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Note

A note on the weakly convex and convex domination numbers of a torus

Joanna Raczek *, Magdalena Lemańska *

Department of Applied Physics and Mathematics, Gdansk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

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ABSTRACT

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest (u, v) path in G. A (u, v) path of length $d_G(u, v)$ is called a (u, v)-*geodesic*. A set $X \subseteq V$ is called *weakly convex* in G if for every two vertices $a, b \in X$, exists an (a, b)-geodesic, all of whose vertices belong to X. A set X is *convex* in G if for all $a, b \in X$ all vertices from every (a, b)-geodesic belong to X. The *weakly convex domination number* of a graph G is the minimum cardinality of a weakly convex dominating set of G, while the *convex domination number* of a graph G is the minimum cardinality of a convex dominating set of G. In this paper we consider weakly convex and convex domination numbers of tori.

1. Definitions

Here we consider simple, undirected and connected graphs $G = (V, E)$ with $|V| = n$. The *open neighbourhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices adjacent to v and the *closed neighbourhood* $N_G[v]$ of a vertex $v \in V$ is the set $N_G(v) \cup \{v\}$. A subset D of V is *dominating* if every vertex of $V - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality of a dominating set of G.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest (u, v) path in G. A (u, v) path of length $d_G(u, v)$ is called a (u, v)-*geodesic*.

A set $X \subseteq V$ is called *weakly convex* in G if for every two vertices $a, b \in X$, exists an (a, b)-geodesic, all of whose vertices belong to X. A set X is *convex* in G if for all $a, b \in X$ all vertices from every (a, b)-geodesic belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* in G if X is weakly convex and dominating. Further, X is a *convex dominating set* if it is convex and dominating. The *weakly convex domination number* $\gamma_{\mathrm{wcon}}(G)$ of a graph G is the minimum cardinality of a weakly convex dominating set of G, while the *convex domination number* $\gamma_{\mathrm{con}}(G)$ of a graph G is the minimum cardinality of a convex dominating set of G. Convex and weakly convex domination numbers were first introduced by Topp [4].

2. Cartesian product

The *Cartesian product* of two graphs G_1 and G_2, denoted by $G = G_1 \Box G_2$, is the graph with vertex set $V(G) = V(G_1) \times V(G_2)$, where two vertices (u_1, u_2), (v_1, v_2) are adjacent in $G_1 \Box G_2$ if and only if the following holds:

(a) $u_1 = v_1$ and $u_2 v_2 \in E(G_1)$,
(b) $u_2 = v_2$ and $u_1 v_1 \in E(G_2)$.

For $v_i \in V(G_2)$, G_1^i denotes the subgraph of $G_1 \Box G_2$ induced by $V(G_1) \times \{v_i\}$ and we call G_1^i the ith copy of G in $G_1 \Box G_2$. If $V(G_2) = \{v_1, \ldots, v_n\}$, then G_1^i and G_1^j are *neighbouring copies* in $G_1 \Box G_2$ if $v_i v_j \in E(G_2)$. The Cartesian product of two cycles C_m and C_n is called a *torus*, if $m \geq 3$ and $n \geq 3$. The problem of domination in a torus was considered in [5]. The bondage

* Corresponding author.
E-mail addresses: gardenia@pg.gda.pl (J. Raczek), magda@mif.pg.gda.pl (M. Lemańska).

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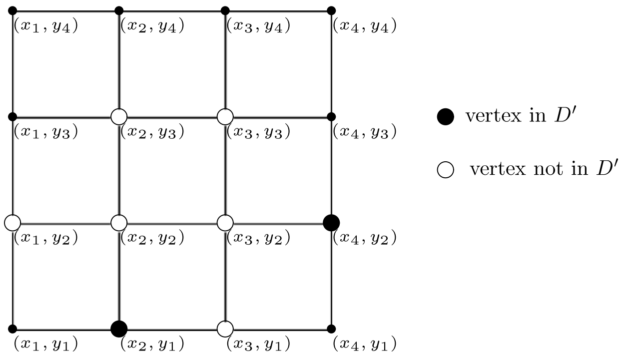

Fig. 1. The case (a).

number and signed domatic number of a torus is studied in [2] and in [3], respectively. Here we consider weakly convex and convex domination numbers of a torus.

In 1963 Vizing conjectured that $\gamma(G_1 \square G_2) \geq \gamma(G_1)\gamma(G_2)$ for every two graphs G_1 and G_2. In [1] it was proven that the following Vizing-type inequality for the convex domination number is true.

Theorem 1 ([1]). *For connected graphs G_1 and G_2,*

$$\gamma_{\text{con}}(G_1)\gamma_{\text{con}}(G_2) \leq \gamma_{\text{con}}(G_1 \square G_2).$$

It is easy to verify that the convex domination number of a cycle C_m on $m \geq 6$ vertices equals m. Hence, Theorem 1 implies what follows.

Corollary 2. *Let C_m and C_n be cycles on $m \geq 6$ and $n \geq 6$ vertices, respectively. Then*

$$\gamma_{\text{con}}(C_m \square C_n) = mn.$$

A similar result may be proven for the weakly convex domination number.

Theorem 3. *Let C_m and C_n be the cycles on $m \geq 7$ and $n \geq 7$ vertices, respectively. Then*

$$\gamma_{\text{wcon}}(C_m \square C_n) = mn.$$

Proof. Let $G = C_m \square C_n$, where $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_n), 1 \leq i \leq m, 1 \leq j \leq n\}$ and $m, n \geq 7$. Suppose $\gamma_{\text{wcon}}(G) < mn$. Let D be a minimum weakly convex dominating set of G. Since $|D| < mn$ we assume, without of loss of generality, that $(x_3, y_3) \notin D$. Then (x_3, y_3) belongs to the $((x_3, y_2), (x_3, y_4))$-geodesic and thus $(x_3, y_2) \notin D$ or $(x_3, y_4) \notin D$. Without loss of generality, let $(x_3, y_2) \notin D$. Similarly, (x_3, y_3) belongs to the $((x_3, y_1), (x_3, y_4))$-geodesic and thus we let $(x_3, y_1) \notin D$. Further, (x_3, y_3) belongs to the $((x_2, y_3), (x_4, y_3))$-geodesic so, without loss of generality, we let $(x_2, y_3) \notin D$.

Each $((x_2, y_2), (x_3, y_4))$-geodesic contains (x_3, y_3) or (x_3, y_2) or (x_2, y_3), so $(x_2, y_2) \notin D$ or $(x_3, y_4) \notin D$.

(a) If $(x_2, y_2) \notin D$, then $(x_4, y_2) \in D$ dominates (x_3, y_2). Moreover, since D is weakly convex, $(x_1, y_2) \notin D$. However this implies that $(x_2, y_1) \in D$ which is a contradiction, because no $((x_2, y_1), (x_4, y_2))$-geodesic is contained in D. (See Fig. 1, some edges in the figures are omitted to make them more clear.)

(b) If $(x_3, y_4) \notin D$, then $(x_4, y_3) \in D$ dominates (x_3, y_3). Moreover, since D is weakly convex, $(x_1, y_3) \notin D$ and $(x_2, y_4) \notin D$. However, this implies that $(x_2, y_2) \in D$ which is a contradiction, because no $((x_2, y_2), (x_4, y_3))$-geodesic is contained in D. (See Fig. 2.) $\square$

The following propositions give exact values for the convex domination number and the weakly convex domination number for a torus $G = C_m \square C_n$, where $n \in \{3, 4, 5, 6\}$. We start with a lemma.

Lemma 4. *Let $G = C_m \square C_n$, where $n \geq 3$ and $m \geq 5$. Let D be a minimum weakly convex dominating set of G such that $(x_3, y_j) \notin D$ for $j \in \{1, 2, 3\}$. Then, without loss of generality, $(x_i, y_j) \notin D$ for $i \in \{2, 3\}, j \in \{1, 2, 3\}$ and $(x_1, y_2) \in D$, $(x_4, y_2) \in D$.*

Proof. Let G and D be defined as above. Since D is dominating, (x_3, y_2) has a neighbour in D. Without loss of generality let $(x_4, y_2) \in D$. Since $m \geq 5$, each $((x_2, y_1), (x_4, y_2))$-geodesic contains (x_3, y_1) or (x_3, y_2), so the weak convexity of D implies that $(x_2, y_1) \notin D$. By similar arguments, $(x_2, y_3) \notin D$ and $(x_2, y_2) \notin D$. Hence, $(x_1, y_2) \in D$ dominates (x_2, y_2). (See Fig. 3.) $\square$

Proposition 5. *Let $G = C_m \square C_3$, where $m \geq 3$. Then*

$$\gamma_{\text{wcon}}(G) = \gamma_{\text{con}}(G) = m.$$

Proof. Since $\gamma_{\text{con}}(G) \geq \gamma_{\text{wcon}}(G) \geq \gamma(G) = 3$ and the set $\{(x_2, y_1), (x_2, y_2), (x_2, y_3)\}$ is a convex dominating set of G, the result is true for $m = 3$. Let $G = C_m \square C_3$, where $m \geq 4$ and denote $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_3), 1 \leq i \leq$

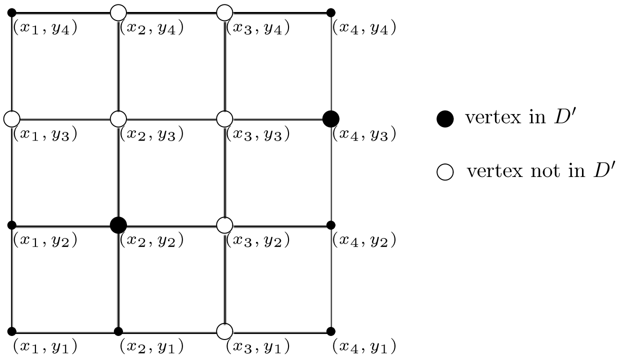

Fig. 2. The case (b).

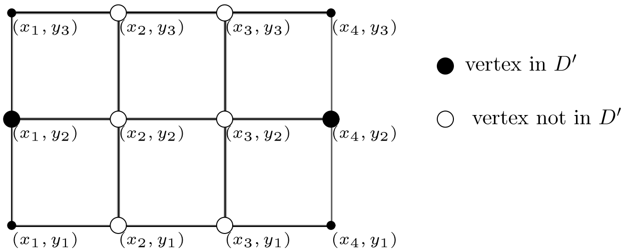

Fig. 3. Illustration for the Lemma 4.

m, $1 \le j \le 3\}$. We show that $D = \{(x_i, y_1) : 1 \le i \le m\}$ is a minimum convex and weakly convex dominating set of G. It is obvious that D is convex and dominating, so $\gamma_{\mathrm{wcon}}(G) \le \gamma_{\mathrm{con}}(G) \le m$. Suppose D is not a minimum weakly convex dominating set of G. Then there exists a weakly convex dominating set $D' \subseteq V(G)$ such that $|D'| < |D| = m$. However, it is possible to verify that for $C_4 \square C_3$ we cannot find a weakly convex dominating set of cardinality at most 3. Hence, in what follows we assume $m \ge 5$. Since $|D'| < m$, there exists an index i such that all three vertices of $V(C_3^i)$ do not belong to D'. Without loss of generality, let $i = 3$. By Lemma 4, we obtain that $(x_l, y_k) \notin D'$ for $l \in \{2, 3\}$, $k \in \{1, 2, 3\}$ and $(x_1, y_2) \in D'$, $(x_4, y_2) \in D'$ (See Fig. 4.)

Recall that $\gamma_{\mathrm{wcon}}(C_k) = k$ for $k \ge 7$. For this reason, since D' is weakly convex and $d_G((x_1, y_2), (x_4, y_2)) \le 3$, we conclude that $m \le 6$. Hence, $|D'| < 6$. Since D' is dominating, (x_1, y_1), (x_1, y_3), (x_4, y_1), $(x_4, y_3) \in D'$. However, this implies that $|D'| \ge 6$, a contradiction. Therefore, $\gamma_{\mathrm{wcon}}(G) = \gamma_{\mathrm{con}}(G) = m$. □

Proposition 6. *Let $G = C_m \square C_4$, where $m \ge 7$. Then*

$$\gamma_{\mathrm{wcon}}(G) = \gamma_{\mathrm{con}}(G) = 2m.$$

Proof. Let $G = C_m \square C_4$, where $m \ge 7$ and denote $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_4), 1 \le i \le m, 1 \le j \le 4\}$. We show that $D = \{(x_i, y_j) : 1 \le i \le m, 1 \le j \le 2\}$ is a minimum weakly convex and convex dominating set of G. It is obvious that D is convex and dominating, so $\gamma_{\mathrm{wcon}}(G) \le \gamma_{\mathrm{con}}(G) \le 2m$. Suppose D is not a minimum weakly convex dominating set of G. Then there exists a weakly convex dominating set $D' \subseteq V(G)$ such that $|D'| < |D| = 2m$. Then there exists an index i such that at least three vertices of $V(C_4^i)$ do not belong to D'. Without loss of generality, let $(x_3, y_1), (x_3, y_2), (x_3, y_3) \notin D'$. Now, by Lemma 4, we obtain that $(x_l, y_k) \notin D'$ for $l \in \{2, 3\}$, $k \in \{1, 2, 3\}$ and $(x_1, y_2) \in D'$, $(x_4, y_2) \in D'$. Now, since D' is weakly convex and $d_G((x_1, y_2), (x_4, y_2)) \le 3$, we conclude that $m \le 6$ which is impossible. Hence, $\gamma_{\mathrm{wcon}}(G) = \gamma_{\mathrm{con}}(G) = 2m$. □

The straightforward, albeit technical proof of the following observation is omitted.

Observation 7. *The weakly convex domination number and convex domination numbers of the Cartesian product of C_m and C_4, where $m \in \{4, 5, 6\}$, are also equal to $2m$.*

Proposition 8. *Let $G = C_m \square C_5$, where $m \ge 5$. Then*

$$\gamma_{\mathrm{wcon}}(G) = \gamma_{\mathrm{con}}(G) = 3m.$$

Proof. Let $G = C_m \square C_5$, where $m \ge 5$ and denote $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_5), 1 \le i \le m, 1 \le j \le 5\}$. We show that $D = \{(x_i, y_j) : 1 \le i \le m, 1 \le j \le 3\}$ is a minimum convex and weakly convex dominating set of G. It is obvious that D is convex and dominating, so $\gamma_{\mathrm{wcon}}(G) \le \gamma_{\mathrm{con}}(G) \le 3m$. Suppose D is not a minimum weakly convex dominating set

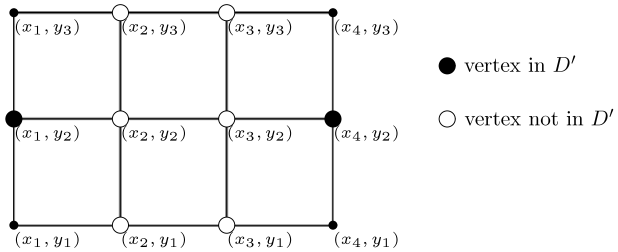

Fig. 4. The $C_m \square C_3$ (some edges are omitted).

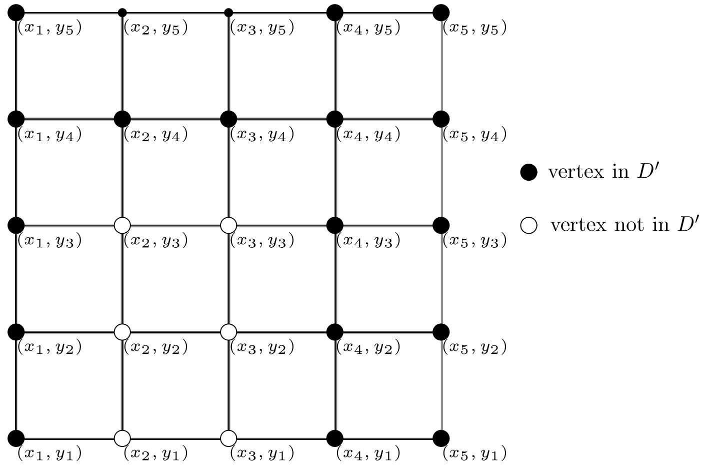

Fig. 5. Case (a) of $C_m \square C_5$.

of G. Then there exists a weakly convex dominating set $D' \subseteq V(G)$ such that $|D'| < |D| = 3m$. Therefore, there exists an index i such that at least three vertices of $V(C_5^i)$ do not belong to D'. Since D' is weakly convex, those three vertices induce a P_3 in G. Without loss of generality, let $(x_3, y_1), (x_3, y_2), (x_3, y_3) \notin D'$. Now, by Lemma 4, we obtain that $(x_l, y_k) \notin D'$ for $l \in \{2, 3\}$, $k \in \{1, 2, 3\}$ and $(x_1, y_2) \in D'$, $(x_4, y_2) \in D'$. Again we conclude that $m \leq 6$.

If $(x_4, y_3) \notin D'$, then $(x_3, y_4) \in D'$ and so no $((x_4, y_2), (x_3, y_4))$-geodesic is contained in D'. Thus we conclude that $(x_4, y_3) \in D'$. By similar reasoning and symmetry we obtain that $(x_1, y_3) \in D'$, $(x_4, y_1) \in D'$ and $(x_1, y_1) \in D'$. Further, since D' is weakly convex, we have that $(x_5, y_k) \in D'$ for $k \in \{1, 2, 3\}$ and if $m = 6$ then, additionally, $(x_6, y_k) \in D'$ for $k \in \{1, 2, 3\}$.

(a) Suppose $(x_3, y_4) \in D'$. Then (x_4, y_4) belongs to the $((x_3, y_4), (x_4, y_3))$-geodesic that does not contain vertices from $V(G) - D'$, so $(x_4, y_4) \in D'$. Similarly, considering the $((x_3, y_4), (x_1, y_3))$-geodesic outside of D' we conclude that $(x_1, y_4) \in D'$ and $(x_2, y_4) \in D'$. This implies that $(x_5, y_4) \in D'$, $(x_5, y_5) \in D'$, $(x_4, y_5) \in D'$ and $(x_1, y_5) \in D'$ (see Fig. 5). Then $|D'| \geq 17$ for $m = 5$ and $|D'| \geq 20$ for $m = 6$. However this contradicts $|D'| < 3m$.

(b) Thus, $(x_3, y_4) \notin D'$. If $(x_2, y_4) \in D'$, then each $((x_2, y_4), (x_4, y_3))$-geodesic would contain vertices from $V(G) - D'$, so $(x_2, y_4) \notin D'$. If $(x_4, y_4) \notin D'$, then $(x_3, y_5) \in D'$ dominates (x_3, y_4). However, each $((x_3, y_5), (x_4, y_3))$-geodesic contains vertices from $V(G) - D'$. Thus, $(x_4, y_4) \in D'$. Similarly we obtain that $(x_1, y_4) \in D'$. Since D' is weakly convex, $(x_5, y_4) \in D'$, $(x_1, y_5) \in D'$, $(x_4, y_5) \in D'$ and $(x_5, y_5) \in D'$ (see Fig. 5). Thus, $|D'| \geq 15$ for $m = 5$. If $m = 6$, then $(x_6, y_k) \in D'$ for $k \in \{1, 2, \ldots, 5\}$ and thus $|D'| \geq 20$. However both cases contradict the fact that $|D'| < 3m$ (Fig. 6). $\square$

Our last result for the Cartesian product of two cycles concerns only weakly convex domination.

Proposition 9. *Let $G = C_m \square C_6$, where $m \geq 6$. Then*

$$\gamma_{\mathrm{wcon}}(G) = 4m.$$

Proof. Let $G = C_m \square C_6$, where $m \geq 6$ and denote $V(G) = \{(x_i, y_j) : x_i \in V(C_m), y_j \in V(C_6), 1 \leq i \leq m, 1 \leq j \leq 6\}$. We show that $D = \{(x_i, y_j) : 1 \leq i \leq m, 1 \leq j \leq 4\}$ is a minimum weakly convex dominating set of G. It is obvious that D is weakly convex and dominating, so $\gamma_{\mathrm{wcon}}(G) \leq 4m$. Suppose D is not a minimum weakly convex dominating set of G. Then there exists a weakly convex dominating set $D' \subseteq V(G)$ such that $|D'| < |D| = 4m$ and therefore there exists an index i such that at least three vertices of $V(C_6^i)$ do not belong to D'. Since D' is weakly convex, those three vertices induce a P_3 in

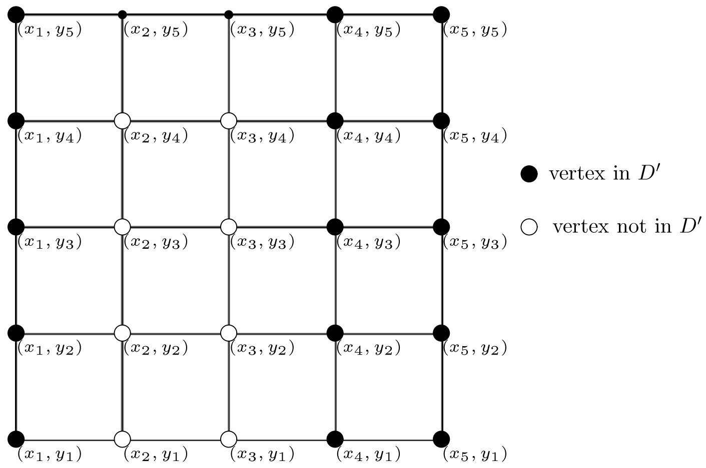

Fig. 6. Case (b) of $C_m \Box C_5$.

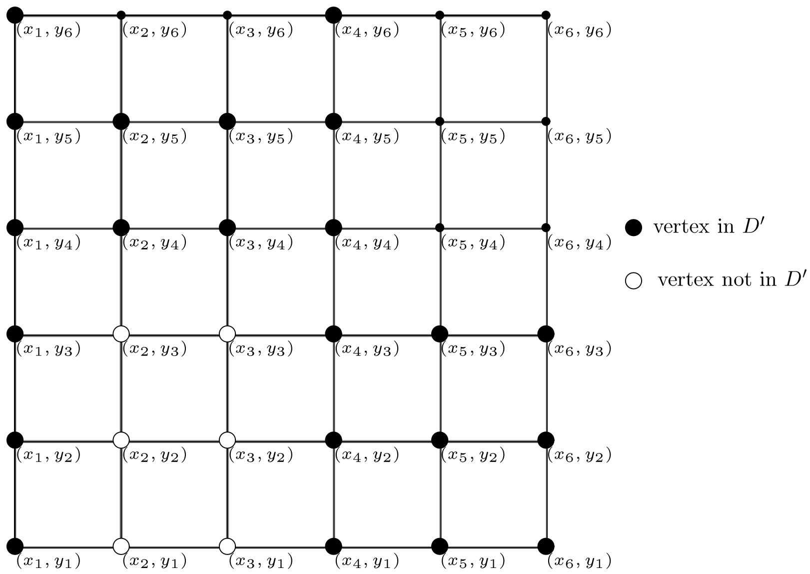

Fig. 7. Case (a): subcase $(x_3, y_5) \in D'$ of $C_6 \Box C_6$.

G. Without loss of generality, let $(x_3, y_1), (x_3, y_2), (x_3, y_3) \notin D'$. Now, by Lemma 4, we obtain that $(x_l, y_k) \notin D'$ for $l \in \{2, 3\}$ and $k \in \{1, 2, 3\}$ and $(x_1, y_2) \in D'$, $(x_4, y_2) \in D'$. Again we conclude that $m \leq 6$.

Moreover, by similar reasoning as in the proof of previous proposition, we obtain that $(x_l, y_k) \in D'$ for $l \in \{1, 4, 5, 6\}$ and $k \in \{1, 2, 3\}$.

(a) If $(x_3, y_4) \in D'$, then by considering $((x_3, y_4), (x_4, y_3))$-geodesics we conclude that $(x_4, y_4) \in D'$. Similarly, (x_1, y_4) and (x_2, y_4) belong to a $((x_3, y_4), (x_1, y_3))$-geodesic and thus they belong to D'. If $(x_3, y_5) \in D'$, then each $((x_3, y_5), (x_4, y_1))$-geodesic in D' contains (x_4, y_6), so $(x_4, y_6) \in D'$ and further $(x_4, y_5) \in D'$. Similarly, each $((x_3, y_5), (x_1, y_1))$-geodesic in D' contains (x_1, y_6), so $(x_1, y_6) \in D'$ and further $(x_1, y_5) \in D'$ and thus $(x_2, y_5) \in D'$ (see Fig. 7). Since D' is weakly convex, $(x_2, y_6), (x_3, y_6) \in D'$ or $(x_5, y_6), (x_6, y_6) \in D'$. In both situations $|D'| \geq 24 = 4m$, a contradiction.

Therefore we conclude $(x_3, y_5) \notin D'$. Since D' is weakly convex, $(x_3, y_6) \notin D'$. If $(x_2, y_5) \in D'$, then each $((x_2, y_5), (x_4, y_1))$-geodesic would contain vertices of $V(G) - D'$ which is impossible. Thus $(x_2, y_5) \notin D'$ and further $(x_2, y_6) \notin D'$. Since D' is dominating, $(x_1, y_6) \in D'$, $(x_4, y_6) \in D'$. Moreover, the weakly convexity of D' implies that $(x_l, y_5) \in D'$ for $l \in \{1, 4, 5, 6\}$ and $(x_l, y_6) \in D'$ for $l \in \{5, 6\}$ (see Fig. 8). However then $|D'| \geq 24$, a contradiction.

(b) If $(x_3, y_4) \notin D'$, then by the similar reasoning as in the proof of Proposition 8, we obtain that $(x_2, y_4) \notin D'$ and $(x_l, y_4) \in D'$ for $l \in \{1, 4, 5, 6\}$. Now, the rest of the proof is similar to the proof of the Case (a) and (b) of Proposition 8. □

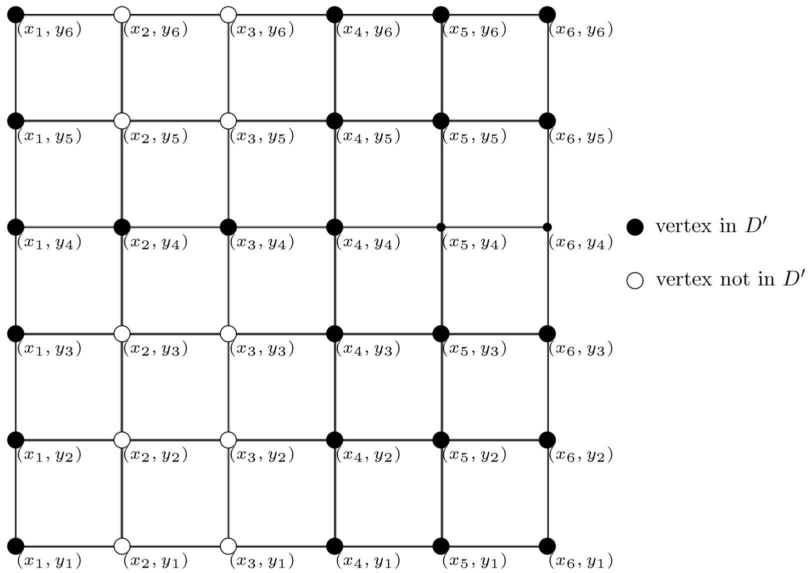

Fig. 8. Case (a): subcase $(x_3, y_5) \notin D'$ of $C_6 \square C_6$.

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