AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **47** (2010), Pages 269–277

Strong weakly connected domination subdivisible graphs

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Abstract

The *weakly connected domination subdivision number* $sd_{\gamma_w}(G)$ of a connected graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the weakly connected domination number. The graph is strong-γ_w-subdivisible if for each edge $uv \in E(G)$ we have $\gamma_w(G_{uv}) > \gamma_w(G)$, where G_{uv} is a graph G with subdivided edge uv. The graph is strong-γ_w-k-subdivisible if $sd_{\gamma_w}(G) = k$ and subdivision of each of the k edges in G changes the weakly connected domination number of the graph obtained by these subdivisions. We constructively characterize all strong-γ_w-1-subdivisible and strong-γ_w-2-subdivisible trees and give some properties of strong-γ_w-k-subdivisible graphs for $k = 1, 2$.

1 Introduction

Let $G = (V, E)$ be a connected simple graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v. For a set $X \subseteq V(G)$, the *open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* is $N_G[X] = N_G(X) \cup X$. We say that a vertex v is *a universal vertex* of G if it is a neighbour of every other vertex of a graph and v is an *end-vertex* of G if v has exactly one neighbour in G. A path with two end-vertices x and y we denote by P_{xy}.

A subset D of $V(G)$ is *dominating* in G if every vertex of $V(G) - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$.

A dominating set $D \subseteq V(G)$ is a *weakly connected dominating set* in G if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by D is connected, where E_w is the set of all edges with at least one vertex in D. Dunbar et al. [2] defined the *weakly* *connected domination number* $\gamma_w(G)$ of a graph G to be the minimum cardinality among all weakly connected dominating sets in G.

We say that a set $D \subseteq V(G)$ has the property $\mathcal{F}$ in G if D contains no end-vertex of G. It is easy to observe that in every tree T different than P_2, there exists a minimum weakly connected dominating set with property $\mathcal{F}$.

Here we consider connected graphs only. If G is a graph, let $n = n(G)$ be the order of G and let $n_1 = n_1(G)$ denote the number of end-vertices of G. The set of all end-vertices in G is denoted by $\Omega(G)$. A vertex v is called a *support* if it is adjacent to an end-vertex. If v is adjacent to more than one end-vertex, then we call v a *strong support*. The set of all supports in a graph we denote by $S(G)$.

The *weakly connected domination subdivision number* $sd_{\gamma_w}(G)$ of a connected graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the weakly connected domination number. The weakly connected domination subdivision number was defined and studied in [3]. The weakly connected domination subdivision number is considered for graphs which have at least three vertices, since the weakly connected domination number of the graph K_2 does not increase when its only edge is subdivided.

The graph is *strong weakly connected domination subdivisible* (strong-γ_w-subdivisible) if for each edge $uv \in E(G)$ is $\gamma_w(G_{uv}) > \gamma_w(G)$, where G_{uv} is a graph G with subdivided edge uv.

The graph is *strong weakly connected domination k-subdivisible* (strong-γ_w-k-subdivisible) if $sd_{\gamma_w}(G) = k$ and subdivision of every k edges in G changes the weakly connected domination number of the graph obtained by these subdivisions. If we subdivide two edges xy and uv in G, then the graph obtained as a result of these subdivisions we denote by $\gamma_w(G_{xy,uv})$.

2 Strong weakly connected domination subdivisible trees

We consider connected graphs of order $n \geq 3$. We begin with the following observations.

Observation 1 *If D is a minimum weakly connected dominating set with property $\mathcal{F}$ of G, then every support belongs to D.*

Observation 2 *If D is a minimum weakly connected dominating set of T, then every edge of T has at least one of its end-vertices in D.*

In [3] the following theorems have been proved.

Theorem 1 *[3] For any tree T of order at least 3,*

$$1 \leq sd_{\gamma_w}(T) \leq 2.$$

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Proposition 2 *[3] For a path P_n on $n \geq 3$ vertices,*

$$\gamma_w(P_n) = \lceil \frac{n-1}{2} \rceil.$$

Proposition 3 *[3] For a path P_n on $n \geq 3$ vertices,*

$$sd_{\gamma_w}(P_n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

We are now in position to constructively characterize all strong-γ_w-1-subdivisible trees. To this aim we define some operations and the family of trees.

Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, \ldots, T_j$ ($j \geq 1$) of trees such that T_1 is a star $K_{1,s}$ for $s \geq 2$ and $T = T_j$, and, if $j \geq 2$, T_{i+1} can be obtained from T_i by operation $\mathcal{Y}$ listed below.

We define the *status* of a vertex v, denoted $\mathrm{sta}(v)$, to be A or B where initially if $T_1 = K_{1,s}$ for $s \geq 2$, then $\mathrm{sta}(v) = A$ for a central vertex of T_1 and $\mathrm{sta}(v) = B$ for every leaf of T_1. Once a vertex is assigned a status, this status remains unchanged as the tree is recursively constructed.

Intuitively, if a vertex v has status A or B in a strong-γ_w-subdivisible tree, then using operation $\mathcal{Y}$ we construct a new strong-γ_w-subdivisible tree.

- **Operation $\mathcal{Y}$** The tree T_{i+1} is obtained from T_i by adding a star $K_{1,r}$ for $r \geq 2$ and an edge ab, where a is the vertex of T_i such that $\mathrm{sta}(a) = B$ and b is a center of a star $K_{1,r}$ and letting $\mathrm{sta}(b) = A$ and $\mathrm{sta}(x) = B$ for each end-vertex x from $K_{1,r}$.

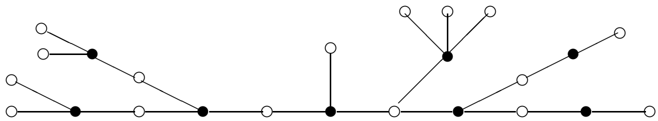

Figure 1: Example of a tree belonging to the family $\mathcal{T}$

By definition of the family $\mathcal{T}$ we immediately have the following observations.

Observation 3 *If T is a tree belonging to the family $\mathcal{T}$, then the distance between any two end-vertices in T is an even number.*

Observation 4 *If T is a tree belonging to the family $\mathcal{T}$ and there are k vertices with status A in T, then $D = \{a_1, \ldots, a_k\}$, where $sta(a_i) = A$ for $i = 1, \ldots, k$ is a unique minimum weakly connected dominating set of T.*

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Observation 5 *If T is a tree belonging to the family $\mathcal{T}$ and D is a unique minimum weakly connected dominating set of T, then D and $V(T) - D$ are independent sets of vertices and $sta(v) = A$ for every vertex v belonging to D and $sta(u) = B$ for every vertex u belonging to $V(T) - D$.*

Observation 6 *If T is a tree belonging to the family $\mathcal{T}$ obtained as a result of p operations $\mathcal{Y}$, then T can be divided into p stars $\{S_1, \ldots, S_p\}$ such that vertices with status A are centers of the stars, end-vertices of T are end-vertices of the stars and $V(S_i) \cap V(S_j) = \emptyset$ or $V(S_i) \cap V(S_j) = \{v\}$, where v is a non-end vertex with status B of T for $i, j \in \{1, \ldots, p\}, i \neq j$.*

Theorem 4 *A tree T is strong-γ_w-1-subdivisible if and only if T belongs to the family $\mathcal{T}$.*

Proof. Let T belong to the family $\mathcal{T}$. Then T is obtained from a star $K_{1,s}, s \geq 2$, using p times operation $\mathcal{Y}$ and, from Observation 6, can be divided into p stars. If we subdivide any edge uv of T, we have T_{uv} which can be divided into $p-1$ stars belonging to $\{S_1, \ldots, S_p\}$ and one star with one edge subdivided. From Observations 4 and 5 we have that $p-1$ vertices are needed to dominate stars and at least two vertices are necessary to dominate a star with one edge subdivided. Thus $\gamma_w(T_{uv}) \geq p+1 > p = \gamma_w(T)$.

Assume now that T is strong-γ_w-1-subdivisible and let D be a minimum weakly connected domination set of T. Since subdividing any edge increases the weakly connected domination number, every vertex belonging to D is an isolate in D and since D is weakly connected, every vertex belonging to $V(T) - D$ is an isolate in $V(T) - D$. Vertices of T can be partitioned into two classes: D, and we can assign a status A for every vertex x belonging to D and $V(T) - D$ with $\mathrm{sta}(y) = B$ for every vertex y belonging to $V(T) - D$. Thus T belongs to the family $\mathcal{T}$. ■

Domke et al. [1] have defined the class ε to be the class of trees obtained from P_2 by a finite sequence of the following operation: attach to any vertex a path P_2. A graph G is a γ_w-*excellent graph* if every vertex of G belongs to some $\gamma_w(G)$-set. In [3] the following theorem and proposition have been proved.

Theorem 5 *[3] Let T be a tree of order at least three. Then the following conditions are equivalent:*

1. *T belongs to the family ε;*
2. *T is a γ_w-excellent tree;*
3. *$sd_{\gamma_w}(T) = 2$;*
4. *$\gamma_w(T) = \frac{n(T)}{2}$;*

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5. $\beta(T) = \frac{n(T)}{2}$, *where* $\beta(T)$ *is a maximum cardinality of an independent set of* T.

Proposition 6 *[3] If a connected graph* G *has a strong support vertex, then* $sd_{\gamma_w(G)} = 1$.

Now we characterize all strong-γ_w-2-subdivisible trees. From Theorem 5, every strong-γ_w-2-subdivisible tree belongs to the family ε.

Lemma 7 *If* T *is strong-*γ_w*-2-subdivisible, then there is no strong support in* T.

Proof. Suppose that there exists a support s in T such that s is a strong support of T. Then, by Proposition 6, $sd_{\gamma_w(T)} = 1$ which contradicts the fact that T is strong-γ_w-2-subdivisible. ∎

Lemma 8 *If* T *is strong-*γ_w*-2-subdivisible and* D *is a minimum weakly connected dominating set with property* $\mathcal{F}$ *in* T, *then* $T[D] = K_2 \cup \overline{K_p}$, $p = \gamma_w(G) - 2$.

Proof. Suppose T is strong-γ_w-2-subdivisible and let D be a minimum weakly connected dominating set of T. If there are two or more edges in $T[D]$, then subdividing these edges does not change a weakly connected domination number, a contradiction. If $T[D] = \overline{K_p}$, $p = \gamma_w(T)$, then $T \in \mathcal{T}$ and subdividing of any edge increases the weakly connected domination number, again a contradiction. Thus every minimum weakly connected dominating set D with property $\mathcal{F}$ in T has $T[D] = K_2 \cup \overline{K_p}$, $p = \gamma_w(T) - 2$. ∎

Theorem 9 T *is strong-*γ_w*-2-subdivisible if and only if* T *is a path* P_n *with* n *vertices, where* n *is even.*

Proof. Let T be a path P_n with n vertices, where n is even. Then $n = 2k$ for some positive integer k. Hence $\gamma_w(P_n) = \lceil \frac{n-1}{2} \rceil = k$, $\gamma_w(P_{n+1}) = \lceil \frac{n}{2} \rceil = k$, and $\gamma_w(P_{n+2}) = \lceil \frac{n+1}{2} \rceil = k + 1$. After subdividing any two edges of P_n we obtain P_{n+2} and $\gamma_w(P_{n+2}) > \gamma_w(P_n)$, so if n is even, then P_n is strong-γ_w-2-subdivisible.

Suppose now T is strong-γ_w-2-subdivisible and T is different than P_n, where n is even. Then $sd_{\gamma_w}(T) = 2$ and from Theorem 5, T belongs to the family ε and T is different than P_n, where n is odd. Thus there is a P_{xy} connecting end-vertices x and y with even number of vertices and there is a vertex $v \in V(T)$ on this path such that $d_T(v) \geq 3$. Moreover, we can choose the third end-vertex z such that there is P_{vz}. Then, one of the P_{xv} and P_{yv} has even order and another one has odd order, what gives that one of the P_{xz} and P_{yz} has even order and another one has odd order.

Let us choose a minimum weakly connected dominating set D with property $\mathcal{F}$ in T. We will choose D in a following way, described below.

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Now, we consider two cases:
Case 1. Let v be a support vertex. A vertex from P_{vz} adjacent to v we denote by u. From Theorem 5 (T is γ_w-excellent tree) we can chose D such that $u \in D$. Obviously, v as a support, belongs to D. So $f = uv$ has both of its end-vertices in D. Moreover, from Observation 2 we have that $\frac{|V(P_{xy})|}{2}$ vertices from P_{xy} belong to D. Thus we have that one edge, let us say an edge $e, e \in E(P_{xy})$ has both of its end-vertices in D.
Case 2. Let v not be a support vertex. Without loss of generality assume that P_{xz} has even number of vertices and P_{yz} has odd order (if not we relabel the vertices x and y). Let us denote by u and w vertices such that $yu, uw \in E(P_{yz})$. From Theorem 5 (T is γ_w-excellent tree) we can chose D such that $w \in D$. Obviously, $u \in D$. Therefore, $f = uw$ has both its ends in D. On the other hand, from Observation 2 we have that $\frac{|V(P_{xz})|}{2}$ vertices from P_{xz} belong to D. Thus we have that one edge, let us say an edge $e, e \in E(P_{xz})$ has both of its end-vertices in D.

Finally, we can conclude that there exists such a minimum weakly connected set D with property $\mathcal{F}$ in T that two edges belong to $T[D]$, what gives a contradiction (see Lemma 8). Thus $T = P_n$, where n is even. ∎

3 Strong weakly connected domination subdivisible graphs

We begin with the following observation.

Observation 7 *If there exists a minimum weakly connected dominating set D in G such that $G[D]$ contains at least k edges, then G is not strong-γ_w-k-subdivisible.*

The following result is obtain in [3].

Proposition 10 *[3] If G is a connected graph of order at least 3 and e is an edge of G, then for the graph G' obtained from G by subdividing e,*

$$\gamma_w(G) \leq \gamma_w(G') \leq \gamma_w(G) + 1$$

Now we try to characterize strong-γ_w-1-subdivisible graphs.

Theorem 11 *Graph G is a strong-γ_w-1-subdivisible graph if and only if for every minimum weakly connected dominating set D with property $\mathcal{F}$ in G we have*

- $G[D] = \overline{K_p}$, $p = \gamma_w(G)$;
- *for every cycle C in G there exists an edge uv belonging to C such that $u \notin D$ and $v \notin D$.*

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Proof. Let G be a strong-γ_w-1-subdivisible graph. Suppose there exists a minimum weakly connected dominating set D with property $\mathcal{F}$ in G such that there is at least one edge, let us say $ab \in G[D]$. Then $\gamma_w(G_{ab}) = \gamma_w(G)$, a contradiction. Thus for every minimum weakly connected dominating set D with property $\mathcal{F}$ in G is $G[D] = \overline{K_p}, p = \gamma_w(G)$. Suppose there exists a cycle C in G such that every edge of C has exactly one vertex in D and the other one in $V - D$, where D is a minimum weakly connected dominating set with property $\mathcal{F}$ in G. Then subdividing of any edge of C does not increase the weakly connected domination number. Thus for every weakly connected dominating set D with property $\mathcal{F}$ in G there exists an edge uv in C such that $u \notin D$ and $v \notin D$.

Now let D be a minimum weakly connected dominating set with property $\mathcal{F}$ in G such that $G[D] = \overline{K_p}$, $p = \gamma_w(G)$ and for every cycle C in G there exists an edge uv belonging to C such that $u \notin D$ and $v \notin D$. Let $e = xy$ be an edge in G and let z be a new vertex in G_{xy} which appeared after subdividing the edge xy. Obviously $D \cup \{z\}$ is a weakly connected dominating set with property $\mathcal{F}$ in G_{xy} and $|D \cup \{z\}| = \gamma_w(G) + 1$. Suppose that $D \cup \{z\}$ is not a minimum weakly connected dominating set of G_{xy}, so there exists a minimum weakly connected dominating set D' such that $|D'| < |D \cup \{z\}|$. From Proposition 10, $|D'| = \gamma_w(G)$. We consider two cases:

Case 1. Let $z \in D'$

Case 1.1. First, let $x \notin D$ and $y \notin D$. It easy to verify that at least one of vertices x and y is adjacent to a vertex in D' different from z, if not then D' would not be a weakly connected set. Without loss of generality assume that x has this property. Then $D' \setminus \{z\} \cup \{x\}$ is a minimum weakly connected dominating set of G and K_2 is a subset of $D' \setminus \{z\} \cup \{x\}$, a contradiction.

Case 1.2. At least one of x and y belongs to D', without loss of generality we can say $x \in D'$. Hence, $D' \setminus \{z\}$ a weakly connected dominating set of G of order $\gamma_w(G) - 1$, a contradition.

Case 2. Now let $z \notin D'$

Case 2.1. Let $x \in D'$ and $y \in D'$. Then D' is a minimum weakly connected dominating set of G and K_2 is a subset of D', a contradiction.

Case 2.2. One of x and y belongs to D', assume that $x \in D'$. Then there is a vertex $w \in D'$ such that $yw \in E(G)$. Since D' is weakly connected, there exists a path P_{xw} connecting x and w not containing y and every edge on this path has one of its end-vertices in D'. Finally, D' is a minimum weakly connected dominating set of G and $P_{xw} \cup xy \cup yw$ is a cycle in G which does not contain an edge having both end-vertices in D', a contradiction.

Summing up we have that $D \cup \{z\}$ is a minimum weakly connected dominating set of G_{xy}. Hence, $\gamma_w(G_{xy}) = \gamma_w(G) + 1 > \gamma_w(G)$ for every edge $xy \in E(G)$ so G is a strong-γ_w-1-subdivisible graph. ■

Graphs shown in Figure 2 are γ_w-1-subdivisible. Moreover, for a graph G_1 we can find such a γ_w-set that K_2 is a subset of D and in a graph G_2 the only γ_w-set D does not have any edge with both of its end-vertices ends out of D. Hence, by

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Theorem 11 they are not strong-γ_w-1-subdivisible. On the other hand, the graph in Figure 3 is strong-γ_w-1-subdivisible.

Figure 2: Graphs G_1 and G_2, $\gamma_w(G_1) = \gamma_w(G_2) = 3$

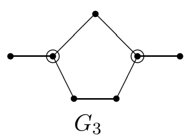

Figure 3: Graph G_3, $\gamma_w(G) = 2$

The following results are obtain in [3].

Proposition 12 *[3] If G is a connected graph of order at least 3 and $\gamma_w(G) = 1$, then $sd_{\gamma_w}(G) = 1$.*

Thus, it is easy to observe that the following holds.

Proposition 13 *If G is a connected graph of order at least 3 and $\gamma_w(G) = 1$, then G is a strong-γ_w-1-subdivisible graph.*

Proof. Let G be a connected graph of order at least 3 and $\gamma_w(G) = 1$. Hence G has an universal vertex. Let uv be an edge of a graph G. Then G_{uv} does not have an universal vertex and $\gamma_w(G_{uv}) = 2 > 1 = \gamma_w(G)$. So G is a strong-γ_w-1-subdivisible graph. ∎

Proposition 14 *[3] For a cycle C_n on $n \geq 3$ vertices,*

$$\gamma_w(C_n) = \lceil \frac{n-1}{2} \rceil.$$

Proposition 15 *[3] For a a cycle C_n on $n \geq 3$ vertices,*

$$sd_{\gamma_w}(C_n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

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Thus we have the following.

Proposition 16 *Cycle C_n is strong-γ_w-1-subdivisible graph if n is odd and is strong-γ_w-2-subdivisible graph if n is even.*

Proof. After subdividing any edge of a cycle C_n we obtain a cycle C_{n+1} and after subdividing any two edges we obtain C_{n+2}. Assume that $n = 2k + 1$ for an integer k. Then $\gamma_w(C_n) = k$ and $\gamma_w(C_{n+1}) = k + 1$, so C_n is strong-γ_w-1-subdivisible graph. Now assume that $n = 2k$ for an integer number k. Then $\gamma_w(C_n) = k$, $\gamma_w(C_{n+1}) = k$ and $\gamma_w(C_{n+2}) = k + 1$, so C_n is a strong-γ_w-2-subdivisible graph. ■

Proposition 17 *[3] Let $G = K_{m_1,m_2,\ldots,m_k}$ be the complete k partite graph, $k \geq 2$ with $m_1 \leq m_2 \leq \ldots \leq m_k$.*

- *If $m_1 = 1$, then $sd_{\gamma_w}(G) = 1$.*
- *If $m_1 \geq 2$, then $sd_{\gamma_w}(G) = 2$.*

Proposition 18 *Let $G = K_{m_1,m_2,\ldots,m_k}$ be the complete k partite graph, $k \geq 2$ with $m_1 \leq m_2 \leq \ldots \leq m_k$.*

- *If $m_1 = 1$, then G is a strong-γ_w-1-subdivisible graph.*
- *$K_{2,2}$ is a strong-γ_w-2-subdivisible graph.*
- *K_{2,m_2} for $m_2 \geq 3$ is not a strong-γ_w-2-subdivisible graph.*
- *If $m_1 \geq 3$ and $k = 2$, then G is a strong-γ_w-2-subdivisible graph.*
- *If $m_1 \geq 2$ and $k \geq 3$, then G is not a strong-γ_w-2-subdivisible graph.*

Conjecture 19 *Graph G is strong-γ_w-2-subdivisible if and only if $G = C_n$, where n is even.*

Conjecture 20 *There is no strong-γ_w-k-subdivisible graph for $k \geq 3$.*

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(Received 23 Dec 2009; revised 18 Feb 2010)

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