=X_4. \tag{57}$$

The resulting equation of the physical model could be solved by a simple numerical method. We however would apply analytical method to construct approximate formulas for the boundary layer thickness δ as a function of variables ϵ and ϵ_m as in the case of a cone in [6].

The coefficients X_i Eqs. (52)–(55) are functions of $\rho(z)$ which in turn depends on the $z(\epsilon,\epsilon_m)$ determining the form of the curve Eq. (27) on a revolution surface via the differential Eq. (26). Let us underline that our choice of the coordinate system allows to consider ϵ_m as a parameter. The boundary conditions of the convective fluid flow problem yields for the function of the boundary layer thickness $\delta(\epsilon,\epsilon_m)$

$$\delta(-\epsilon_m,\epsilon_m)=0 \tag{58}$$

$$\delta(0,\epsilon_m)<\infty \tag{59}$$

The analysis of the Eq. (57) shows that the sign of the coefficient X_3 changes in the point $\epsilon = 0$, that means that at this point the equation has singularity (see the function $\sin\varepsilon$ in Eq. (54)), and the order of this equation is reduced. We used this opportunity to build the solution as a power series of ϵ at the vicinity of the point $\epsilon = 0$. The construction of the series coefficients hence could be made by means of the Eq. (57) however the only boundary condition (58) should be applied for the reason of the singularity. The power series expansion is considered as asymptotic one. The singular term with the second derivative is taken into account when the expansion is substituted to the Eq. (57).

6. On verification of the resulting equation

To verify the form of the Eq. (57) we choose two typical cases of the isothermal lateral surface of axisymmetrical horizontal body. The first of them is conical surface ($\rho = z\cot\alpha$, (2)) and the next one is the round vertical plate. The vertical round plate is the degenerated case of the lateral axisymmetrical surface ($h = 0$) of the horizontal cone.

The Eq. (26) in the case of a horizontal cone gives: $\rho(\varepsilon)=h\cot\alpha\left(\frac{\cos\varepsilon_m}{\cos\varepsilon}\right)^{\cos^2\alpha}$. The substitution into Eq. (37) gives the expression for the average tangential component of the fluid velocity inside boundary layer:

$$\overline{W_\tau} = \frac{g\beta\Delta T\delta^2(\cos\varepsilon)^{\cos^2\alpha+1}}{\nu\sqrt{(1-\sin^2\varepsilon\sin^2\alpha)}}\left(\frac{\delta\cos\varepsilon\sin\alpha}{180\rho_0} + \frac{\partial\delta(\varepsilon)}{\partial\varepsilon}\frac{\sin\alpha\sin\varepsilon}{72\rho_0} - \frac{1-\sin^2\varepsilon\sin^2\alpha}{40(\cos\varepsilon)^{\cos^2\alpha+1}}\right). \quad (60)$$

This expression coincides with one derived in our previous works [6,7].

To verify the resulting equation for the boundary layer form Eq. (57) it is enough to plug the expression for the function $\rho(\varepsilon) = \rho_0(\cos\varepsilon)^{-\cos^2\alpha}$ corespondent to the conical surface into the coefficients X_i:

$$X_{c1} = -\frac{\rho_0(\cos\varepsilon)^{-\cos^2\alpha}\rho'\cos\varepsilon}{40\cdot\sqrt{1+(\rho')^2}}, \quad (61)$$

$$X_{c2} = \frac{\rho'\cos^2\varepsilon}{180\cdot((\rho')^2+\cos^2\varepsilon)}, \quad (62)$$

$$X_{c3} = \frac{\rho'\sin\varepsilon\cos^2\varepsilon}{72\cdot((\rho')^2+\cos^2\varepsilon)}, \quad (63)$$

$$X_{c4} = -\frac{(\rho_0(\cos\varepsilon)^{-\cos^2\alpha})^2\rho'}{\sqrt{1+\rho'^2}}, \quad (64)$$

where $\rho_0 = h\cot\alpha(\cos\varepsilon_m)^{\cos^2\alpha}$ and $\rho' = \cot\alpha$.

These expressions and as a corollary the resulting Eq. (57) also coincide with ones derived in our previous works [6,7].

In the case of a vertical round plate ($\alpha = 0$) the coefficient by the second derivative in the basic equation for the boundary layer thickness Eq. (57)

$$X_3 = \frac{\rho'\sin\varepsilon\cos^2\varepsilon}{72\cdot((\rho')^2+\cos^2\varepsilon)} = \frac{\sin\alpha\sin\varepsilon\cos^2\varepsilon\cos\alpha}{72(1-\sin^2\varepsilon\sin^2\alpha)} = 0.$$

The resulting equation is hence of the first order and is integrating in explicit form

$$\delta = C(\epsilon - \epsilon_m)^{1/4}.$$

For the theoretical treatment and experimental verification see [4,9].

7. Conclusion

Solutions of the resulting equations for the boundary layer thickness Eq. (57) allow to evaluate the heat transfer from arbitrary revolution surface with horizontal axes of symmetry. The calculation of the heat transfer is based on Newton law $Q = \alpha A\Delta T$ in which $\alpha = 2\lambda/\delta$ Eq. (42) in the vicinity of each point of the surface. The total heat transfer is found by integration as in [6]. To proceed with such calculation the differential Eq. (26) should be solved for a given surface defined by the function $\rho = \rho(z)$. The solution yields the expression for the function $z = z(\epsilon)$ that describes the curvilinear coordinate system on the surface used in our approach.

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