DISCREPANCIES OF ENERGY VALUES IN DYNAMICS OF THREE INTERSECTING PLATES

J. Chróścielewski, W. Witkowski
Faculty of Civil and Environmental Engineering, Gdansk University of Technology, Narutowicza 11/12, 80-952 Gdańsk, Poland

Corresponding author: Wojciech Witkowski, wojwit@pg.gda.pl,
phone +48 58 347 21 74, fax +48 58 347 16 70

ABSTRACT

In this paper we discuss the discrepancies between results reported in the literature in the context of dynamics of shell structure composed of three intersecting plates. The shell structure is subjected to a system of spatially uniformly distributed dead loads of prescribed time variation. Once the forces die out the structure experiences free motion. The comparison of reported solutions from literature shows that the total energy in free motion has different values, despite the fact that the same material, geometry and loads have been used. The aim of study is to address the issue by referring own results to that known from literature.

1. THE PROBLEM

The example discussed here belong to the tumbling flexible structures. In literature a spaghetti problem was first modeled and analyzed in [1] and then in [2], [3], [4] using rod formulation. These works gave an impulse for analyzing this type of problems using various rod and shell formulations. The ultimate goal and the culmination of the works by Simo and Vu-Quoc on flying flexible structures, with application to satellite dynamics, can be found in [5].

The example discussed here, see Fig. 1, is concerned with motion (without gravity loads) of irregular shell structure subjected to spatial system of uniformly distributed dead loads of given time variation. After the driving loads die out the structure experiences free motion where, by definition, the total energy, linear momentum vector and angular momentum vector of the structure must remain constant. The problem has been studied, among others, in [6], [7] and [8], yet in the latter reference the load was defined differently than in the two former papers. Consequently, the results from [8] can not be discussed directly here. The comparison of results reported in [6] and [7] indicates significant discrepancies between computed total energy of the structure. The reported values are respectively: approximately 70 and approximately 20. What makes the matter interesting is that in both papers the same geometrical, material and load parameters are reported to have been used.

That the structure is composed of orthogonally intersecting plates and experiences unlimited rotations and translations make the example a challenging task for every shell theory incorporating the sixth degree of freedom, the finite elements elaborated within that theory and temporal integration schemes. The lack of boundary constraints excludes the use of finite elements with uniform reduced or selective reduced integration of matrices, with uncontrolled spurious zero-energy forms. Consequently higher order elements, or elements specially formulated to minimize locking effect, should be employed.

We have tried to study the example on the grounds of 6-parameter geometrically and statically exact nonlinear shell theory and temporal integration schemes that have been discussed in the following references: [9][10][11][12][13][14][15][16]. The main aspects of the underlying theory of shells are briefly presented in Appendix A.

Nonetheless, despite the fact that we have used the material, load and geometrical parameters as reported in [6] and [7] our computed values of total energy in free motion do not coincide with any value reported in the literature. This fact has been already addressed in [17].

2. SOLUTIONS

The shell in general is a three dimensional body merely of specific geometry. As such it obeys usual balance laws of continuum mechanics expressed by, among others, quantities: mass m, resultant force vector $\boldsymbol{F}$, linear momentum vector $\boldsymbol{L}$, resultant moment vector $\boldsymbol{M}$ and moment of momentum vector $\boldsymbol{J}$. In shell structures, as in every mechanical system, the universal laws of mechanics must be satisfied. These laws are: balance of mass

$$\left[m(t)\right]_{t_1}^{t_2}=0 \tag{1}$$

balance of linear momentum

$$\left[\boldsymbol{L}(t)\right]_{t_1}^{t_2}=\int_{t_1}^{t_2}\boldsymbol{F}(\tau)d\tau \tag{2}$$

balance of angular momentum

$$\left[\boldsymbol{J}(t)\right]_{t_1}^{t_2}=\int_{t_1}^{t_2}\boldsymbol{M}(\tau)d\tau \tag{3}$$

where $\left[\mathsf{f}(\tau)\right]_{t_1}^{t_2}\equiv\mathsf{f}(t_2)-\mathsf{f}(t_1)$. In structural dynamics problems the mass is preserved and thus will be not dealt with here. The laws of conservation of momenta may be rewritten as

$$\boldsymbol{L}(t_2)-\boldsymbol{L}(t_1)-\int_{t_1}^{t_2}\boldsymbol{F}(t)dt=\boldsymbol{0}\quad\text{and for }\boldsymbol{L}(0)=\boldsymbol{0}\,,\quad\boldsymbol{L}(t)-\int_0^t\boldsymbol{F}(\tau)d\tau=\boldsymbol{0} \tag{4}$$

$$\boldsymbol{J}(t_2)-\boldsymbol{J}(t_1)-\int_{t_1}^{t_2}\boldsymbol{M}(t)dt=\boldsymbol{0}\quad\text{and for }\boldsymbol{J}(0)=\boldsymbol{0}\,,\quad\boldsymbol{J}(t)-\int_0^t\boldsymbol{M}(\tau)d\tau=\boldsymbol{0} \tag{5}$$

In the above forms the balance of linear and angular momenta may be viewed as additional error estimation of the time integration scheme. Moreover, in the error context, the energy of the non-dissipative structure may be also used, writing the energy equation in the form analogical to (4) or (5)

$$U(t)+K(t)-G_{ext}(t)=0 \tag{6}$$

Here U denotes potential energy, K stands for kinetic energy and G_{ext} denotes work done by external loads. Note that in case of dead loads $\boldsymbol{F}(\tau)\neq\boldsymbol{F}\left(\boldsymbol{x}(\tau),\tau\right)$ the second component of equation (4) and right hand side of (2) are independent of deformation and hence independent of mesh discretization and may be integrated exactly immediately. Then equation (4) may be treated as one of criteria of verification of the theory and volume (mass) integration scheme. We employ this fact in our study.

In the present study two families of shell finite elements have been used: the 4–, 9– and 16–node, displacement/rotation based CAM (Computer and Analytical Modeling) elements [9], [10], [11] with full integration (FI) and 4– and 9–node semi-mixed elements (SEM) (FI) [9], [11]. While the CAM elements are used with energy conserving algorithm (ECA, [15], [16]) the SEM elements are used with Newmark type scheme described in [9] and [14]. Appendix B contains the example presenting the essential verification of the ECA scheme.

The parameters of the example used in the present study follow from [6]. For completeness they are set in Table 1 together with original values reported by other researchers. It is stressed that we are only sure about our parameters used in this communication. In particular we use the following

$$I_0 = \rho_I \frac{h_0^3}{12} = 50\frac{0.02^3}{12} = 3.333\times 10^{-5}\,,\ m_0 = \rho_m h_0 = 1.0\cdot 0.02 = 0.02 \tag{7}$$

In equations (7) following [6], we have introduced different mass densities ρ_I and ρ_m for rotational component I_0 and translational component m_0 respectively. Such case may exists for composite shells.

Time history of load is prescribed by function (cf. Fig.1)

$$p(t) = \begin{cases} \frac{1}{2}t & for \quad 0 \le t \le \frac{1}{2}\,s \\ \frac{1}{2}(1-t) & for \quad \frac{1}{2} \le t \le 1\,s \\ 0 & for \qquad t \ge 1\,s \end{cases} \tag{8}$$

The spatial distribution of the reference load along structure edges reported in the literature is problematic. The question arises whether the load should be applied as the uniformly distributed or in the form of point forces. In [6] the load acting on structure has been described as „…nodal loading at…" (cf. Fig. 6 of [6]). References [7] and [8] do not throw additional light on this matter. Analysis of FEM mesh used in [6] and [7] reveals that along edges C and D (cf. Fig. 6 of [6] and Fig. 9 of [7]) three 4-node elements have been used. This suggests that load can not be unambiguously applied as the point force situated in the half-length of respective edge $s_{(i)}$. It suggests therefore, that the load is treated as the piecewise uniformly distributed $\boldsymbol{q}_{ref\,(i)}(s_{(i)})$ along directions of global axes along appropriate edges $s_{(i)}$ of i^{th} plate. Such approach has been used here – cf. Fig. 1. For completeness we report the values of load resultants from each edge with the coordinates

$$R_{AB}(x;y;z) = (8;0;-8)\,, \qquad (x^0_{AB};y^0_{AB};z^0_{AB}) = (-5;-3;0) \tag{9}$$

$$R_{CD}(x;y;z) = (-8;-16;8)\,, \qquad (x^0_{CD};y^0_{CD};z^0_{CD}) = (-5;6;0) \tag{10}$$

$$R_{EF}(x;y;z) = (30;18;-3)\,, \qquad (x^0_{EF};y^0_{EF};z^0_{EF}) = (5;1.5;7) \tag{11}$$

$$R_{GH}(x;y;z) = (3;-3;-3)\,, \qquad (x^0_{GH};y^0_{GH};z^0_{GH}) = (5;1.5;-7) \tag{12}$$

The components of resultant force vector $\boldsymbol{F}$ and resultant moment vector $\boldsymbol{M}$ (computed about the origin of the assumed coordinate system, cf. Fig. 1) and their Euclidean norms are therefore

$$F_x = 33\,, \qquad F_y = -1\,, \qquad F_z = -6\,, \qquad |F| = 33.556 \tag{13}$$

$$M_x = -84\,, \qquad M_y = 219\,, \qquad M_z = 177.5\,, \qquad |M| = 294.15 \tag{14}$$

In the light of the addressed discrepancies between known reference solutions we have carried out *h*–type and *p*–type convergence analysis of total energy of the structure in free motion. In the analysis the following rules have been applied:

(1) every plate is divided into the same number of finite elements,

(2) within a plate the same regular division $N \times 3N$ with respect to the shorter edge N is used,

(3) N takes on even values $N = 2, 4, 6, ...$

The justification of using the energy as the measure of the solution correctness follows from the following facts. The total energy of the system is well defined scalar value. Moreover, with unambiguously defined parameters of: material, geometric and load when the latter die out the energy is constant during the free motion. As such it may be regarded as physical property of the system, understood as the structure and loads. Therefore, it is independent of time step, used temporal approximation scheme and spatial approximation (finite elements used, mesh density).

In [15] and [16] the time histories of energy have been presented, computed using meshes 8×24 CAMe16 (FI) and 4×12 CAMe9(FI) elements (for each plate). The values from [15] and [16] for $\Delta t = 0.002\,s$ computed with the former mesh are repeated here in Fig. 2 with results known from literature. The figure shows also the result corresponding to (almost) rigid body solution obtained with Young's modulus ($E \to E \times 10^6$) in coarse mesh 4×12 CAMe4(FI). This discretization strongly "locks" the results as a consequence of locking effect associated with classical displacement formulation of 4–node element of Lagrange type with full integration. This (almost) rigid body results has been confirmed by the solution of the structure treated analytically by assumption as the rigid body [13], obtained using algorithms independent of those discussed here.

Fig. 2 shows that in our solutions the energy is bounded between approximately 60 and approximately 19 (19.6069). The lower bound is computed as the (almost) rigid body solution The figure clearly indicates different energy from [6] and [7]. What is interesting, the solution obtained in the present (almost) rigid body solution is very close to value from [7].

In the subsequent analysis, apart from CAM elements [9], [10], the semi-mixed SEM elements [9], [11] resulting from displacement/stress formulation have been used. The use of two types of elements stemming from different and independent variational formulations, and partially different concept of interpolation, has enabled us to detect possible error contributed by the element itself. The computed values of total energy $U + K$ of the system at $t = 1s$ i.e. the at the transition to free motion as the function of number of nodes on the shorter edge of the plate are depicted in Fig. 3 and Fig. 4. In the figures $n = 3$ nodes corresponds to 342 degrees of freedom of the whole structure while $n = 25$ nodes corresponds to 32550 degrees of freedom of the whole structure.

From Fig. 3, depicting *p*–type convergence analysis, it may be inferred that the curves are bounded by value around. 60. The curves monotonically converge to the least upper bound with the increase of number of degrees of freedom. Such convergence is typical for compatible elements, for instance CAM. Fig. 4 shows somewhat different exposition of convergence analysis carried out using 4–, 9– and 16–node CAM elements employing bilinear, biquadratic and bicubic shape functions with full integration. The values computed using 4–node elements CAMe4(FI) are evidently polluted by locking effect. All the computed energy values, for the sake of clarity, are set in Table 2. The values have been found using full Gaussian quadrature rule i.e.: 2×2 for 4–node elements, 3×3 for 9– node elements and 4×4 for 16– node elements.

Analyzing the error of preservation of linear and angular momenta i.e. equations (4) and (5), we have verified that the equation (4) is recovered exactly by the numerical integration scheme, yielding the analytical values from Table 3. For angular momentum components we have performed the *p*–type and *h*–type convergence analysis. The results are depicted in Fig. 5, Fig. 6 and Fig. 7 and set in Table 4. It is seen that the convergence is achieved better than in case of energy.

In addition, Fig. 8. depicts some representative configurations at selected time instances. Since originally the configurations would overlap, they have been moved away from each other.

3. CONCLUSIONS

All own results have been obtained on the grounds of 6–parameter geometrically and statically exact nonlinear shell theory. The spatial discretization has been carried out with the help of finite elements based on two different independent variational principles. Furthermore, different interpolation schemes have been used in the elements. On the basis of the presented own energy convergence analysis, the authors venture to say that the total energy of the analyzed structure is bounded between 19 and 60. The obtained convergence curves exhibit typical for compatible elements monotonical characteristic.

Unfortunately, the computed values are different than those known from the literature. It is worth mentioning that the discussed example, due to reasons given in Section 1 (three orthogonally intersecting plates, finite rotations and translations in free motion) is relatively seldom undertaken by other researchers. Another possible explanation is an imprecise definition of load in the source paper [6] and vague definition of material properties.

Based on own study it has been also found that the use in computations of lumped mass matrix instead of consistent mass formulation is of negligible influence on the obtained results. In addition, assuming $I_{\rho_I} = 0$ ($\rho_I = 0$) instead of $I_{\rho_I} = 3.333\times10^{-5}$ ($\rho_I = 50$) yields also negligible effect.

In the conclusion therefore a question should be asked about the properties of this interesting example. In this connection another question arises what is the total energy of the structure. If these issues are not answered the analyzed problem in the present form can not be regarded as the repeatable example.

The authors will be grateful for further discussion, remarks and suggestions that help to remove the presented controversies.

REFERENCES

[1] Vu-Quoc L, Dynamics of Flexible Structures Performing Large Overall Motions: Geometrically Nonlinear Approach, Ph.D. Dissertation, ERL Memorandum UCB/ERL M86/36, UC Berkeley, May 1986.

[2] Simo JC, Vu-Quoc L. On the dynamics of flexible beams under large overall motions. The plane case: Part I. *ASME Journal of Applied Mechanics* 1986; **53**: 849-854.

[3] Simo JC, Vu-Quoc L. On the dynamics of flexible beams under large overall motions. The plane case: Part II. *ASME Journal of Applied Mechanics* 1986; **53**: 855-863.

[4] Simo JC, Vu-Quoc L. On the dynamics in space of rods undergoing large motions. A geometrically exact approach. *Computer Methods in Applied Mechanics and Engineering*, 1988; **66**: 125-161.

[5] Vu-Quoc L, Simo, JC. On the dynamics of Earth-orbiting flexible satellites with multibody components. *AIAA Journal of Guidance, Control, and Dynamics*, 1987; **10**(6), 549-558, paper G87-088.

[6] Simo JC, Tarnow N. A new energy and momentum conserving algorithm for the non-linear dynamics of shells. *International Journal for Numerical Methods in Engineering* 1994; **37**: 2527–2549.

[7] Miehe C, Shroeder J. Energy and momentum conserving elastodynamics of non-linear brick-type mixed finite shell element. *International Journal for Numerical Methods in Engineering* 2001; **50**: 1801–1823.

[8] Zhong HG, Crisfield MA. An energy-conserving co-rotational procedure for the dynamics of shell structures, *Engineering Computations* 1998; **15**(5): 552–576.

[9] Chróścielewski J, Makowski W, Pietraszkiewicz W. Statics and Dynamics of Multifold Shells: Nonlinear Theory and Finite Element Method. IPPT PAN, Warsaw, 2004 (in Polish).

[10] Chróścielewski J, Makowski J, Stumpf H. Genuinely resultant shell finite elements accounting for geometric and material non-linearity. *International Journal for Numerical Methods in Engineering* 1992; **35**: 63–94.

[11] Chróścielewski J, Makowski J, Stumpf H. Finite element analysis of smooth, folded and multi-shell structures. *Computer Methods in Applied Mechanics and Engineering*, 1997; **141**: 1–46.

[12] Chróścielewski J, Witkowski W. 4-node semi-EAS element in 6–field nonlinear theory of shells. *International Journal for Numerical Methods in Engineering* 2006; **68**: 1137–1179, DOI: 10.1002/nme.1740.

[13] Lubowiecka I. Integration of nonlinear dynamics equations of rigid body and elastic shells (in Polish). Gdansk University of Technology Publishers, Gdańsk, 2004.

[14] Lubowiecka I, Chróścielewski J. On dynamics of flexible branched shell structures undergoing large overall motion using finite elements. *Computers and Structures* 2002; **80**: 891–898.

[15] Lubowiecka I, Chróścielewski J. *Energy-conserving time integration algorithm for six-field irregular shell dynamics*, ECCOMAS Thematic Conference: Advances in Computational Multibody Dynamics: Proceedings of Multibody Dynamics 2005: Madrid, 21-24, June 2005: 1-16, Madrid: ETS Ingenieros de Caminos Canales y Puertos; Univ. Politec. Madrid.

[16] Chróścielewski J, Lubowiecka I, Witkowski W. *Dynamics of six-field shells in the context of energy-conserving scheme*, Shell Structures: Theory and Applications. W. Pietraszkiewicz, C. Szymczak (eds.) Proceedings of the 8th SSTA Conference. Gdańsk-Jurata, Poland, 12-14 October 2005: 303-307. Taylor & Francis/Balkema, 2005.

[17] Chróscielewski J, Witkowski W. *Energy of structural shells by example of dynamics of three intersecting plates*. (in Polish). MES2007 "Programy MES we Wspomaganiu Analizy, Projektowania i Wytwarzania" X Jubileuszowa Konferencja Naukowo-Techniczna, Kazimierz Dolny (Poland), November 13-16, p.24, pp.1-10, (CD-ROM), 2007.

[18] Chróścielewski J, Lubowiecka I, Pietraszkiewicz W. *FEM and time stepping procedures in non-linear dynamics of flexible branched shell structures*. Theories of plates and shells. Eds.: R. Kienzler, H. Altenbach, I. Ott. Berlin - Tokyo: Critical Review and New Applications in Lecture Notes in Applied and Computational Mechanics, vol. 16:21-28, Springer 2004.

[19] Libai A., Simmonds JG.: The Nonlinear Theory of Elastic Shells, 2nd ed. Cambridge University Press, Cambridge (1998).

[20] Kuhl D, Ramm E. Generalized Energy-Momentum Method for non-linear adaptive shell dynamics. *Computer Methods in Applied Mechanics and Engineering* 1999; **178**:343–366.

[21] Vu-Quoc, L, Tan, XG. Optimal solid shells for nonlinear analyses of multilayer composites. Part II: Dynamics. *Computer Methods in Applied Mechanics and Engineering*, 2003; **192**: 1017–1059.

[22] Vu-Quoc, L, Tan, XG. Optimal solid shells for nonlinear analyses of multilayer composites. Part I: Statics. *Computer Methods in Applied Mechanics and Engineering*, 2003; **192**: 975-1016.

[23] Tan XG, Vu-Quoc L. Efficient and accurate multilayer solidshell element: Nonlinear materials at finite strain. *International Journal for Numerical Methods in Engineering*, 2005; **63**: 2124-2170

[24] Appendix A. Outline of the shell theory

The shell theory employed here and associated various FEM formulations have been discussed in depth in [9][10][11][12][13][14][15][16]. For sake of completeness, we recapitulate some important facts about the formulation, cf. also [18].

The motion of the shell structure, understood here as a 2D Cosserat continuum M having a boundary ∂M with an attached structure tensor $\boldsymbol{T}_0(\boldsymbol{x})$, can be described by the translation vector field

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{y}(\boldsymbol{x},t)-\boldsymbol{x} \tag{A.1}$$

and the rotation tensor field

$$\boldsymbol{T}(\boldsymbol{x},t)=\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{T}_0(\boldsymbol{x}), \quad \boldsymbol{Q}(\boldsymbol{x},t)\in SO(3) \tag{A.2}$$

Here $\boldsymbol{x}\in M$ and $\boldsymbol{y}(\boldsymbol{x},t)$ are position vectors of the undeformed and the current base surface of the shell, respectively, and $\boldsymbol{Q}(\boldsymbol{x},t)$ is an independent proper orthogonal tensor field representing the mean rotary deformations of the shell attached structure tensor $\boldsymbol{T}_0(\boldsymbol{x})$. The fields $\boldsymbol{y}(\boldsymbol{x},t)$ and $\boldsymbol{Q}(\boldsymbol{x},t)$ are assumed to be continuous during the motion. The linear velocity vector is defined as

$$\boldsymbol{\upsilon}(\boldsymbol{x},t)=\dot{\boldsymbol{y}}(\boldsymbol{x},t)=\dot{\boldsymbol{u}}(\boldsymbol{x},t) \tag{A.3}$$

whereas the angular velocity vector $\boldsymbol{\omega}(\boldsymbol{x},t)$ in the spatial representation is

$$\mathrm{ad}\,\boldsymbol{\omega}=\dot{\boldsymbol{Q}}\boldsymbol{Q}^T, \quad \mathrm{ad}: E^3\rightarrow so(3) \tag{A.4}$$

The strain measures are defined as follows: stretching vector

$$\boldsymbol{\varepsilon}_\beta=\boldsymbol{y},_\beta-\boldsymbol{Q}\boldsymbol{x},_\beta, \quad \beta=1,2 \tag{A.5}$$

and bending vector

$$\boldsymbol{\kappa}_\beta=\mathrm{ad}^{-1}(\boldsymbol{Q},_\beta\boldsymbol{Q}^{\mathrm{T}}) \tag{A.6}$$

The kinetic constitutive relations for the linear $\boldsymbol{p}(\boldsymbol{x},t)$ and angular $\boldsymbol{m}(\boldsymbol{x},t)$ momentum surface vectors may be taken respectively as (cf. [19])

$$\boldsymbol{p}(\boldsymbol{x},t)=m_0\boldsymbol{\upsilon}=\rho_m h_0\boldsymbol{\upsilon}, \quad \boldsymbol{m}(\boldsymbol{x},t)=I_0\boldsymbol{\omega}=(\rho_I h_0^3/12)\boldsymbol{\omega} \tag{A.7}$$

where $\rho_m(\boldsymbol{x})$ is the initial shell mass density of translational motion, $\rho_I(\boldsymbol{x})$ is the initial shell mass density of rotary motion and $h_0(\boldsymbol{x})$ is the initial shell thickness. Hence

$$\boldsymbol{L}(t)=\iint_M \boldsymbol{p}(\boldsymbol{x},t)da, \quad \boldsymbol{J}(t)=\iint_M \boldsymbol{m}(\boldsymbol{x},t)da \tag{A.8}$$

Assuming that there exists a two dimensional strain energy $W(\boldsymbol{\varepsilon}_\beta,\boldsymbol{\kappa}_\beta;\boldsymbol{x})$ of the shell strain measures (A.5) and (A.6) the constitutive relations of the shell material are given by

$$\boldsymbol{n}^\beta=\partial W/\partial\boldsymbol{\varepsilon}_\beta, \quad \boldsymbol{m}^\beta=\partial W/\partial\boldsymbol{\kappa}_\beta \tag{A.9}$$

Here $\boldsymbol{n}^\beta(\boldsymbol{x},t)$ and $\boldsymbol{m}^\beta(\boldsymbol{x},t)$ are the internal stress and couple resultant vectors, respectively. Therefore the internal energy and kinetic energy are defined as respectively

$$U = \iint_M W da\,, \qquad K = \frac{1}{2}\iint_M (m_0 \boldsymbol{v}\cdot\boldsymbol{v} + I_0 \boldsymbol{\omega}\cdot\boldsymbol{\omega}) da \tag{A.10}$$

When expressed in the weak form, the initial-boundary value problem for the structural shell can is stated as follows.

Given:

- the external resultant force and couple vector fields $\boldsymbol{f}(\boldsymbol{x},t)$ and $\boldsymbol{c}(\boldsymbol{x},t)$ on $\boldsymbol{x} \in M$, $\boldsymbol{n}^*(\boldsymbol{x},t)$ and $\boldsymbol{m}^*(\boldsymbol{x},t)$ along ∂M_f,
- the kinematic boundary conditions $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}^*(\boldsymbol{x},t)$ and $\boldsymbol{Q}(\boldsymbol{x},t) = \boldsymbol{Q}^*(\boldsymbol{x},t)$ along $\partial M_d = \partial M \backslash \partial M_f$
- and the initial values $\boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{Q}_0(\boldsymbol{x})$, $\dot{\boldsymbol{u}}_0(\boldsymbol{x})$, $\dot{\boldsymbol{Q}}_0(\boldsymbol{x})$ at $t = 0$,

find a curve $\mathbf{u}(\boldsymbol{x},t) = (\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{Q}(\boldsymbol{x},t))$ on the configuration space $C(M, E^3 \times SO(3))$ such that for any continuous kinematically admissible virtual vector field $\mathbf{w}(\boldsymbol{x}) \equiv (\boldsymbol{v}(\boldsymbol{x}), \boldsymbol{w}(\boldsymbol{x}))$ the following principle of virtual work is satisfied

$$\begin{aligned} G[\mathbf{u};\mathbf{w}] = \iint_M [m_0 \dot{\boldsymbol{v}}\cdot\boldsymbol{v} + I_0 \dot{\boldsymbol{\omega}}\cdot\boldsymbol{w}] da + \iint_M [\boldsymbol{n}^\beta \cdot(\boldsymbol{v},_\beta + \boldsymbol{y},_\beta \times \boldsymbol{w}) + \boldsymbol{m}^\beta \cdot \boldsymbol{w},_\beta] da \\ - \iint_M (\boldsymbol{f}\cdot\boldsymbol{v} + \boldsymbol{c}\cdot\boldsymbol{w})\, da - \int_{\partial M_f} (\boldsymbol{n}^*\cdot\boldsymbol{v} + \boldsymbol{m}^*\cdot\boldsymbol{w}) ds = 0\,. \end{aligned} \tag{A.11}$$

In equation (A.11) it is implicitly assumed that virtual vector fields are kinematically admissible if $\boldsymbol{v}(\boldsymbol{x}) = \mathbf{0}$ and $\boldsymbol{w}(\boldsymbol{x}) = \mathbf{0}$ along ∂M_d. The solution of (A.11) is obtained in the course of iterative procedure reducing the problem to a sequence of solutions of linearized problems. Each linearized problem is formulated at discrete values of both temporal (ECA) and spatial variables (FEM). The main difficulty of such solution procedure is associated with the structure of the configuration space $C(M, E^3 \times SO(3))$ which does not possess the structure of linear space since it contains $SO(3)$.

In connection with equation (6) the external work is

$$G_{ext}(t) = \int_0^t \left[\iint_M (\boldsymbol{f}\cdot\boldsymbol{v} + \boldsymbol{c}\cdot\boldsymbol{\omega}) da + \int_{\partial M_f} (\boldsymbol{n}^*\cdot\boldsymbol{v} + \boldsymbol{m}^*\cdot\boldsymbol{\omega}) ds \right] d\tau \tag{A.12}$$

Appendix B. Validation of the algorithm – free flying plate

To validate our formulation we have run well-known example of the free flying plate, cf. [20] for shell formulation or [21] for in-depth analysis using solid-shell elements (see also [22] for statics and important results about satisfaction about patch-test). These solid-shell elements provide good alternative for modeling shell junctions, see for instance [23].

The geometry and loads are presented in Fig. 9. The material properties are

$$E = 206\, GPa\,, \qquad \nu = 0\,, \qquad \rho_m = \rho_I \equiv \rho = 7800\, kg/m^3 \tag{B.1}$$

To discretize the shell we use two meshes: 2×15 and 4×30 CAMe16 (FI) elements. The time step was $\Delta t = 5 \cdot 10^{-5} s$. Some deformed configurations are presented in Fig. 10. The conservation of energy is depicted in Fig. 11. We have noticed that there is no significant difference between the results obtained from 2×15 and 4×30 mesh and in both cases our results are consistent with reference solutions [20], [21]. The conservation of momenta is portrayed in Fig. 12. As in the case of energy, the obtained results are also in agreement with reference solutions. Our computed values of energy, linear and angular momenta at time $t = 0.004s$ are set in Table 5, Table 6 and Table 7 respectively. We note, however, that there may be differences inf sign of linear momentum components (Table 6) and also differences of values of angular momentum components (Table 7) depending on orientation and origins of assumed coordinate systems.

In passing we would like to point out that although we have not observed the differences between solutions obtained with 2×15 and 4×30 CAMe16 (FI) meshes, we have noticed that these solutions are obtained with different number (decreasing with mesh refinement) of iterations – see Fig. 13. Similar effect has been discussed in reference papers [20], [21].

Table 1. Three intersecting plates, properties of example reported in literature

	Simo and Tarnow [1994]	Zhong and Crisfield [1998]	Miehe and Shroeder [2001]	Present study
Material properties	$\lambda = 8\times10^6$, $\mu = 8\times10^6$	$E = 2\times10^7$, $\nu = 0.25$	?	$E = 2\times10^7$, $\nu = 0.25$
Mass density	$\rho_m = 1$	3.333×10^{-5}	?	$\rho_m = 1$
Nominal rotational inertia	$I_0 = 3.333\times10^{-5}$	?	?	$I_0 = 3.333\times10^{-5}$ ($\rho_I = 50$)
Thickness	$h_0 = 0.02$	?	?	$h_0 = 0.02$
Nominal translation al mass	?	?	?	$m_0 = 0.02$ ($\rho_m = 1$)
Time variation	$p(t) = \begin{cases} 0.5t & for \quad 0 \le t \le 0.5\ s \\ 0.25 - 0.5t & for \quad 0.5 \le t \le 1\ s \\ 0 & for \quad t \ge 1\ s \end{cases}$ * (original function is provided)	$p(t) = \begin{cases} 0.5t & for \quad 0 \le t \le 0.02\ s \\ 0.02 - 0.5t & for \quad 0.02 \le t \le 0.04\ s \\ 0 & for \quad t \ge 0.04\ s \end{cases}$	$p(t) = \begin{cases} \frac{1}{2}t & for \quad 0 \le t \le \frac{1}{2}\ s \\ \frac{1}{2}(1-t) & for \quad \frac{1}{2} \le t \le 1\ s \\ 0 & for \quad t \ge 1\ s \end{cases}$	$p(t) = \begin{cases} \frac{1}{2}t & for \quad 0 \le t \le \frac{1}{2}\ s \\ \frac{1}{2}(1-t) & for \quad \frac{1}{2} \le t \le 1\ s \\ 0 & for \quad t \ge 1\ s \end{cases}$

Table 2. Three intersecting plates, energies of the structure at $t = 1.0s$

Element (plate discretization)	U	K	$U+K$
CAMe4 (2× 6) FI, $E \to E \times 10^6$ (rigid body)	0.0002	19.65	19.65
CAMe4 (2× 6) FI	0.057	21.93	22.99
CAMe4 (4×12) FI	0.148	24.04	24.18
CAMe4 (6×18) FI	0.714	24.57	25.28
CAMe4 (8×24) FI	1.264	24.66	25.92
CAMe4 (10×30) FI	1.714	24.81	26.52
CAMe4 (12×36) FI	2.166	25.21	27.38
CAMe4 (14×42) FI	2.263	26.45	28.71
CAMe4 (16×48) FI	2.751	27.62	30.37
CAMe4 (18×54) FI	4.543	27.50	32.04
CAMe4 (20×60) FI	7.255	26.29	33.55
CAMe4 (22×66) FI	10.24	24.66	34.90
CAMe4 (24×72) FI	13.08	23.04	36.12
CAMe9 (2× 6) FI	17.62	26.06	43.68
CAMe9 (4×12) FI	13.45	43.14	56.59
CAMe9 (6×18) FI	25.01	33.65	58.66
CAMe9 (8×24) FI	27.68	31.31	58.99
CAMe9 (10×30) FI	28.04	31.11	59.15
CAMe9 (12×36) FI	28.03	31.23	59.26
CAMe16 (2× 6) FI	27.75	30.76	58.51
CAMe16 (4×12) FI	27.73	31.51	59.24
CAMe16 (6×18) FI	27.89	31.51	59.40
CAMe16 (8×24) FI	27.98	31.48	59.46
SEMe4 (4×12)	28.25	31.32	59.57
SEMe4 (6×18)	28.06	31.32	59.38
SEMe4 (8×24)	28.04	31.37	59.41
SEMe4 (10×30)	28.01	31.42	59.43
SEMe4 (12×36)	28.00	31.45	59.45
SEMe4 (14×42)	28.01	31.45	59.46
SEMe4 (16×48)	28.02	31.45	59.47
SEMe4 (18×54)	28.03	31.45	59.48
SEMe4 (20×60)	28.04	31.45	59.49
SEMe4 (22×66)	28.05	31.44	59.49
SEMe4 (24×72)	28.06	31.44	59.50
SEMe9 (2× 6)	28.17	31.19	59.36
SEMe9 (4×12)	27.97	31.45	59.42
SEMe9 (6×18)	28.01	31.46	59.47
SEMe9 (8×24)	28.04	31.45	59.49
SEMe9 (10×30)	28.06	31.44	59.50
SEMe9 (12×36)	28.08	31.43	59.51

Table 3. Three intersecting plates, linear momentum values at $t = 1.0s$

Element (plate discretization)	L_x	$-L_y$	$-L_z$
All meshes	4.125	0.125	0.75

Table 4. Three intersecting plates, angular momentum of the structure at $t = 1.0s$

Element (plate discretization)	$-J_x$	J_y	J_z
CAMe4 (2× 6) FI, $E \to E\times10^6$ (rigid body)	10.04	27.20	20.99
CAMe4 (2× 6) FI	8.940	25.99	19.83
CAMe4 (4×12) FI	8.663	25.64	19.50
CAMe4 (6×18) FI	8.563	25.51	19.43
CAMe4 (8×24) FI	8.525	25.44	19.48
CAMe4 (10×30) FI	8.519	25.42	19.62
CAMe4 (12×36) FI	8.530	25.42	19.84
CAMe4 (14×42) FI	8.546	25.42	20.10
CAMe4 (16×48) FI	8.556	25.42	20.35
CAMe4 (18×54) FI	8.559	25.41	20.59
CAMe4 (20×60) FI	8.554	25.39	20.80
CAMe4 (22×66) FI	8.542	25.37	20.99
CAMe4 (24×72) FI	8.526	25.35	21.17
CAMe9 (2× 6) FI	8.025	25.03	23.27
CAMe9 (4×12) FI	6.736	23.51	24.94
CAMe9 (6×18) FI	6.183	22.80	25.43
CAMe9 (8×24) FI	6.004	22.55	25.57
CAMe9 (10×30) FI	5.933	22.46	25.62
CAMe9 (12×36) FI	5.904	22.41	25.64
CAMe16 (2× 6) FI	5.889	22.36	25.64
CAMe16 (4×12) FI	5.868	22.35	25.65
CAMe16 (6×18) FI	5.873	22.37	25.66
CAMe16 (8×24) FI	5.876	22.38	25.67
SEMe4 (4×12)	5.934	22.47	25.68
SEMe4 (6×18)	5.899	22.41	25.66
SEMe4 (8×24)	5.889	22.40	25.66
SEMe4 (10×30)	5.885	22.40	25.66
SEMe4 (12×36)	5.883	22.39	25.66
SEMe4 (14×42)	5.882	22.39	25.67
SEMe4 (16×48)	5.882	22.39	25.67
SEMe4 (18×54)	5.881	22.39	25.67
SEMe4 (20×60)	5.881	22.39	25.67
SEMe4 (22×66)	5.881	22.39	25.67
SEMe4 (24×72)	5.881	22.39	25.67
SEMe9 (2× 6)	5.899	22.43	25.70
SEMe9 (4×12)	5.880	22.39	25.67
SEMe9 (6×18)	5.879	22.39	25.67
SEMe9 (8×24)	5.879	22.39	25.67
SEMe9 (10×30)	5.880	22.39	25.67
SEMe9 (12×36)	5.880	22.39	25.68

Table 5. Free flying plate, energies of the structure at $t = 0.004s$

Element (plate discretization)	U	K	$U+K$
CAMe16 (2× 15) FI	105.785	141.057	246.842
CAMe16 (4× 30) FI	105.825	141.038	246.863

Table 6. Free flying plate, linear momentum values at $t = 0.004s$

Element (plate discretization)	L_x	L_y	L_z
All meshes	4.8015	3.201	3.201

Table 7. Free flying plate, angular momentum values at $t = 0.004s$

Element (plate discretization)	J_x	J_y	J_z
CAMe16 (2× 15) FI	–0.0673232	–0.0387033	0.108168
CAMe16 (4× 30) FI	–0.0673351	–0.0387041	0.108223

Figure captions

Fig. 1. Geometry of the structure and loads

Fig. 2. Energy history, reference and own solutions

Fig. 3. *h*–type convergence analysis of total energy of the structure, Δt = 0.002*s* CAM (ECA), SEM (Newmark)

Fig. 4. *p*–type and *h*–type convergence analysis of total energy of the structure, Δt = 0.002*s*

Fig. 5. *p*–type and *h*–type convergence analysis of *x* component of angular momentum vector

Fig. 6. *p*–type and *h*–type convergence analysis of *y* component of angular momentum vector

Fig. 7. *p*–type and *h*–type convergence analysis of z component of angular momentum vector

Fig. 8. Some representative configurations, mesh 3×(8×24) CAM 16(FI), 32550 DOFs

Fig. 9. Free–flying plate, geometry, loads and material data

Fig. 10. Free–flying plate, motion trajectories of points (a) and (b) and some configurations plotted every 4 *ms*

Fig. 11. Free–flying plate, energy versus time

Fig. 12. Free–flying plate, conservation of linear and angular momenta

Fig. 13. Free–flying plate, number of iterations for different mesh density per time step