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Generation of the Vorticity Mode by Sound in a Relaxing Maxwell Fluid

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Summary
This study develops a new theory of nonlinear acoustics investigating interactions between acoustical and other non-acoustical modes, such as vorticity modes, in a fluid. The ideas proposed by the authors make possible to derive instantaneous equations describing interaction between different modes in a relaxing Maxwell fluid. The procedure of deriving of a new dynamic equation governing the vorticity mode which is generated by sound, is discussed in details. It uses only instantaneous quantities and does not include averaging over sound period. The resulting equation applies to both periodic and aperiodic sound of any waveform as the origin of the vorticity mode. The theory is illustrated by two representative examples of generation of the vorticity mode in a relaxing Maxwell fluid, caused by periodic sound beam and a sound beam with a stationary but aperiodic waveform.

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1. Introduction

The theory of sound propagation and phenomena associated with it in a thermoviscous nonlinear flow began to develop rapidly in the fifties of the 20th century and has achieved undoubted success. The most widely used model equation for describing the combined effects of diffraction, absorption and nonlinearity in directional sound beams is the KZK (Khokhlov-Zabolotskaya-Kuznetsov) equation. It was derived in 1971 [1]. Analytical methods for solving the fully nonlinear form of the Khokhlov-Zabolotskaya equation (that is, the lossless form of the KZK equation) have been proposed only recently. One method incorporates analytical techiques used in nonlinear geometrical acoustics [2]. An approximate axial solution is derived for the preshock region of a beam radiated by a monofrequency source. The second method is based on combining the perturbation technique referred to as renormalization with what amounts to an application of weak shock theory [3]. This method is more general in that it applies to pulses, both on and off axis, and takes into account shock formation. Gaussian amplitude shading at the source is assumed in both approaches.

However, in a thermoviscous flow, secondary nonlinear effects induced by sound are also of great importance for medical and technical applications. These effects include acoustic streaming and heating, e.g. nonlinear generation of vorticity and entropy modes by an acoustic mode. Acoustic streaming is the mean motion of a fluid caused by acoustic waves. Extensive reviews on this subject exist in the references [4, 5, 6]. More recent discussion of acoustic streaming can be found in [7, 8, 9, 10]. The authors of [9, 10] suggest that there is an unresolved issue concerning acoustic streaming, the effect of compressibility: indeed, the starting point is usually equations describing incompressible liquids. The usual method to identify the different modes of motion (acoustic, entropy and vorticity modes) consists in two successive steps: first, implementation of averaging the continuity and momentum equations over the sound period, and, second, the linear combination of appropriate equations [5, 11]. It does not account for energy balance and, therefore, discards thermal conductivity. It is well-understood, however, that the streaming velocity depends on the total attenuation, including thermal conductivity [9, 12]. The effects of compressibility and heat conduction upon acoustic streaming are exhibited by noting the dependence of the streaming velocities on the Prandtl number and the specific heat ratio of the fluid. Larger streaming velocities are obtained for compressible fluids and are more evident in a gas than in a liquid [9]. The usual method includes also many intermediate discussions of an individual contribution of every term forming an acoustic force, the quantity averaged over the sound period [5, 11]. The next weak point of the theory is inconsistency when distinguishing between the vorticity and entropy modes. Indeed, the formerly applied methodology assumes zero temporal average over the sound period of partial derivative of total density with respect to time, $\partial\rho/\partial t$. However, in the thermoviscous flow, an excess density includes, besides an acoustic term, a slowly decreasing part belonging to the isobaric entropy mode. So that an averaged value of $\partial\rho/\partial t$ can no longer be zero. The

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part of mean velocity attributable to the entropy mode is also non-zero if the heat conduction of a fluid differs from zero. We can avoid these inconsistencies by means of immediate projection of the initial equations onto dynamic equations governing each specific mode (section 2).

We first need to determine every branch of acoustic and non-acoustic types of motion in a Maxwell fluid. That allows to derive individual dynamic equations governing every mode in a weakly nonlinear flow. The procedure was proposed and applied by one of the authors in analysis of acoustic heating and streaming in fluids with the standard attenuation [13, 14]. This method is valid for both periodic and aperiodic sound. It uses instantaneous quantities and therefore does not require averaging over the sound period at any stage. Actually, the method is useful in the general problems of modes interactions, not only those studying acoustic streaming and dealing with the mean fields. Relatively to generation of the vortical mode in the field of sound, the method makes it possible to derive dynamic equations considering every branch of acoustic and vortical motions individually. It applies in a weakly nonlinear flow and distributes the nonlinear terms between dynamic equations of different modes correctly. The method is free from inconsistencies of the traditional scheme. An acoustic driving force of the vorticity mode transforms into the well-known formula in the case of periodic sound being averaged over the sound period. The first ones who derived the equations of interacting modes in a viscous heat-conducting compressible gas, were Chu and Kovasznay [15]. Unfortunately, Chu and Kovasznay could not conclude about the acoustic force of the vorticity or entropy modes. The reason for that was an insufficiently high order of evaluations in the important paper [15] (section 2).

In the present study, we consider a relaxing Maxwell fluid, whose attenuation differs from the standard one. Relaxation of a fluid to the equilibrium state is described by an integral operator. Agreement of the final equation with the well-known result in the field of nonlinear acoustics (case of standard attenuation in a thermoviscous fluid and periodic sound as an origin of the vorticity mode) may be easily verified. It is sufficient to replace the integral operator by a constant factor and to consider a periodic sound (section 4). Note this study investigates only flows within an unbounded volume of fluid.

2. Decomposition of the vorticity and sound modes for a relaxing Maxwell fluid

2.1. Basic equations describing motion of a relaxing and heat conducting fluid

[illegible] momentum and energy equations and the mass con[illegible]vation equation in the relaxing and heat conducting [illegible]d flow without external forces are

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \frac{1}{\rho}\left(-\vec{\nabla}p + \mathrm{Div}\,\mathbf{P}\right),$$
$$\frac{\partial e}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)e = \quad (1)$$
$$\frac{1}{\rho}\left(-p\left(\vec{\nabla}\cdot\vec{v}\right) + \chi\Delta T + \mathbf{P} : grad\ \vec{v}\right),$$
$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0,$$

where $\vec{v}$ denotes velocity of the fluid, ρ, p are its density and pressure, e, T mark the specific internal energy (per unit mass) and temperature, χ is the thermal conductivity, x_i, t denote spatial Cartesian coordinates and time. Operator Div denotes the tensor divergence, $grad$ is the dyad gradient, and $\mathbf{P}$ is the viscous stress tensor. The equation relating the viscous viscous stress tensor and the vector of particle displacements $\vec{u}(\vec{r}, t)$ fits the Maxwell viscous model, in two following equivalent forms

$$\frac{\partial \mathbf{P}_{i,k}}{\partial t} + \frac{1}{\tau_R}\mathbf{P}_{i,k} = \mu\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \quad (2)$$
$$\mathbf{P}_{i,k} = \mu\int_{-\infty}^{t}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\mathrm{e}^{-(t-t')/\tau_R}\,\mathrm{d}t',$$

where τ_R denotes time of relaxation to the equilibrium state, and μ which is a generalized viscosity, measures the magnitude order of the relaxation process. Note that, when the relaxation time τ_R tends to infinity (or in the high frequency regime $\omega\tau_R \to \infty$) the Maxwell fluid behaves as a classical Newtonian fluid. The system (1) is supplemented by the two thermodynamic state equations $e(p, \rho)$ and $T(p, \rho)$ which can be expanded as a Taylor series up to quadratic nonlinear terms,

$$e' = \frac{E_1}{\rho_0}p' + \frac{E_2 p_0}{\rho_0^2}\rho' + \frac{E_3}{p_0\rho_0}p'^2 + \frac{E_4 p_0}{\rho_0^3}\rho'^2 + \frac{E_5}{\rho_0^2}\rho' p', \quad (3)$$
$$T' = \frac{\Theta_1}{\rho_0 C_v}p' + \frac{\Theta_2 p_0}{\rho_0^2 C_v}\rho' + \frac{\Theta_3}{p_0\rho_0 C_v}p'^2 + \frac{\Theta_4 p_0}{\rho_0^3 C_v}\rho'^2 + \frac{\Theta_5}{\rho_0^2 C_v}\rho' p'.$$

Primes denote perturbations, the ambient quantities are marked by index 0, and C_v is the specific heat coefficient at constant volume. We assume that the fluid is homogeneous in composition, that its unperturbed density and pressure are uniform, and the thermal conductivity is constant. The series expansion for the excess internal energy and temperature (3) allow to consider thermodynamic state of a fluid in the most general form, where $E_1, \ldots \Theta_5$ are dimensionless coefficients. Noting that a small change in entropy is a total differential, provides the ratio of the first coefficients in the series (3)

$$\Theta_2 = \frac{C_v\rho_0 T_0}{E_1 p_0} - \frac{(1-E_2)\Theta_1}{E_1}. \quad (4)$$

The common practice in nonlinear acoustics is to focus on the equations of the second order of acoustic Mach number $M = v_0/c_0$, where v_0 is a typical particle velocity magnitude, $c_0 = \sqrt{(1-E_2)p_0/(E_1\rho_0)}$ is an infinitely small signal velocity. We use also small dimensionless parameters responsible for relaxation $m = \mu/(\rho_0 c_0^2)$ and thermal conductivity $\delta = \chi T_0/(c_0^3\Lambda E_1^2\rho_0)$ (Λ is a characteristic scale of a flow). We shall discard $O(M^3)$ terms in all expansions. To facilitate implementation of the ordering scheme, we introduce a generic parameter $\widetilde{\delta}$ that characterizes the smallness of m and δ. Our primary objective is to derive dynamic equations valid at order $\widetilde{\delta}M^2$.

The dispersion relations describing the three independent modes follow from the linearized version of equations (1). They are: acoustic (two branches), thermal (or entropy), and vorticity (two branches) modes. From these dispersion relations we are able to suggest a linear model propagation equation for every mode in an unbounded fluid. In general, each of the field variables contains contributions from each of three modes, for example, $\vec{v} = \vec{v}_a + \vec{v}_{ent} + \vec{v}_{vort}$. That allows to decompose not only overall vector of perturbations into specific parts, but also to split the governing equations themselves. The method proposed in [13] and developed by the authors in the present study, enables splitting of the initial system (1) into specific dynamic equations for every mode (and, moreover, for every branch of acoustic or vorticity modes) using specific properties of each mode in a weakly nonlinear and thermoviscous flow. It is convenient to rearrange formulae in the dimensionless quantities as follows

$$\begin{aligned} p^{nd} &= \frac{p'}{c_0^2 \cdot \rho_0}, \quad \rho^{nd} = \frac{\rho'}{\rho_0}, \quad \vec{v}^{nd} = \frac{\vec{v}}{c_0}, \\ x^{nd} &= \frac{x}{\Lambda}, \quad y^{nd} = \frac{y}{\Lambda}, \quad z^{nd} = \frac{z}{\Lambda}, \\ t^{nd} &= \frac{c_0}{\Lambda} t, \quad \tau^{nd} = \frac{c_0}{\Lambda} \tau_R. \end{aligned} \tag{5}$$

Everywhere below in the text, superscripts by dimensionless quantities will be omitted. In the dimensionless quantities, equations (1) read

$$\begin{aligned} &\frac{\partial v_x}{\partial t} + \frac{\partial p}{\partial x} - \widehat{A}\left(\triangle v_x + \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v})\right) = \\ &\quad -(\vec{v}\cdot\vec{\nabla})v_x + \rho\frac{\partial p}{\partial x} - \rho\widehat{A}\left(\triangle v_x + \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v})\right), \\ &\frac{\partial v_y}{\partial t} + \frac{\partial p}{\partial y} - \widehat{A}\left(\triangle v_y + \frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{v})\right) = \\ &\quad -(\vec{v}\cdot\vec{\nabla})v_y + \rho\frac{\partial p}{\partial y} - \rho\widehat{A}\left(\triangle v_y + \frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{v})\right), \\ &\frac{\partial v_z}{\partial t} + \frac{\partial p}{\partial z} - \widehat{A}\left(\triangle v_z + \frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{v})\right) = \\ &\quad -(\vec{v}\cdot\vec{\nabla})v_z + \rho\frac{\partial p}{\partial z} - \rho\widehat{A}\left(\triangle v_z + \frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{v})\right), \\ &\frac{\partial p}{\partial t} + \vec{\nabla}\cdot\vec{v} - \delta_1\Delta p - \delta_2\Delta\rho = \\ &\quad -(\vec{v}\cdot\vec{\nabla})p + \left(D_1 p + D_2\rho\right)(\vec{\nabla}\cdot\vec{v}) \\ &\quad + \frac{1}{E_1}\left(2\frac{\partial v_x}{\partial x}\widehat{A}\frac{\partial v_x}{\partial x} + 2\frac{\partial v_y}{\partial y}\widehat{A}\frac{\partial v_y}{\partial y} + 2\frac{\partial v_z}{\partial z}\widehat{A}\frac{\partial v_z}{\partial z}\right) \\ &\quad + \frac{1}{E_1}\left[\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\widehat{A}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right. \\ &\qquad\left. + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\widehat{A}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] \\ &\quad + \frac{1}{E_1}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\widehat{A}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \\ &\quad + \delta_3\Delta(p^2) + \delta_4\Delta(\rho^2) + \delta_5\Delta(\rho p), \\ &\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{v} = -(\vec{v}\cdot\vec{\nabla})\rho - \rho(\vec{\nabla}\cdot\vec{v}), \end{aligned} \tag{6}$$

where $\vec{\nabla} = (\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z)$, $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, and $\widehat{A}$ denotes the dimensionless operator which is applied on a scalar function $\phi(x, y, z, t)$,

$$\widehat{A}\phi = m\int_{-\infty}^{t} \phi \mathrm{e}^{-(t-t')/\tau}\,\mathrm{d}t'. \tag{7}$$

Equations (6) include the dimensionless quantities

$$\begin{aligned} \delta_1 &= \frac{\chi\Theta_1}{\rho_0 c_0 \Lambda C_v E_1}, \quad \delta_2 = \frac{\chi\Theta_2}{\rho_0 c_0 \Lambda C_v (1-E_2)}, \\ \delta_3 &= \frac{\Theta_3\chi}{E_1\rho_0 c_0 \Lambda C_v}\frac{1-E_2}{E_1}, \delta_4 = \frac{\Theta_4\chi}{(1-E_2)\rho_0 c\Lambda C_v}, \\ \delta_5 &= \frac{\Theta_5\chi}{E_1\rho_0 c\Lambda C_v}, \\ D_1 &= \frac{1}{E_1}\left(-1 + 2\frac{1-E_2}{E_1}E_3 + E_5\right), \\ D_2 &= \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right). \end{aligned} \tag{8}$$

The sum of δ_1 and δ_2 is the sound attenuation due to the thermal conductivity,

$$\delta = \delta_1 + \delta_2. \tag{9}$$

The parameter of nonlinearity B/A in any fluid equals $-D_1 - D_2 - 1$, that for ideal gases gives $\gamma - 1$ (γ denotes the ratio of specific heats, $D_1 = -\gamma$, $D_2 = 0$).

2.2. Modes in a flow of infinitely-small magnitude. Projection onto the vorticity mode

The equivalent form of the system (6) is

$$\frac{\partial \Psi}{\partial t} + L\Psi = \Psi_{nl}, \tag{10}$$

where $\Psi = (\,v_x \; v_y \; v_z \; p \; \rho\,)^T$, L is a linear matrix operator including spatial derivatives, Ψ_{nl} denotes a nonlinear vector.

Studies of motions of infinitely-small amplitudes begin usually by representing all perturbations as a sum of planar waves,

$$f(\vec{r}, t) = \int_{R^3} \widetilde{f}(\vec{k}, t)\exp(-\mathrm{i}\vec{k}\vec{r})\,\mathrm{d}\vec{k}, \tag{11}$$

($\widetilde{f}(\vec{k}, t)$ denotes the Fourier transform of $f(\vec{r}, t)$, $\widetilde{f}(\vec{k}, t) = 1/(2\pi)^3 \int_{R^3} f(\vec{r}, t)\mathrm{e}^{\mathrm{i}\vec{k}\vec{r}}\,\mathrm{d}\vec{r}$). Five eigenvalues of the linearized version of equations (10) (when $\Psi_{nl} = 0$), λ_n ($n = 1, \ldots, 5$), determine wave (two branches of sound, $n = 1$ and $n = 2$) and non-wave (the entropy mode, $n = 3$, and two vorticity branches, $n = 4$ and $n = 5$) types of motion that may exist in a fluid. The relations between the

Fourier transforms of perturbations of both acoustic modes in the leading order are following ($n = 1, 2$):

$$\widetilde{\Psi}_n = \left(\widetilde{v}_{x,n}\ \widetilde{v}_{y,n}\ \widetilde{v}_{z,n}\ \widetilde{p}_n\ \widetilde{\rho}_n \right)^T = \left(-\frac{\mathrm{i}\widetilde{\lambda}_n k_x}{\widetilde{\Delta}},\ -\frac{\mathrm{i}\widetilde{\lambda}_n k_y}{\widetilde{\Delta}},\ -\frac{\mathrm{i}\widetilde{\lambda}_n k_z}{\widetilde{\Delta}},\ 1 - \delta\lambda_n,\ 1 \right)^T \widetilde{\rho}_n,$$

$$\widetilde{\lambda}_1 = -\sqrt{\widetilde{\Delta}} - \frac{\widehat{\beta}}{2}\widetilde{\Delta}, \qquad \widetilde{\lambda}_2 = \sqrt{\widetilde{\Delta}} - \frac{\widehat{\beta}}{2}\widetilde{\Delta}, \tag{12}$$

where $\widetilde{\Delta}$ corresponds to the Laplacian ($\widetilde{\Delta}$ is applied in the Fourier transforms space), $\widehat{\beta}$ reflects the attenuating properties of a fluid due to relaxation and heat conduction,

$$\widetilde{\Delta} = -k_x^2 - k_y^2 - k_z^2, \quad \sqrt{\widetilde{\Delta}} = \mathrm{i}\sqrt{k_x^2 + k_y^2 + k_z^2},$$
$$\widehat{\beta} = 2\widehat{A} + \delta. \tag{13}$$

Multiplying by $-\mathrm{i}k_l$ in the space of Fourier transforms corresponds to evaluating of partial derivative $\partial/\partial l$. Among others, equation (12) implies the important property satisfied by sound velocity in the $(\vec{r}, t)$ space,

$$\vec{\nabla} \times \vec{v}_{a,1} = \vec{0}, \qquad \vec{\nabla} \times \vec{v}_{a,2} = \vec{0}. \tag{14}$$

That is the well-known property of sound velocity to be irrotational field.

The entropy type of motion in the considered formulation of the problem specifies non-zero velocity only if the thermal conductivity differs from zero. Velocity of the entropy mode is also a potential field, its excess pressure is zero, but its excess density differs from zero:

$$\vec{\nabla} \times \vec{v}_{ent} = \vec{0}, \quad p_{ent} = 0, \quad \vec{v}_{ent} = \delta_2 \vec{\nabla}\rho_{ent}. \tag{15}$$

There exist also two branches of the solenoidal vorticity flow determined by relations as

$$\vec{\nabla} \cdot \vec{v}_{vort,4} = 0, \quad p_{vort,4} = 0, \quad \rho_{vort,4} = 0,$$
$$\vec{\nabla} \cdot \vec{v}_{vort,5} = 0, \quad p_{vort,5} = 0, \quad \rho_{vort,5} = 0. \tag{16}$$

The projection matrix operators may be determined by use of equations (14)–(16). Every individual mode may be obtained by applying the appropriate projector on the vector Ψ. For example, applying the operator P_{vort} on the vector Ψ, decomposes it into a sum of the two vorticity modes,

$$P_{vort}\Psi = \Psi_{vort} = \Psi_{vort,4} + \Psi_{vort,5}. \tag{17}$$

The sum of all projectors is the unit matrix, projectors are orthogonal to one another, and, if squared, they are all equals to themselves [13, 14]. Every projector is a matrix of spatial operators consisting of five rows and five columns. The part of the vorticity projector, which is applied on the velocity vector, $P_{vort,\vec{v}}$, consists of three rows and three columns. As a result, it decomposes the vorticity part of velocity,

$$\Delta^{-1} \begin{pmatrix} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} & -\frac{\partial^2}{\partial x \partial y} & -\frac{\partial^2}{\partial x \partial z} \\ -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} & -\frac{\partial^2}{\partial y \partial z} \\ -\frac{\partial^2}{\partial x \partial z} & -\frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{pmatrix} \tag{18}$$
$$\begin{pmatrix} v_{x,a} + v_{x,ent} + v_{x,vort} \\ v_{y,a} + v_{y,ent} + v_{y,vort} \\ v_{z,a} + v_{z,ent} + v_{z,vort} \end{pmatrix} = \begin{pmatrix} v_{x,vort} \\ v_{y,vort} \\ v_{z,vort} \end{pmatrix}.$$

Equation (18) manifests in fact a certain way of application of the Helmholtz vector decomposition theorem, which enables us to decompose irrotational and solenoidal vector fields. It is easy to prove that $P_{vort,\vec{v}}$ satisfies the equality

$$-\vec{\nabla} \times (\vec{\nabla} \times \vec{\varphi}) = \Delta P_{vort,\vec{v}}\vec{\varphi}, \tag{19}$$

where $\vec{\varphi}$ is any three-component vector.

2.3. Equation governing the vorticity mode induced by sound

The linearized theory does not indicate any interaction among the modes as long as the domain of interest is far from solid boundaries and it may be valid as long as the fluctuations are weak. However, the projection method can be applied also in the study of a weakly nonlinear flow as well. It is of importance, that applying the projector which decomposes a certain mode, reduces all other modes in the linear part of equations. The projection distributes the nonlinear terms between dynamic equations in the correct manner. Applying P_{vort} on the system (10) cancels all acoustic and entropy terms in the linear part, but gives rise to a nonlinear acoustic source in its right-hand nonlinear part. The nonlinear part of resulting equations includes mixed quadratic terms of all modes, but only acoustic ones will be kept in the frames of this study.

Applying of $P_{vort,\vec{v}}$ on the first three equations from the system (10) (they represent the momentum equation), results in the dynamic equation governing the vorticity mode,

$$\frac{\partial \vec{v}_{vort}}{\partial t} - \widehat{A}\Delta\vec{v}_{vort} = \tag{20}$$
$$P_{vort,\vec{v}} \begin{pmatrix} -(\vec{v}\cdot\vec{\nabla})v_x + \rho\frac{\partial p}{\partial x} - \rho\widehat{A}\left(\Delta v_x + \frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{v})\right) \\ -(\vec{v}\cdot\vec{\nabla})v_y + \rho\frac{\partial p}{\partial y} - \rho\widehat{A}\left(\Delta v_y + \frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{v})\right) \\ -(\vec{v}\cdot\vec{\nabla})v_z + \rho\frac{\partial p}{\partial z} - \rho\widehat{A}\left(\Delta v_z + \frac{\partial}{\partial z}(\vec{\nabla}\cdot\vec{v})\right) \end{pmatrix}_a,$$

where in the right-hand side only acoustic terms should be considered. It may be rearranged into

$$\frac{\partial \vec{v}_{vort}}{\partial t} - \widehat{A}\Delta\vec{v}_{vort} = \tag{21}$$
$$P_{vort,\vec{v}}\left(-\rho_a\frac{\partial}{\partial t}\vec{v}_a\right) = P_{vort,\vec{v}}\left(-\sum_{n=1}^{2}\rho_{a,n}\cdot\frac{\partial}{\partial t}\sum_{n=1}^{2}\vec{v}_{a,n}\right)$$
$$= \vec{F}_a.$$

One can obtain another form of equation (21) in terms of vorticity $\vec{\Omega} = \vec{\nabla} \times \vec{v}_{vort}$:

$$\frac{\partial\vec{\Omega}}{\partial t} - \widehat{A}\Delta\vec{\Omega} = \vec{\nabla} \times \left(-\rho_a\frac{\partial}{\partial t}\vec{v}_a\right). \tag{22}$$

It is useful to compare the right-hand side of equation (22) with that derived by Chu and Kovasznay [15] in the standard thermoviscous flows (the shear viscosity stands on the left-hand side of the equation instead of operator $\widehat{A}$ in the formula (3.5a) of the cited paper). The right-hand acoustic source from [15] in the dimensionless quantities takes the form

$$\vec{\nabla}\times\left(-p_a\frac{\partial}{\partial t}\vec{v}_a-\frac{1}{2}\vec{\nabla}(\vec{v}_a^2)\right)+O(\widetilde{\delta}M^2) = \tag{23}$$
$$-\frac{1}{2}\vec{\nabla}\times\vec{\nabla}\left(p_a^2+\vec{v}_a^2\right)+O(\widetilde{\delta}M^2) = O(\widetilde{\delta}M^2).$$

and exceeds accuracy of the ordering scheme used in [15] (terms of order $\widetilde{\delta}M^2$ were not considered, only those proportional to M^2). However, the terms of order $\widetilde{\delta}M^2$ are of importance, they reflect actually the origins of the acoustic force inducing the vorticity mode, such as nonlinearity and absorption. By the use of relations (12), equation (22) may be rearranged into the following one with a non-zero acoustic source only if operator $\widehat{\beta}$ differs from zero:

$$\frac{\partial\vec{\Omega}}{\partial t}-\widehat{A}\Delta\vec{\Omega} = \vec{\nabla}\rho_a\times\vec{\nabla}\left(\widehat{\beta}\frac{\partial\rho_a}{\partial t}\right) \tag{24}$$
$$= -\vec{\nabla}\rho_a\times\widehat{\beta}\vec{\nabla}(\vec{\nabla}\cdot\vec{v}_a) = \vec{\nabla}\rho_a\times\widehat{\beta}\Delta\vec{v}_a.$$

An acoustic excess density itself satisfies the analog of the Westervelt equation [16], which describes dynamics of both acoustic branches and therefore includes the second-order partial derivative with respect to time in its linear part of order $\widetilde{\delta}^0$. The dynamic equation includes also absorption due to relaxation:

$$\frac{\partial^2\rho_a}{\partial t^2}-\Delta\rho_a-\widehat{\beta}\frac{\partial^3\rho_a}{\partial t^3}-\frac{1-D_1-D_2}{2}\frac{\partial^2\rho_a^2}{\partial t^2} = 0. \tag{25}$$

The projection of the system (6) onto dynamic equations for every branch of sound results in two equations for each acoustic mode individually. The governing equation for the first acoustic branch takes the form

$$\frac{\partial\rho_{a,1}}{\partial t}+\sqrt{\Delta}\rho_{a,1}-\frac{\widehat{\beta}}{2}\Delta\rho_{a,1} = \tag{26}$$
$$\frac{1}{2}\left((D_1+D_2)\rho_{a,1}\left(\vec{\nabla}\cdot\vec{v}_{a,1}\right)-\vec{v}_{a,1}\cdot\left(\vec{\nabla}\rho_{a,1}\right)\right).$$

2.4. Quasi-planar geometry of a flow

Until this point, no restriction on a type of flow geometry was done. Let y designate the nominal axis of the sound beam pointing in the propagation direction, and let x, z be the coordinates perpendicular to that axis. The following assumptions will be made regarding the source: it is defined at the plane $y = 0$, it has a characteristic radius a, and it radiates at frequencies satisfying inequality $|\vec{k}|a \gg 1$. The last assumption ensures that the beam is reasonably directional. Introducing one more small parameter $\epsilon = 1/(|\vec{k}|a)^2$, responsible for diffraction, considerably simplifies modes and projectors, allowing to expand the Laplacian in a power series in the small parameter ϵ:

$$\Delta = \partial^2/\partial y^2+\epsilon\Delta_\perp, \quad \sqrt{\Delta}\approx\partial/\partial y+0.5\epsilon\Delta_\perp\int\mathrm{d}y, \tag{27}$$

where $\Delta_\perp = \partial^2/\partial x^2+\partial^2/\partial z^2$. Accounting for relations specific for the acoustic modes from section 2.2 (equation 12), equation (21) may be readily rearranged into

$$\frac{\partial\vec{v}_{vort}}{\partial t}-\widehat{A}\frac{\partial^2\vec{v}_{vort}}{\partial y^2} = \tag{28}$$
$$P_{vort,\vec{v}}\left((\rho_{a,1}+\rho_{a,1})\widehat{\beta}\vec{\nabla}\frac{\partial}{\partial y}\left(\rho_{a,2}-\rho_{a,1}\right)\right).$$

If sound is associated with a beam propagating in the positive direction of axis y only ($\rho_a = \rho_{a,1}$), one gets the equation governing the longitudinal component of the vorticity mode velocity $v_{y,vort}$:

$$\frac{\partial v_{y,vort}}{\partial t}-\widehat{A}\frac{\partial^2 v_{y,vort}}{\partial y^2}$$
$$= -\epsilon\begin{pmatrix}-\frac{\partial}{\partial x}\int\mathrm{d}y\\ \Delta_\perp\int\mathrm{d}y\int\mathrm{d}y\\ -\frac{\partial}{\partial z}\int\mathrm{d}y\end{pmatrix}^T\left(\rho_a\widehat{\beta}\vec{\nabla}\frac{\partial\rho_a}{\partial y}\right)$$
$$= \epsilon\left[\frac{\partial}{\partial x}\int\mathrm{d}y\left(\rho_a\widehat{\beta}\frac{\partial^2\rho_a}{\partial x\partial y}\right)\right. \tag{29}$$
$$\left.-\Delta_\perp\int\mathrm{d}y\int\mathrm{d}y\left(\rho_a\widehat{\beta}\frac{\partial^2\rho_a}{\partial y^2}\right)+\frac{\partial}{\partial z}\int\mathrm{d}y\left(\rho_a\widehat{\beta}\frac{\partial^2\rho_a}{\partial y\partial z}\right)\right].$$

In the practically important case of a cylindrically symmetric flow, ρ_a is a function of two spatial coordinates, $\rho_a(x,y,z,t) = \rho_a(r = \sqrt{x^2+z^2}, y, t)$. That allows us to write the previous formula as

$$\frac{\partial v_{y,vort}}{\partial t}-\widehat{A}\frac{\partial^2 v_{y,vort}}{\partial y^2}$$
$$= \epsilon\int\mathrm{d}y\left(\frac{\partial\rho_a}{\partial r}\widehat{\beta}\frac{\partial^2\rho_a}{\partial r\partial y}+\rho_a\widehat{\beta}\Delta_\perp\frac{\partial\rho_a}{\partial y}\right)$$
$$-\epsilon\Delta_\perp\int\mathrm{d}y\int\mathrm{d}y\left(\rho_a\widehat{\beta}\frac{\partial^2\rho_a}{\partial y^2}\right), \tag{30}$$

where $\Delta_\perp = 1/r\partial/\partial r+\partial^2/\partial r^2$. It is the diffusion equation for the vorticity part of velocity induced by nonlinear losses in acoustic momentum. An excess acoustic density in the beam propagating in the positive direction of axis y, is a solution of equation

$$\frac{\partial\rho_{a,1}}{\partial t}+\frac{\partial\rho_{a,1}}{\partial y}+\frac{\epsilon}{2}\int\Delta_\perp\rho_{a,1}\,\mathrm{d}y \tag{31}$$
$$-\frac{\widehat{\beta}}{2}\frac{\partial^2\rho_{a,1}}{\partial y^2}+\frac{1-D_1-D_2}{2}\rho_{a,1}\frac{\partial\rho_{a,1}}{\partial y} = 0.$$

Equation (31) is analogous to the famous KZK equation [1, 5] with $\widehat{\beta}$ replacing the standard attenuation. It may also be derived by projecting of the initial system (6) onto dynamic equation of the first acoustic branch by means of the appropriate projector [13] and performing the paraxial approximations (equation 27).

3. Examples

In the previous sections, two forms of equation have been derived, that govern acoustic streaming in a heat conducting relaxing Maxwell fluid, one which does not refer to quasi-planar flows, equation (21) (and its modified form, equation 24), and the second one valid in a quasi-planar flow, equation (30). It includes a double integral, originating from the series expansion of the Laplacian (equation 27). That makes necessary to determine integration constants corresponding to the physical setting of a problem. In general, evaluation of acoustic force of the vorticity mode is a very difficult problem because an excess acoustic density itself should satisfy equation (26) or equation (31).

3.1. Periodic sound

It is possible to evaluate the right-hand side of equation (21) in some simple cases analytically. Let an excess acoustic density take the form ($r = \sqrt{x^2 + z^2}$)

$$\rho_a = M \exp\big(-r^2\big) \sin\big(t - y\big). \tag{32}$$

It may be considered as an approximate solution of equation (31) in the limiting case of very weak attenuation, nonlinearity and diffraction. An acoustic force of streaming in accordance with equation (21) takes the form

$$\vec{F}_a = P_{vort,\vec{v}} \left(\rho_a \widehat{\beta} \vec{\nabla} \frac{\partial \rho_a}{\partial t} \right), \tag{33}$$

which gives the longitudinal acoustic force

$$F_{a,y} = M^2 \left(\frac{m\tau}{1+\tau^2} + \frac{\delta}{2} \right) \exp\big(-2r^2\big). \tag{34}$$

This example belongs to the field of interest of acoustic streaming in its classical meaning, because the acoustic acoustic force in the right-hand side of equation (34) represents the mean field. In its derivation, however, we did not use averaging over sound period at any stage.

3.2. Stationary sound

As the next example we consider the stationary sound waveform depending exclusively on the retarded time $\xi = t - x$ in a pure relaxing fluid without heat conduction. The one-dimensional form of equation (31) is

$$\begin{aligned} &\frac{1 - D_1 - D_2}{2} \rho_a \frac{\mathrm{d}\rho_a}{\mathrm{d}\xi} \\ &+ m \int_{-\infty}^{\xi} \frac{\mathrm{d}^2 \rho_a}{\mathrm{d}\xi'^2} \exp\big(-(\xi - \xi')/\tau\big) \mathrm{d}\xi' = 0. \end{aligned} \tag{35}$$

ıation (35) can be integrated to obtain [17]

$$\left(\rho_a + \frac{m}{\gamma + 1} \right) \frac{\mathrm{d}\rho_a}{\mathrm{d}\xi} + \frac{\rho_a^2}{2\tau} = \text{const.} \tag{36}$$

consider a density jump $2M$ from $-M$ ahead of the ve front to M behind. Applying the boundary conditions at $\xi = -\infty$: $\rho_a = -M$, $\mathrm{d}\rho_a/\mathrm{d}\xi = 0$ and $\xi = \infty$: $= M$, $\mathrm{d}\rho_a/\mathrm{d}\xi = 0$, gives the integration constant $M^2/2\tau$ and results after separating of variables and integrating to the expression for ξ [18, 19],

$$\xi = \tau \ln \frac{(1 + \rho_a/M)^{G-1}}{(1 - \rho_a/M)^{G+1}}, \tag{37}$$

where $G = 2m/(1 - D_1 - D_2)M$ measures the ratio of relaxation effects to nonlinear effects. In the limit of weak nonlinearity $G \gg 1$, the solution (37) can be inverted analytically [5, 17]:

$$\rho_a(\xi) = M \tanh\big(\xi/2G\tau\big). \tag{38}$$

The waveform

$$\rho_a(\xi) = M \exp\big(-r^2\big) \tanh\big(\xi/2G\tau\big) \tag{39}$$

may be considered as an approximate formula describing weakly diffracting acoustic beam. The corresponding longitudinal acoustic force of the vorticity mode is found to be

$$\begin{aligned} F_{a,y} &= \frac{M^2}{4\tau G^2} \left(m + \frac{\delta}{2\tau} \right) \exp\big(-2r^2\big) \\ &\quad \cdot \cosh\left(\frac{\xi}{\tau G} \right) \cosh^{-4}\left(\frac{\xi}{2\tau G} \right). \end{aligned} \tag{40}$$

It is positive and decreases rapidly with enlarging of $|\xi|$.

4. Conclusions

New instantaneous equations governing a secondary solenoidal flow nonlinearly induced by sound, equations (21), (24) and equation (30), are derived (the latter one is valid in a quasi-planar geometry of a flow). Each equation takes the form of a diffusion equation with a nonlinear acoustic force in the right-hand side. The operator $\widehat{A}$ describes relaxation to the equilibrium state, and $\widehat{\beta}$ accounts for the relaxation and heat conduction, δ. It is of importance, that equations (21), (24), (30) may be easily rearranged in the case of the standard attenuation, by means of replacing of $\widehat{A}$ by dimensionless shear viscosity, $\frac{\eta}{\rho_0 c_0 \lambda}$, and $\widehat{\beta}$ by the diffusivity, consisting of shear, bulk viscosity and thermal conductivity,

$$\frac{4\eta}{3\rho_0 c_0 \lambda} + \frac{\eta_B}{\rho_0 c_0 \lambda} + \delta.$$

Equation (34) agrees with the well-known formula on acoustic radiation force caused by periodic sound (equation 32) in a fluid flow with the standard attenuation

$$\big\langle F_{a,y} \big\rangle = \left(\frac{4\eta}{3\rho_0 c_0 \lambda} + \frac{\eta_B}{\rho_0 c_0 \lambda} + \delta \right) \left\langle \left(\frac{\partial \rho_a}{\partial t} \right)^2 \right\rangle, \tag{41}$$

where angular brackets denote temporal averaging over the sound period. It is remarkable, that the acoustic force depends on the total attenuation, including the thermal one. There are many articles including experimental data confirming equation (41) for periodic sound as the origin of the vorticity mode (among other, [20]).

Coefficients in the series of the internal energy (first equation from equations 3) participate in equation governing the vorticity mode by means of the sound which is its origin. Equations (26), (31) include these coefficients.

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References

[1] V. P. Kuznetsov: Equations of nonlinear acoustics. Sov. Phys. Acoust. **16** (1971) 467–470.

[2] M. Hamilton, V. Khokhlova, O. V. Rudenko: Analytical method for describing the paraxial region of finite amplitude sound beams. J. Acoust. Soc. Am. **101** (1997) 1298–1308.

[3] K. E. Frøysa, F. Coulouvrat: A renormalization method for nonlinear pulsed sound beams. J. Acoust. Soc. Am. **99** (1996) 3319–3328.

[4] W. L. Nyborg: Acoustic streaming. – In: Physical Acoustics. W. P. Manson (ed.). Academic Press, New York, 1965, Vol. II. part B, 265–331.

[5] O. V. Rudenko, S. I. Soluyan: Theoretical foundations of nonlinear acoustics. Plenum, New York, 1977.

[6] M. J. Lighthill: Acoustic streaming. J. Sound. Vib. **61** (1978) 391–418.

[7] J. N. Tjøtta, S. Tjøtta: Nonlinear equations of acoustics. Proceedings of the 12th International Symposium on Nonlinear Acoustics, 1990, 80–97.

[8] S. Tjøtta, J. N. Tjøtta: Acoustic streaming in ultrasound beams. Advances in Nonlinear Acoustics, Proceedings of the 13th International Symposium on Nonlinear Acoustics, 1993, 601–606.

[9] Q. Qi: The effect of compressibility on acoustic streaming near a rigid boundary for a plane traveling wave. J. Acoust. Soc. Am. **94** (1993) 1090–1098.

[10] L. Menguy, J. Gilbert: Non-linear acoustic streaming accompanying a plane stationary wave in a guide. Acustica united with Acta Acustica **86** (2000) 249–259.

[11] S. Makarov, M. Ochmann: Nonlinear and thermoviscous phenomena in acoustics. Part I. Acustica **82** (1996) 579–606.

[12] O. V. Rudenko, A. P. Sarvazyan, S. Y. Emelianov: Acoustic radiation force and streaming induced by focused nonlinear ultrasound in a dissipative medium. J. Acoust. Soc. Am. **99** (1996) 2791–2798.

[13] A. Perelomova: Acoustic radiation force and streaming caused by non-periodic acoustic source. Acta Acustica united with Acustica **89** (2003) 754–763.

[14] A. Perelomova: Development of linear projecting in studies of non-linear flow. Acoustic heating induced by non-periodic sound. Physics Letters A **357** (2006) 42–47.

[15] B.-T. Chu, L. S. G. Kovasznay: Nonlinear interactions in a viscous heat-conducting compressible gas. Journ. Fluid. Mech. **3** (1958) 494–514.

[16] P. J. Westervelt: Parametric acoustic array. J. Acoust. Soc. Am. **35** (1963) 535–537.

[17] M. Hamilton, Y. Il'inskii, E. Zabolotskaya: Dispersion. – In: Nonlinear Acoustics. M. Hamilton, D. Blackstock (eds.). Academic Press, 1998, 151–175.

[18] M. J. Lighthill: Viscosity effects in sound waves of finite amplitude. – In: Surveys of mechanics. G. K. Batchelor, R. M. Davies (eds.). Cambridge University Press, Cambridge, 1956, 250–351.

[19] A. L. Polyakova, S. I. Soluyan, R. V. Khokhlov: Propagation of finite distrubances in a relaxing medium. Sov. Phys. Acoust. **8** (1962) 78–82.

[20] T. Kamakura, T. Sudo, K. Matsuda, Y. Kumamoto: Time evolution of acoustic streaming from a planar ultrasonic source. J. Acoust. Soc. Am. **100** (1996) 132–137.