Computers and Mathematics with Applications 59 (2010) 1937–1943

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

First-order functional difference equations with nonlinear boundary value problems

Tadeusz Jankowski

Gdansk University of Technology, Department of Differential Equations, 11/12 G.Narutowicz Street, 80-952 Gdańsk, Poland

ARTICLE INFO

Article history:
Received 26 May 2009
Received in revised form 9 October 2009
Accepted 9 November 2009

Keywords:
Difference equations with delayed arguments
Difference inequalities
Lower and upper solutions
Monotone method
Existence of solutions

ABSTRACT

The monotone iterative method is used to show that corresponding difference problems with boundary conditions have extremal solutions in the region bounded by lower and upper solutions. It is important to indicate that the right-hand sides of problems depend on r delayed arguments. Difference inequalities of such types are also discussed. Two examples satisfying the assumptions are presented.

1. Introduction

Put

$$(\mathcal{F}y)(k) = f(k, y(k), y(\alpha_1(k)), y(\alpha_2(k)), \ldots, y(\alpha_r(k))).$$

In this paper, we consider first-order delayed difference equations with nonlinear boundary value problems:

$$\begin{cases} \Delta y(k-1) = (\mathcal{F}y)(k), \quad k \in Z[1, T] = \{1, 2, \ldots, T\}, \\ g(y(0), y(T)) = 0, \end{cases} \tag{1}$$

where $\Delta y(k-1) = y(k) - y(k-1)$ with assumption
H_1: $f \in C(Z[1, T] \times \mathbb{R}^{r+1}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha_i \in C(Z[1, T], Z[0, T])$, $\alpha_i(k) \le k$, $i = 1, 2, \ldots, r$,
and the next type of equations:

$$\begin{cases} \Delta y(k) = (\mathcal{F}y)(k), \quad k \in Z[0, T-1], \\ g(y(0), y(T)) = 0, \end{cases} \tag{2}$$

where $\Delta y(k) = y(k+1) - y(k)$ with assumption
H_2: $f \in C(Z[0, T-1] \times \mathbb{R}^{r+1}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha_i \in C(Z[0, T-1], Z[0, T-1])$, $\alpha_i(k) \le k$, $i = 1, 2, \ldots, r$.

Difference equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields, see for example [1–4] and the references therein. Mathematical modeling of problems in physical sciences relies heavily on the use of differential equations together with initial or boundary conditions. Discrete boundary value problems are also natural consequences of discretization techniques of differential boundary problems including also problems with delayed arguments. Recently,

E-mail address: tjank@mif.pg.gda.pl.

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2009.11.008

some papers devoted to the study of the existence of solutions for nonlinear difference problems have appeared. There are several techniques which are often employed in boundary value problems. One of the methods employed is the method of upper and lower solutions coupled with iterative methods. It provides constructive schemes for calculating the desired solutions.

It well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations. It can be used both initial and boundary problems. There are only a few papers when the monotone iterative technique is used to first-order differential problems with delayed arguments, see for example [5–7]. This technique can also be used to first-order difference equations with delayed arguments and I known only one such paper [8]. Note that problem (1) represents a discrete analogue of the delay differential equation of the form

$$x'(t) = f(t, x(t), x(\alpha_1(t)), x(\alpha_2(t)), \ldots, x(\alpha_r(t))),$$

where $\alpha_i(t) \le t$ for $i = 1, 2, \ldots, r$ see for example [5,6]. In [8], special cases of (1) and (2) are discussed. Moreover, in paper [8], it is assumed that the right-hand side of difference problem satisfies a one sided Lipschitz conditions with constant coefficients while in my paper f satisfies such condition with functional coefficients. This paper generalizes results of [8].

Let $\mathcal{C} = C(Z[0, T], \mathbb{R})$ denote the class of maps w continuous on $Z[0, T]$ (discrete topology) with the norm

$$\|w\| = \max_{k \in Z[0,T]} |w(k)|.$$

Note that $\mathcal{C}$ is a Banach space. By a solution of (1), we mean $w \in \mathcal{C}$ such that it satisfies problem (1). Similarly we define the solution of problem (2).

2. Difference inequalities

In this section, we present difference inequalities which are needed later.

Lemma 1. *Assume that* $\alpha \in C(Z[1, T], Z[0, T])$, $\alpha(k) \le k$, $k \in Z[0, T]$, $y \in C(Z[0, T], \mathbb{R})$, $M, L \in C(Z[1, T], \mathbb{R}_+)$, $R_+ = [0, \infty)$ *and*

$$\begin{cases} \Delta y(k-1) \le -M(k)y(k) - L(k)y(\alpha(k)), & k \in Z[1, T], \\ y(0) \le 0. \end{cases}$$

In addition, we assume that

$$\rho_1 \equiv \sum_{i=1}^{T} L(i) \prod_{j=1}^{i-1} [1 + M(j)] \le 1, \tag{3}$$

where $\prod_{i=1}^{0} \cdots = 1$.

Then $y(k) \le 0$, $k \in Z[0, T]$.

Proof. Assume that the assertion is not true. Then, there exists $k_0 \in Z(0, T]$ such that $y(k_0) > 0$. Put

$$y(k_1) = \min_{k \in Z[0,k_0]} y(k) = \lambda \le 0.$$

It results

$$[1 + M(k)]y(k) - y(k-1) \le -\lambda L(k), \quad k \in Z[1, k_0],$$

and

$$y(k)S_k - y(k-1)S_{k-1} \le -\lambda L(k)S_{k-1}, \quad k \in Z[1, k_0]$$

with

$$S_0 = 1, \qquad S_k = \prod_{i=1}^{k} [1 + M(i)], \quad k \in Z[1, T].$$

Hence

$$\Delta[y(k-1)S_{k-1}] \le -\lambda L(k)S_{k-1}, \quad k \in Z[1, k_0].$$

Summing it from $k_1 + 1$ to k_0 gives

$$y(k_0)S_{k_0} - \lambda S_{k_1} \le -\lambda \sum_{i=k_1+1}^{k_0} L(i)S_{i-1}. \tag{4}$$

Hence

$$-\lambda S_{k_1} < -\lambda \sum_{i=k_1+1}^{k_0} L(i) S_{i-1},$$

because $y(k_0) > 0$. It yields

$$1 < \sum_{i=k_1+1}^{k_0} L(i) S_{i-1} \le \rho_1$$

because $\lambda < 0,\ S_{k_1} \ge 1$. It is a contradiction. The same conclusion we have for $\lambda = 0$. This ends the proof. ■

Remark 1. Note that if $M(i) = 0,\ i \in Z[0, T]$, then

$$\rho_1 = \sum_{i=1}^{T} L(i).$$

Remark 2. Let $L(i) = L,\ M(i) = M$. Then

$$\rho_1 = \frac{L(1+M)^T}{(L+M)}.$$

In this case Lemma 1 reduces to Theorem 2.1 [8].

Lemma 2. *Assume that* $\alpha \in C(Z[0, T-1], Z[0, T-1])$, $\alpha(k) \le k$, $k \in Z[0, T-1]$, $y \in C(Z[0, T], \mathbb{R})$, $M \in C(Z[1, T], [0, 1))$, $L \in C(Z[1, T], \mathbb{R}_+)$ *and*

$$\begin{cases} \Delta y(k) \le -M(k+1)y(k) - L(k+1)y(\alpha(k)), & k \in Z[0, T-1], \\ y(0) \le 0. \end{cases}$$

In addition, we assume that

$$\rho_2 \equiv \sum_{i=1}^{T} L(i) \frac{1}{\prod_{j=1}^{i}[1 - M(j)]} \le 1. \tag{5}$$

Then $y(k) \le 0,\ k \in Z[0, T]$.

Proof. The proof is similar to the proof of Lemma 1. Assume that the assertion is not true. Then, there exists $k_0 \in Z(0, T]$ such that $y(k_0) > 0$. Put

$$y(k_1) = \min_{k \in Z[0,k_0]} y(k) = \lambda \le 0.$$

It results

$$\Delta[y(k)P_k] \le -\lambda L(k+1)P_{k+1}, \quad k \in Z[0, k_0]$$

with

$$P_i = \prod_{i=1}^{k} \frac{1}{1 - M(i)}, \quad k \in Z[1, T].$$

Summing it from k_1 to $k_0 - 1$ gives

$$y(k_0)P_{k_0} - \lambda P_{k_1} \le -\lambda \sum_{i=k_1}^{k_0-1} L(i+1)P_{i+1}.$$

Hence

$$-\lambda P_{k_1} < -\lambda \sum_{i=k_1}^{k_0-1} L(i+1)P_{i+1},$$

because $y(k_0) > 0$. It yields

$$1 < \sum_{i=1}^{k_0-1} L(i+1)P_{i+1} \le \rho_2$$

because $\lambda < 0,\ P_{k_1} > 1$. It is a contradiction. This ends the proof. ■

Remark 3. Note that if $M(i) = 0,\ i \in Z[0, T]$, then

$$\rho_2 = \sum_{i=1}^{T} L(i).$$

Remark 4. Let $L(i) = L,\ M(i) = M$. Then

$$\rho_2 = \frac{L}{(1-M)^T(L+M)}.$$

In this case Lemma 2 reduces to Theorem 2.2 [8].

The next two lemmas we formulate without any proofs since they are similar to ones of Lemmas 1 and 2, respectively.

Lemma 3. *Assume that* $\alpha_i \in C(Z[1,T], Z[0,T]), \alpha_i(k) \le k, k \in Z[0,T], i \in Z[1,r], y \in C(Z[0,T], \mathbb{R}), M, L_i \in C(Z[1,T], \mathbb{R}_+), i \in Z[1,r]$ *and*

$$\begin{cases} \Delta y(k-1) \le -M(k)y(k) - \sum_{i=1}^{r} L_i(k)y(\alpha_i(k)), & k \in Z[1,T], \\ y(0) \le 0. \end{cases}$$

In addition, we assume that

$$\rho_3 \equiv \sum_{i=1}^{T} \mathcal{L}(i) \prod_{j=1}^{i-1} [1 + M(j)] \le 1, \tag{6}$$

where $\mathcal{L}(i) = \sum_{s=1}^{r} L_s(i)$.

Then $y(k) \le 0,\ k \in Z[0, T]$.

Lemma 4. *Assume that* $\alpha_i \in C(Z[0, T-1], Z[0, T-1]),\ \alpha_i(k) \le k,\ k \in Z[0, T-1],\ i \in Z[1,r],\ y \in C(Z[0,T], \mathbb{R}),\ M \in C(Z[1,T], [0,1)),\ L_i \in C(Z[1,T], \mathbb{R}_+)$ *and*

$$\begin{cases} \Delta y(k) \le -M(k+1)y(k) - \sum_{i=1}^{r} L_i(k+1)y(\alpha_i(k)), & k \in Z[0, T-1], \\ y(0) \le 0. \end{cases}$$

In addition, we assume that

$$\rho_4 \equiv \sum_{i=1}^{T} \mathcal{L}(i) \frac{1}{\prod_{j=1}^{i} [1 - M(j)]} \le 1 \quad \text{with } \mathcal{L}(i) = \sum_{s=1}^{r} L_s(i). \tag{7}$$

Then $y(k) \le 0,\ k \in Z[0, T]$.

3. Monotone iterative method

Theorem 1 (*Discrete Arzela–Ascoli Theorem, [9]*). *Let* $\mathcal{A}$ *be a closed subset of* C. *If* $\mathcal{A}$ *is uniformly bounded and the set* $\{u(k) : u \in \mathcal{A}\}$ *is relatively compact for each* $k \in Z[0, T]$, *then* $\mathcal{A}$ *is compact.*

Now we consider the following linear problem

$$\begin{cases} \Delta y(k-1) = -M(k)y(k) - \sum_{i=1}^{r} L_i(k)y(\alpha_i(k)) + h(k), \\ y(0) = \xi \in \mathbb{R}, \end{cases} \tag{8}$$

where $h \in C(Z[1,T], \mathbb{R})$ and bounded.

Theorem 2. *Assume that $M, L_i \in C(Z[1,T], \mathbb{R}_+)$, $\alpha_i \in C(Z[0,T], Z[0,T])$, $\alpha_i(k) \le k$, $i = 1, 2, \dots, r$. Let $h \in C(Z[0,T], \mathbb{R})$ and be bounded. Let condition (6) hold. Then problem (8) has a unique solution.*

Proof. We first show that solving (8) is equivalent to solving a fixed point problem. Let y be any solution of problem (8). Note that the equation of problem (8) can be also written in the following form

$$\Delta\left[y(k-1)\prod_{i=1}^{k-1}(1+M(i))\right] = \left[-\sum_{j=1}^{r} L_j(k)y(\alpha_j(k)) + h(k)\right]\prod_{i=1}^{k-1}(1+M(i))$$

for $k \in Z[1,T]$. Summing it from 1 to s gives

$$\begin{aligned} y(s) &= \left[\xi + \sum_{i=1}^{s}\left(-\sum_{\nu=1}^{r} L_\nu(i)y(\alpha_\nu(i)) + h(i)\right)\prod_{j=1}^{i-1}(1+M(j))\right]\left(\prod_{j=1}^{s}[1+M(j)]\right)^{-1} \\ &\equiv (A_h y)(s), \end{aligned}$$

for $s \in Z[0,T]$. Similarly, it is easy to see that if y is any solution of $y = A_h y$, then y is a solution of problem (8).

The continuity of M, L_ν, h imply that $A_h : \mathcal{C} \to \mathcal{C}$ is continuous and bounded. This and Theorem 1 imply that A_h is compact. Now, Schauder's fixed point implies that A_h has a fixed point, i.e. problem (8) has a solution.

Now we show that (8) has the unique solution. Assume that (8) has two solutions u, v and $u \neq v$. Put $p = u - v$. Then $p(0) = 0$, and

$$\Delta p(k-1) = -M(k)p(k) - \sum_{i=1}^{r} L_i(k)p(\alpha_i(k)), \quad k \in Z[1,T].$$

Hence $p(k) \le 0$ for $k \in Z[0,T]$, by Lemma 3. It shows that $u \le v$. Now, if we put $p = v - u$, then using again Lemma 3 we see that $v \le u$. It shows that problem (8) has the unique solution. This ends the proof. ∎

We say that y_0 is called a lower solution of problem (1) if

$$\begin{cases} \Delta y_0(k-1) \le (\mathcal{F}y_0)(k), & k \in Z[1,T], \\ g(y_0(0), y_0(T)) \le 0 \end{cases}$$

and it is an upper solution of (1) if the above inequalities are reversed.

Theorem 3. *Suppose that assumption H_1 holds. Let y_0, z_0 be lower and upper solutions of problem (1), respectively, and $y_0 \le z_0$. In addition, we assume that*

H_3*: there exist functions $M, L_i \in C(Z[1,T], \mathbb{R}_+)$, $i = 1, 2, \dots, r$ such that condition (6) holds and*

$$f(k, u, v_1, \dots, v_r) - f(k, \bar{u}, \bar{v}_1, \dots, \bar{v}_r) \le M(k)[\bar{u} - u] + \sum_{i=1}^{r} L_i(k)[\bar{v}_i - v_i]$$

for $y_0 \le u \le \bar{u} \le z_0$, $y_0(\alpha_i(k)) \le v_i \le \bar{v}_i \le z_0(\alpha_i(k))$, $i = 1, 2, \dots, r$,

H_4*: g is nonincreasing with respect to the second variable, and there exists a constant $a > 0$ such that*

$$g(\bar{u}, v) - g(u, v) \le a(\bar{u} - u)$$

for $y_0(0) \le u \le \bar{u} \le z_0(0)$, $y_0(T) \le v \le z_0(T)$.

Then problem (1) has, in the sector $[y_0, z_0]_*$, minimal and maximal solutions, where

$$[y_0, z_0]_* = \{w \in \mathcal{C} : \; y_0 \le w \le z_0\}.$$

Proof. Let us define $\mathcal{G}$ by

$$(\mathcal{G}(u,v))(k) = M(k)[u(k) - v(k)] + \sum_{i=1}^{r} L_i(k)[u(\alpha_i(k)) - v(\alpha_i(k))].$$

For $k \in Z[1,T]$ and $n = 0, 1, \dots$, let

$$\begin{cases} \Delta y_{n+1}(k-1) = (\mathcal{F}y_n)(k) - (\mathcal{G}(y_{n+1}, y_n))(k), \\ y_{n+1}(0) = y_n(0) - \dfrac{1}{a}g(y_n(0), y_n(T)), \end{cases}$$

and

$$\begin{cases} \Delta z_{n+1}(k-1) = (\mathcal{F}z_n)(k) - (\mathcal{G}(z_{n+1}, z_n))(k), \\ z_{n+1}(0) = z_n(0) - \dfrac{1}{a}g(z_n(0), z_n(T)). \end{cases}$$

Note that y_1, z_1 are well defined, by Theorem 2. First we need to show that

$$y_0 \le y_1 \le z_1 \le z_0. \tag{9}$$

Put $p = y_0 - y_1$. Then

$$\begin{aligned} p(0) &= y_0(0) - y_0(0) + \frac{1}{a} g(y_0(0), y_0(T)) \le 0, \\ \Delta p(k-1) &\le (\mathcal{F} y_0)(k) - (\mathcal{F} y_0)(k) + (\mathcal{G}(y_1, y_0))(k) \\ &= M(k)p(k) - \sum_{i=1}^{r} L_i(k) p(\alpha_i(k)). \end{aligned}$$

This and Lemma 3 imply that $p \le 0$, so $y_0 \le y_1$. Similarly, we can show that $z_1 \le z_0$. To show that $y_1 \le z_1$, we put $p = y_1 - z_1$. Then

$$\begin{aligned} p(0) &= y_0(0) - z_0(0) + \frac{1}{a}[g(z_0(0), z_0(T)) - g(y_0(0), y_0(T))] \le 0, \\ \Delta p(k-1) &= (\mathcal{F} y_0)(k) - (\mathcal{F} z_0)(k) - (\mathcal{G}(y_1, y_0))(k) + (\mathcal{G}(z_1, z_0))(k) \\ &\le -M(k)p(k) - \sum_{i=1}^{r} L_i(k) p(\alpha_i(k)), \end{aligned}$$

by assumptions H_3, H_4. It shows that (9) holds.

Using the mathematical induction we can show that

$$y_0 \le y_1 \le \cdots \le y_n \le z_n \le \cdots \le z_1 \le z_0.$$

Since $\{y_n\}$ is increasing and bounded then $\{y_n\}$ converging to y uniformly on $Z[0, T]$. Similarly, $\{z_n\}$ converging to z uniformly on $Z[0, T]$. Moreover, $y_0 \le y \le z \le z_0$. Since f and g are continuous, y and z are solutions of problem (1).

Now we need to show that y, z are extremal solutions of problem (1) in the sector $[y_0, z_0]_*$. Let $u \in [y_0, z_0]_*$ be any solution of (1) in that sector. Assume that $y_i \le u \le z_i$ for some positive integer i. Let $p = y_i - u,\ \ q = u - z_i$. Then $p(0) \le 0,\ q(0) \le 0$ and

$$\begin{aligned} \Delta p(k-1) &= (\mathcal{F} y_{i-1})(k) - (\mathcal{G}(y_i, y_{i-1}))(k) - (\mathcal{F} u)(k) \\ &\le -M(k)p(k) - \sum_{i=1}^{r} L_i(k) p(\alpha_i(k)), \\ \Delta q(k-1) &= (\mathcal{F} u)(k) - (\mathcal{F} z_i)(k) + (\mathcal{G}(z_i, z_{i-1}))(k) \\ &\le -M(k)q(k) - \sum_{i=1}^{r} L_i(k) q(\alpha_i(k)). \end{aligned}$$

This and Lemma 3 give $y_i \le u \le z_i$. By induction, we can show that $y_n \le u \le z_n$. Now if $n \to \infty$, then $y \le u \le z$, so we have the assertion. This ends the proof. ■

Now we discuss problem (2). We say that y_0 is called a lower solution of (2) if

$$\begin{cases} \Delta y_0(k) \le (\mathcal{F} y_0)(k), & k \in Z[0, T-1], \\ g(y_0(0), y_0(T)) \le 0 \end{cases}$$

and it is an upper solution of (2) if the above inequalities are reversed.

The proof of the next theorem is similar to the proof of Theorem 3 therefore it is omitted. By H_3' we denote assumption H_3 with condition (7) instead of (6).

Theorem 4. *Suppose that assumptions H_2, H_3', H_4 hold. Let y_0, z_0 be lower and upper solutions of problem (2), respectively, and $y_0 \le z_0$.*

Then problem (2) has, in the sector $[y_0, z_0]_*$, minimal and maximal solutions.

Example 1. Let $T \le 23$ and $0 < \xi < .001$. We consider the following problem

$$\begin{cases} \Delta x(k-1) = -\dfrac{2k}{1000} x(k) + \dfrac{k}{1000} x(k-1) + \xi \equiv (\mathcal{F} x)(k), & k \in Z[1, T], \\ x(0) = x(T), \end{cases} \tag{10}$$

so $g(u, v) = u - v,\ \alpha(0) = 0$ and $\alpha(k) = k - 1$ if $k \in Z[1, T]$. Note that

$$M(k) = \frac{2k}{1000}, \quad L_1(k) = \frac{k}{1000}, \quad a = 1,$$

and

$$\rho_3 = \frac{1}{1000}\sum_{i=1}^{T} i \prod_{j=1}^{i-1}\left(1+\frac{2j}{1000}\right) \le \frac{23}{1000}\sum_{i=1}^{23}\left(1+\frac{46}{1000}\right)^{i-1}$$
$$= \frac{1}{2}\left[(1.046)^{23}-1\right] \approx 0.90669 < 1.$$

It shows that assumptions H_1, H_3, H_4 are satisfied.

Now we put $y_0(k) = 0, z_0(k) = 1,\ k \in Z[0,T]$. It is easy to show that y_0, z_0 are lower and upper solutions of problem (10), respectively. Hence problem (10) has extremal solutions in the region $[y_0, z_0]_*$, by Theorem 3.

Example 2. We consider the problem

$$\begin{cases} \Delta y(k) = -\beta_1(k)y(\alpha_1(k)) - \beta_2(k)y(\alpha_2(k)) + h(k), & k \in Z[0, T-1], \\ y(0) = y^2(T), \end{cases} \tag{11}$$

where $\beta_1, \beta_2, h \in C[Z[0, T-1], \mathbb{R}_+)$ and

$$\alpha_1(k) = \begin{cases} 0, & k = 0, \\ k-1, & k \in Z[1,T], \end{cases} \qquad \alpha_2(k) = \begin{cases} 0, & k = 0, 1, \\ k-2, & k \in Z[2,T], \end{cases}$$

$$L_1(k) + L_2(k) \ge h(k) > 0, \qquad \sum_{i=1}^{T}[L_1(i) + L_2(i)] \le 1. \tag{12}$$

Assumptions $H_2,\ H_3',\ H_4$ are satisfied with $M(k) = 0,\ L_1(k) = \beta_1(k),\ L_2(k) = \beta_2(k),\ a = 1$.

Put $y_0 = 0,\ z_0 = 1$. Then y_0, z_0 are lower and upper solutions of problem (11), respectively. Problem (11) has extremal solutions in $[y_0, z_0]_*$, by Theorem 4.

For example, if we take $h(k) = \frac{1}{100},\ L_1(k) + L_2(k) = L,\ \frac{1}{100} \le L \le \frac{1}{T}$, then conditions (12) hold.

References

[1] S.N. Elaydi, An introduction to difference equations, in: Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1996.
[2] S. Goldberg, Introduction to Difference Equations, John Wiley and Sons, New York, 1960.
[3] W.G. Kelley, A.C. Peterson, Difference Equations. An Introduction with Applications, second ed., Academic Press, Inc., Tokyo, 1991.
[4] V. Lakshmikantham, D. Trigiante, Theory of difference equations. Numerical methods and applications, in: Mathematics in Science and Engineering, vol. 181, Academic Press, Inc., Boston, MA, 1988.
[5] T. Jankowski, Existence of solutions of boundary value problems for differential equations with delayed arguments, J. Comput. Appl. Math. 156 (2003) 239–252.
[6] T. Jankowski, On delay differential equations with nonlinear boundary conditions, Bound. Value Probl. (2) (2005) 201–214.
[7] D. Jiang, J. Wei, Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations, Nonlinear Anal. 50 (2002) 885–895.
[8] P. Wang, S. Tian, Y. Wu, Monotone iterative method for first-order functional difference equations with nonlinear boundary value conditions, Appl. Math. Comput. 203 (2008) 266–272.
[9] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, London, 1999.