BEARING ERRORS IN PASSIVE SONAR STATIONARY TARGET LOCATION

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The paper presents a method for determining the coordinates of stationary targets using passive sonar bearing. It identifies the requirements sonar movements must meet to keep the incorrect determination of coordinates to a minimum. It gives the relations, which help determine coordinate errors analytically. Numerical experiments are used to demonstrate the success of the analysis.

INTRODUCTION

Because passive sonars cannot determine distances, target determination can only be done by bearing. For stationary targets, the crossing points of the bearing lines are the target's coordinates. With only a limited accuracy of the bearing, there is a degree of error in target coordinate determinations. How big the error is depends on the bearing accuracy of sonar coordinates at points where bearings are taken to identify their location versus the target. The article presents an algorithm for determining target location designed to minimise coordinate error. It is assumed that inaccurate bearing is the only source of error. Errors in sonar positioning were not taken into account. These are negligible given how accurate today's satellite navigation systems are.

The analysis is the first step in further exploring methods for passive sonar target determination. The next stages will involve cases of target steady rectilinear and curvilinear motion. In these more complicated cases, the stationary target hypothesis is the first and preliminary assumption to be verified in further tests.

1. TARGET DETERMINATION PRINCIPLE

We assume that the coordinates of a stationary target are x_0, y_0. The bearings are taken at n points. The sonar array's coordinates at these points are $X(n)$, $Y(n)$. This is where the sonar determines the bearings of a stationary target marked as $\alpha(n)$. The sines and cosines of the bearings are equal to:

$$\cos[\alpha(n)] = [y_0 - Y(n)]/R(n), \tag{1}$$

$$\sin[\alpha(n)] = [x_0 - X(n)]/R(n), \tag{2}$$

where

$$R(n) = \sqrt{[x_0 - X(n)]^2 + [y_0 - Y(n)]^2}. \tag{3}$$

By eliminating distance $R(n)$ from the equations (1) and (2), after some simple transformations we get:

$$\cos[\alpha(n)]x_0 - \sin[\alpha(n)]y_0 = \sin[\alpha(n)]Y(n)] - \cos[\alpha(n)]X(n)]. \tag{4}$$

The above equation has two unknowns x_0 and y_0. To determine these we have to solve a system of two equations. By denoting:

$$\begin{aligned} c_n &= \cos[\alpha(n)] \\ s_n &= \sin[\alpha(n)] \end{aligned} \tag{5}$$

we get:

$$\begin{bmatrix} c_1 & -s_1 \\ c_2 & -s_2 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c_1X_1 - s_1Y_1 \\ c_2X_2 - s_2Y_2 \end{bmatrix}. \tag{6}$$

which we write as:

$$P \cdot x = b. \tag{7}$$

The solution to this matrix equation is the vector x:

$$x = P^{-1} \cdot b. \tag{8}$$

The value of the determinant is equal to:

$$|P| = -c_1s_2 + s_1c_2, \tag{9}$$

hence:

$$|P| = \sin(\alpha_1 - \alpha_2). \tag{10}$$

The system of equations (6) has one solution, if the determinant $|P|$ is different from zero, i.e. when the bearings differ. So for a clear determination of the location of the target, the sonar cannot move along a straight line passing through the target.

The value of the determinant $|P|$ changes from –1 to 1. At the extreme ends of this scale the bearings are perpendicular to one another.

Where the differences between the bearings are small, the equation (6) is incorrectly conditioned. Conditioning can be denoted with the number u which is the quotient of the highest and lowest value of the singular value of s_v matrix P , that is:

$$u = \frac{s_{v\max}}{s_{v\min}}. \tag{11}$$

The minimal value of the number u is equal to 1, when all bearings are perpendicular. Matrix P has this form:

$$P = \begin{Bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{Bmatrix} . \qquad (12)$$

Singular values are then equal and amount to 1, while the value of the determinant of matrix P is also equal to 1. For small differences between the bearings, the number u reaches very high values. As an example for $\alpha_1=30^0$ and $\alpha_2=31^0$ we have:

$$P = \begin{Bmatrix} 0.8660 & -0.5000 \\ 0.8572 & -0.5150 \end{Bmatrix} .$$

Singular values of the matrix are equal to s_{vmax}=1.4142 and s_{vmin}=0.0123, and the number u is equal to about 115. The value of the determinant of matrix **P** is in this case $|P|$= - 0.0175.

Generally, the lower the absolute value of the determinant, the higher the number u. A high u suggests that the system of equations is badly conditioned (6). As a consequence, the results of target coordinate determination are highly sensitive to errors in the bearings.

To ensure the best accuracy of target location, the variances between the bearings should be significantly big. However this cannot be easily attained because of the ship's maximal speed and its manoeuvring ability. We can leave the second factor aside temporarily and instead try to optimise the direction of the ship (sonar array). To that end let us assume that the ship's maximal speed is V. Our task is to determine the ship's direction (course β_1), when the first bearing is α_1. Let us select the system of coordinates so that the ship's initial position is X_1=0 and Y_1=0. After time T the ship's coordinates are:

$$\begin{aligned} X_2 &= VT\sin\beta_1 \\ Y_2 &= VT\cos\beta_1 \end{aligned} \qquad (13)$$

By using equations (1), (2) and (3) we get:

$$\begin{aligned} s_1 &= x_0 / R_1 \\ c_1 &= y_0 / R_1 \\ s_2 &= (x_0 - VT\sin\beta_1)/R_2 \\ c_2 &= (y_0 - VT\cos\beta_1)/R_2 \end{aligned} \qquad (14)$$

where R_1 and R_2 are the distances between the ship and target.

The value of the determinant of matrix P with the elements identified above is equal to:

$$|P| = -\frac{y_0(x_0 - VT\sin\beta_1)}{R_1R_2} + \frac{x_0(y_0 - VT\cos\beta_1)}{R_1R_2} . \qquad (15)$$

Following some elementary transformations of the above formula we get:

$$|P| = \frac{VT}{R_2}\sin(\beta_1 - \alpha_1) = \frac{VT}{\sqrt{(x_0 - VT\sin\beta_1)^2 + (y_0 - VT\cos\beta_1)^2}}\sin(\beta_1 - \alpha_1) . \qquad (16)$$

While it is possible to determine angle β1 at which the above determinant reaches its maximum, doing it would be superfluous because the determinant depends on two unknown target coordinates x_0 and y_0. [1)] If we assume that $VT<<R_1=(x_0^2+y_0^2)^{1/2}$ we can accept that the value of the denominator does

[1)] It can be proved that under the conditions the biggest variance between the bearings is when the ship's course is perpendicular to the second bearing line. This criterion, however, is useless because at the first bearing point, the second bearing is not known.

not depend on the angle $\beta 1$, and then we get the maximal absolute value of the determinant:

$$\beta_1 = \alpha_1 \pm 90^0 . \tag{17}$$

that is when the ship's course is perpendicular to the first bearing line.

Equation (16) shows that the value of the determinant is then equal to:

$$|P| = \pm \frac{VT}{R_2} . \tag{18}$$

By using the above relation we can write formulas (14) as follows:

$$\begin{aligned} s_1 &= x_0 / R_1 \\ c_1 &= y_0 / R_1 \\ s_2 &= (x_0 + VTc_1 / R_2 \\ c_2 &= (y_0 - VTs_1) / R_2 \end{aligned} \quad \text{or} \quad \begin{aligned} s_1 &= x_0 / R_1 \\ c_1 &= y_0 / R_1 \\ s_2 &= (x_0 - VTc_1 / R_2 \\ c_2 &= (y_0 + VTs_1) / R_2 \end{aligned} \tag{19}$$

The results of numerical calculations shown in Fig. 1 confirm the above conclusions. The figure shows the relation between the number *u* (the degree of matrix *P* conditioning) in the function of angle β_1 (course) under the assumption that the first bearing $\alpha_1=45^0$. The parameter is the relation VT/R_1, i.e. the quotient of the distance covered by the ship from the first to the second bearing and the distance between the target and the ship as the first bearing was being taken.

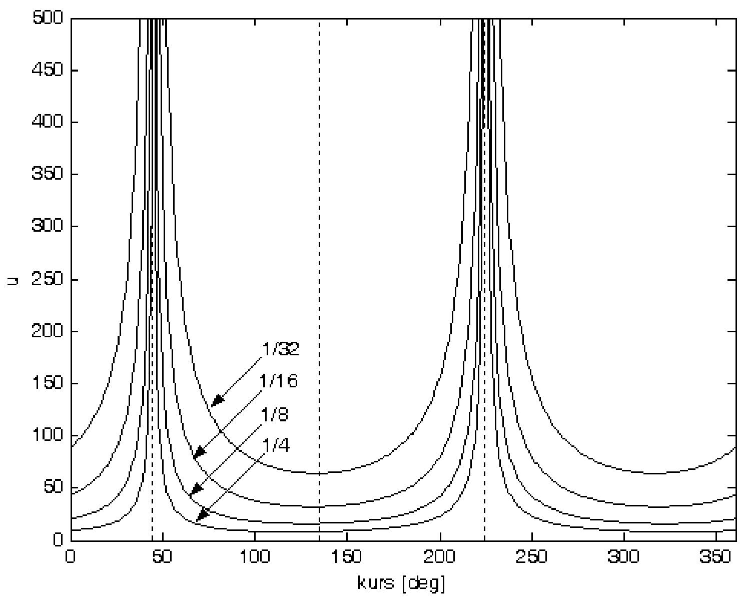

Fig.1. Relation between the degree of conditioning u and the ship's course

The graphs in Fig. 1 show that the conditioning of the system of equations (6) improves as the difference between the bearings becomes greater. The variance becomes greater as the distance to the bearing target becomes smaller and as the distance between bearing points covered by the ship becomes longer. In either situation it is best if the ship's course is perpendicular to the first bearing line so that even relatively high deviations from the optimal course do not affect the degree of equation conditioning (6).

Once solved, the system of equations (6) will give the exact position of the stationary target provided that the bearings are perfectly accurate which in practice is obviously not possible. Because bearing accuracy is limited, target location is determined using a number of successive bearings which are taken as the sonar moves. The relations (18) can then be used for the pairs of successive bearings. The sonar will be moving perpendicularly to the current bearing. If sustained, this principle, however, will cause the sonar to continuously move away from the target. As a result, the differences between neighbouring bearings will decrease. As has been shown above this is known to worsen the degree of equation conditioning (6).

After two bearings are taken, we can determine the target location and change how we define the sonar course. What we can do is determine a course perpendicular to the third bearing line which gives us a higher value of the determinant $|P|$ and the best possible degree of conditioning of the system of equations (1.1.4). As we continue to follow this principle in further bearings, there is a systematic reduction in the distance between sonar and target. Consequently, this improves the degree of conditioning of the system of equations. Obviously for perfectly accurate bearings this procedure is not necessary at all, but may prove useful if the bearings have accidental errors in them. Fig. 2 shows the geometric situation for the modified principle of course setting.

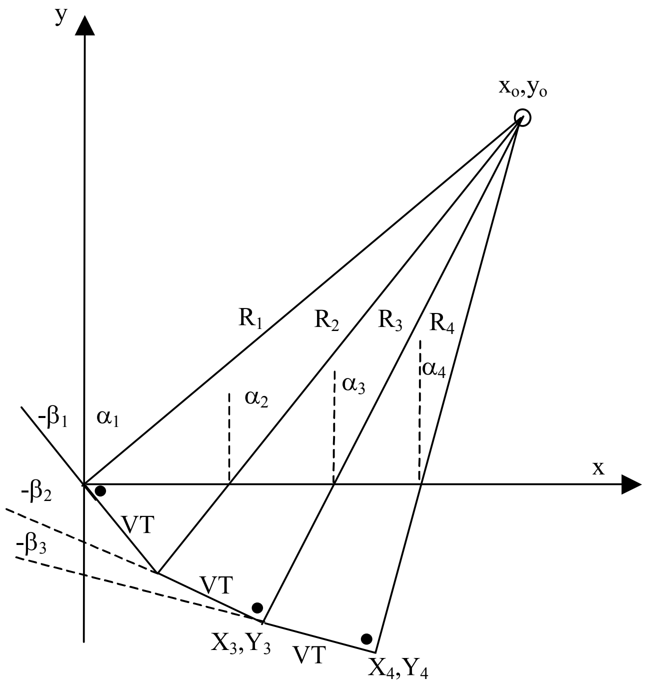

Fig.2. The geometric situation when the sonar is in motion

The two triangles resulting from the bearings α_1, α_2 and α_3 are identical because they are right triangles with two equilateral sides. Consequently, $R_3=R_1$ and $\alpha_1-\alpha_2 = \alpha_2 - \alpha_3$. As a result, formula (10) shows that the determinants of the system of equations for bearing one and two and two and three are equal. The degrees of conditioning u are also equal which means that equation conditioning is not getting worse for the next bearing. The course β_2 perpendicular to the bearing line α_3 is equal to:

$$\beta_2 = 90^0 + \alpha_3 = 2\alpha_2 - \alpha_1 \,. \tag{20}$$

The coordinates of the third point of bearing are described with these relations:

$$\begin{aligned} X_3 &= X_2 + \cos(\alpha_3) = X_2 + \cos(2\alpha_2 - \alpha_1) \\ Y_3 &= Y_2 - \sin(\alpha_3) = Y_2 - \sin(2\alpha_2 - \alpha_1) \end{aligned} \tag{21}$$

After the bearing is complete α_3 we calculate the cosine and sinus of that bearing and obtain the following system of equations:

$$\begin{bmatrix} c_2 & -s_2 \\ c_3 & -s_3 \end{bmatrix} \cdot \begin{bmatrix} x_{02} \\ y_{02} \end{bmatrix} = \begin{bmatrix} c_2 X_2 - s_2 Y_2 \\ c_3 X_3 - s_3 Y_3 \end{bmatrix}. \tag{22}$$

For a perfectly accurate bearing $x_{02}=x_0$ and $y_{02}=y_0$, but it does not if errors occur.

Starting from the third bearing, the following are the relations which describe the algorithm for determining target coordinates:

$$\begin{aligned}
&1) \quad R_n = \sqrt{(x_{0n-1} - X_n)^2 + (y_{0n-1} - Y_n)^2} \\
&2) \quad \alpha_{n+1} = \alpha_n - \arcsin\frac{VT}{R_n} \\
&3) \quad \beta_n = 90^0 + \alpha_{n+1} \\
&4) \quad X_{n+1} = X_n + VT\cos(\alpha_{n+1}) \\
&5) \quad Y_{n+1} = Y_n - VT\sin(\alpha_{n+1}) \\
&6) \quad \begin{bmatrix} c_n & -s_n \\ c_{n+1} & -s_{n+1} \end{bmatrix} \cdot \begin{bmatrix} x_{0n} \\ y_{0n} \end{bmatrix} = \begin{bmatrix} c_n X_n - s_n Y_n \\ c_{n+1} X_{n+1} - s_{n+1} Y_{n+1} \end{bmatrix}
\end{aligned} \tag{23}$$

where n=2,3,4,5,....

In the first step we determine the distance R_n between target and sonar. In the second step we calculate the anticipated bearing α_{n+1}, and in the third the ship's course β_n. When distance VT is covered, we take a target bearing and calculate the cosine c_{n+1} and sine s_{n+1} and bearing α_{n+1}. We then determine the coordinates of the ship's position at the point of the bearing. As the last step we solve the system of equations 6) and as a result obtain the target coordinates x_{0n} and y_{0n}. We increase the number n by one and start the procedure from step one. Note that the angle α_{n+1} determined in step two is the anticipated bearing and given the errors it is usually different from the measured angle α_{n+1} inserted in equations 4) and 5) and 6).

Fig. 3 shows the sonar route determined using the above algorithm and Fig. 4 shows the values of the number u (the degree of matrix P conditioning) and the inverse of the determinant $|P|$ for bearings taken at Fig. 3 points. As you can see, as the distance between sonar and target systematically decreases, the variance between the consecutive bearings increases improving equation conditioning (6). If the ship moves around the circle shown in Fig. 3, the degree of conditioning is constant and equal to about *u(1).*

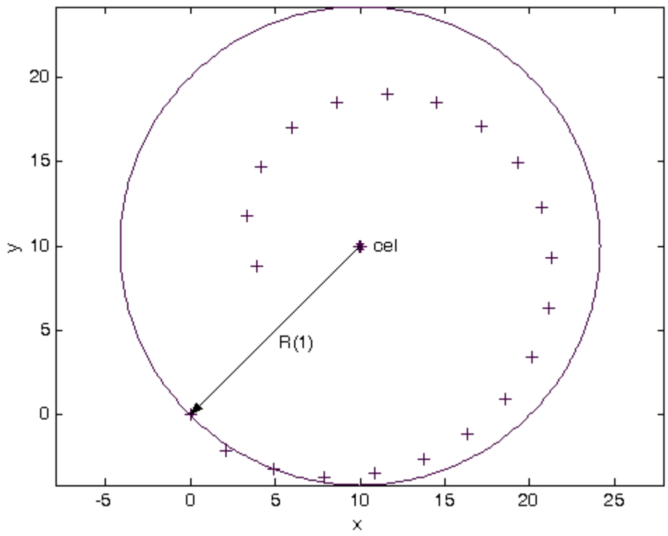

Fig.3. Sonar route (points +) determined using the modified algorithm for 30 consecutive bearings (x_0=10, y_0=10, VT=3)

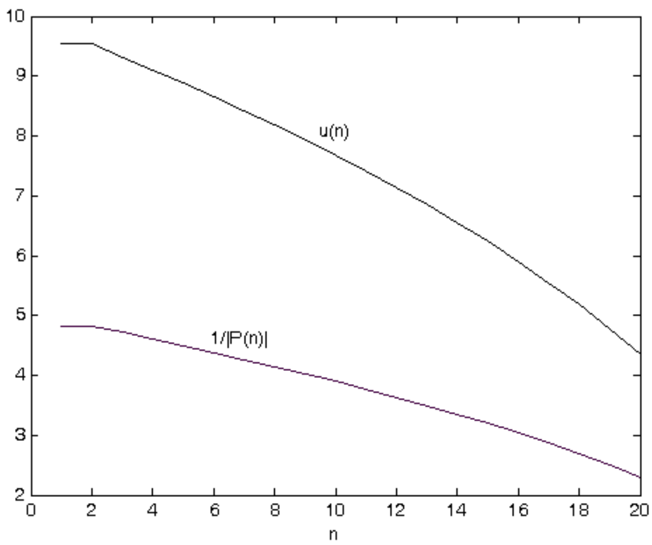

Fig.4. Degree of conditioning u of matrix P and the inverse of the matrix determinant for 30 consecutive bearings (x_0=10, y_0=10, VT=3)

2. ERRORS IN DETERMINING TARGET COORDINATES

The main cause of target estimation error in passive sonars is the poor accuracy of bearings. We are going to study the effects of bearing errors on target estimation using the above method. Let us assume that there is an error $d\alpha_\mathrm{n}$ in bearing α_n and the distribution of error probability is equally distributed and ranges from α_n-$D\alpha$ to α_n+$D\alpha$. [2)] This was introduced into algorithm (23) to numerically determine target coordinates for 1000 pairs of bearings. The results of the calculations for

[2)] The assumption that bearing error boundary values are constant is usually a simplification compared to the real conditions because sonar angular resolution deteriorates as the beam deflects from the perpendicular direction to the array.

two characteristic cases are given in Fig. 5 and Fig. 6.

In both cases target coordinates are contained inside a quadrilateral figure. The reason for the complicated shape of the figure is that the second bearing is taken at different points depending on the current bearing error. The shape and size of the field defined by target coordinates with an error in them as shown above depend on the difference between the bearings. These in turn are the result of the distance covered by the sonar. You can observe a similar effect for a constant distance *VT* and changing boundary values of the bearing error $D\alpha$.

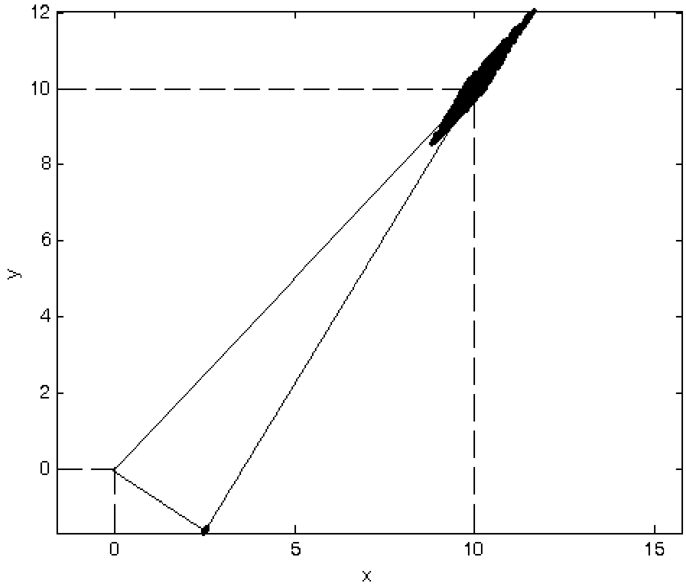

Fig.5. Target coordinates for bearing errors (VT=3, Dα=10, xo=10, y0=10, number of attempts - 1000)

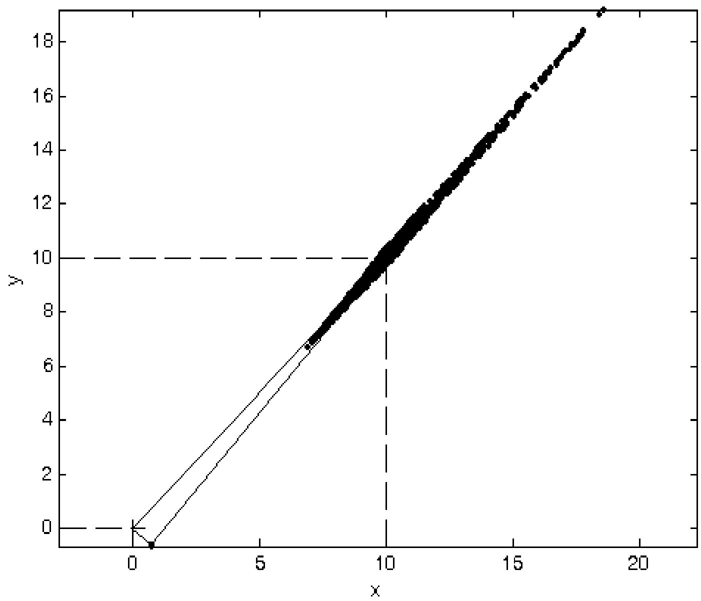

Fig.6. Target coordinates for bearing errors. (VT=1, Dα=10, xo=10, y0=10, number of attempts - 1000)

To simplify the analysis we will assume that the figure is formed by the extreme bearing lines and the points at which they intersect as shown in Fig. 7.

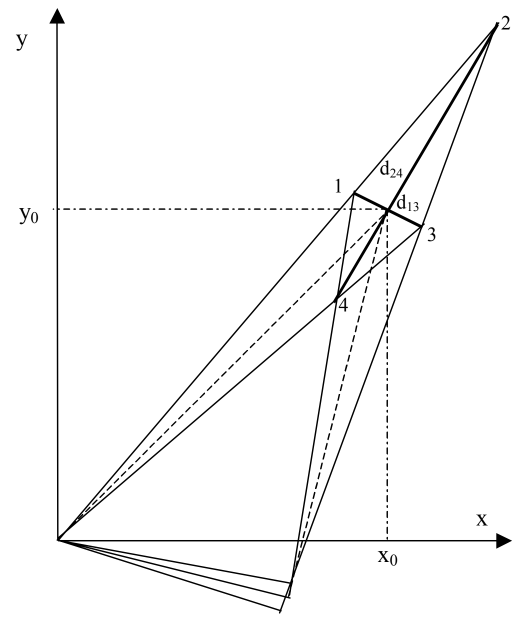

Fig.7. Defining the figure containing target coordinates with bearing error

We are now going to determine analytically the diagonal d_{13} and d_{24} shown in Fig. 7. To that end we need to find the vertex coordinates of the figure containing the calculated target coordinates. Using equations 4), 5) and 6) in formulas (23) or using the geometric relations, we can describe target coordinates with error in them as follows:

$$x_{0n} = X_n + \sin(\alpha_n + d\alpha_n)\frac{VT}{\sin(\alpha_n + d\alpha_n - \alpha_{n+1} - d\alpha_{n+1})}$$
$$y_{0n} = Y_n + \cos(\alpha_n + d\alpha_n)\frac{VT}{\sin(\alpha_n + d\alpha_n - \alpha_{n+1} - d\alpha_{n+1})} \tag{24}$$

After inserting into the above equations the bearing error $\pm D\alpha$ we determine the coordinates of the vertexes and following the transformations we obtain:

$$d_{13} = 2\sin D\alpha \frac{VT}{\sin(\alpha_n - \alpha_{n+1})} \tag{25}$$

Please note that the length of the diagonal depends on the difference between the bearings rather than on their values. This is understandable because the value of the bearing depends on the choice of coordinates which is something that cannot influence system parameters.

The length of the second diagonal is expressed in a complex formula and one that is difficult to interpret. We will give a simplified formula under the assumption that the maximal bearing error $D\alpha$ does not exceed one digit degrees. We then have:

$$d_{24} \cong 4\sin D\alpha \frac{VT}{\sin(\alpha_n - \alpha_{n+1} - 2D\alpha)\sin(\alpha_n - \alpha_{n+1} + 2D\alpha)} \cong$$
$$\cong 4\sin D\alpha \frac{VT}{\sin^2(\alpha_n - \alpha_{n+1}) - \sin^2(2D\alpha)} \qquad (26)$$

The length of the diagonal increases quickly, when the bearing error is comparable to the differences in the bearings. In general the diagonal d_{24} is significantly longer than diagonal d_{12}. If the maximal error in bearing is significantly lower than the differences in the bearings, length can be determined from this simple formula:

$$d_{24} \cong 4\sin D\alpha \frac{VT}{\sin^2(\alpha_n - \alpha_{n+1})}. \qquad (27)$$

The relation between the length of the diagonals is approximately equal to:

$$\frac{d_{24}}{d_{12}} \cong \frac{2}{\sin(\alpha_n - \alpha_{n+1})}. \qquad (28)$$

The figures below illustrate the correctness of the assumption and the relations derived. Fig. 8 shows the figure formed by coordinates determined from the above formulas.

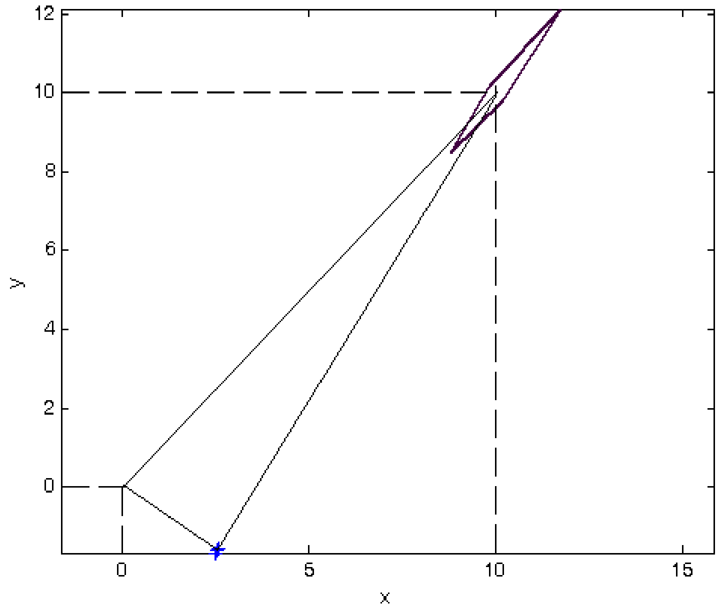

Fig.8. Quadrilateral containing target coordinates determined with bearing error.
(VT=3, Dα=10, xo=10, y0=10)

As anticipated the diagonals of the quadrangle differ in length. And so diagonal d_{13} calculated from formula (25) is 0.4936 long. The second diagonal d_{24} is much longer and when determined from the exact geometric relations it is 4.6992 long and when calculated from the formula (27) it is 4.6662 long. As you can see, the differences between numerically and analytically determined values have no practical significance for the parameters adopted. The differences in diagonal length d_{24} become significant when the maximal bearing error $D\alpha$ has several degrees. The direction of the diagonal d_{24} is close to the midperpendicular of the bearing triangle and the direction of the diagonal d_{13} is perpendicular to it. The diagonals intersect practically at point x_0,y_0. The surface of the quadrangle shown in Fig. 8 is 1.16. If we assign a unit [km] to the abstract numbers, the area of the quadrangle is equal to 1.16 km^2.

With the parameters from Fig. 8, Fig. 9 shows a magnified picture of the target coordinates and the matching quadrilateral. The same figure shows the results of the numerical simulation based on 1000 attempts assuming an equal distribution of bearing error probability density.

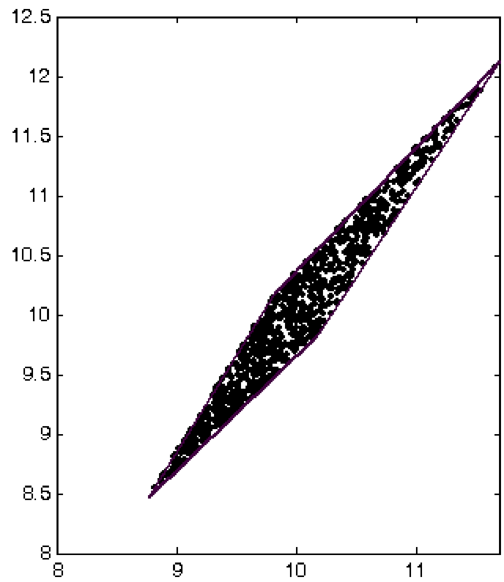

Fig.9. Magnified area containing target coordinates from Fig.8

The figure shows that practically all coordinates are contained in the quadrangle determined from the above formulas. As you can see, the distribution of coordinate probability density is not constant; there are more bearings in the lower part of the quadrilateral. As a result, the area's centre of gravity corresponds to point x_0,y_0 despite the figure's asymmetry towards diagonal d_{13}. For multiple bearings, target coordinates are estimated by these coordinates:

$$\overline{x_0} = \frac{1}{K}\sum_{k=1}^{K} x_{0k}$$
$$\overline{y_0} = \frac{1}{K}\sum_{k=1}^{K} y_{0k} \tag{29}$$

The results of the numerical experiment are given in Fig. 9. The values determined using the above formulas are $\overline{x_0} = 10.0531$ and $\overline{y_0} = 10.0385$. What this proves is that mean values described with formulas (29) are the correct estimators of the coordinates of a stationary target.

3. CONCLUSIONS

Stationary targets as discussed in the article are a rare occurrence in hydrological practice. The purpose of this analysis has been mainly to identify the rules of procedure in real cases of rectilinear motion and other more complicated types of motion. In particular, we can look forward to promising results by basing the error minimisation method on sonar trajectory optimisation and studying the degree of conditioning of systems of equations.

REFERENCES

[1] J.Giertowski, T. Meissner, Podstawy nawigacji morskiej, Wydawnictwo Morskie, Gdańsk, 1969.
[2] M. Jurdziński, Podstawy nawigacji morskiej, Wydawnictwo AM Gdynia, 2003.
[3] S. C. Nardone, A. G. Lindgren, K. F. Gong, Fundamental properties and performance of conventional bearings-only target motion analysis, IEEE Trans. Automat. control 29 (9), pp. 775-787, 1984.
[4] D. T. Pham, Some quick and efficient methods for bearing only target motion analysis, IEEE Trans. signal Process. 41 (9), pp. 2737 – 2751, 1993.
[5] R. Salamon, Systemy hydrolokacyjne, Gdańskie Towarzystwo Naukowe, Gdańsk, 2006.