

# Interpolation properties of domination parameters of a graph

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## Abstract

An integer-valued graph function  $\pi$  is an interpolating function if a set  $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$  consists of consecutive integers, where  $\mathcal{T}(G)$  is the set of all spanning trees of a connected graph  $G$ . We consider the interpolation properties of domination related parameters.

## 1 Introduction

The interpolation properties of different graph parameters were studied in a number of papers. In particular, the interpolating character of domination related parameters were investigated: domination number, lower and upper distance  $k$ -domination numbers [8, 14], global and total domination numbers and  $n$ -domination number [14] and  $(r, s)$ -domination number [15]. In this paper we establish the interpolation properties of other types of domination numbers of a graph.

In general, we use the terminology and notation of [9]. For the sake of completeness we now give a few definitions. Let  $G$  be a graph and  $v$  be a vertex of  $G$ . The *neighbourhood* of  $v$  in  $G$  is the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The *closed neighbourhood* of  $v$  in  $G$  is the set  $N_G[v] = N_G(v) \cup \{v\}$ . More generally, for a subset  $S \subseteq V(G)$ ,  $N_G(S)$  and  $N_G[S]$  are defined to be  $\bigcup_{v \in S} N_G(v)$  and  $N_G(S) \cup S$ , respectively. The degree of  $v$  in  $G$  is  $d_G(v) = |N_G(v)|$ .

A set  $D \subseteq V(G)$  is said to be independent if no two vertices of  $D$  are adjacent. The *independence number* of  $G$ , denoted by  $\alpha(G)$ , is the maximum cardinality of an independent set in  $G$ .

A unicyclic graph is a connected graph containing precisely one cycle.

A well-known method of transforming one spanning tree into another spanning tree of a graph is very useful in our research. We describe this transformation here.

For a connected graph  $G$ , let  $\mathcal{T}(G)$  be the set of all spanning trees of  $G$  and let  $T \in \mathcal{T}(G)$ . Let  $e$  be an edge of  $G$  which is not in  $T$ . Then  $T + e$  is a graph which has a unique cycle. If  $f$  is an edge which belongs to the cycle of  $T + e$ , then  $T + e - f$  is a spanning tree of  $G$ . The transformation of  $T$  to  $T + e - f$  is called a *simple edge-exchange*. A simple edge-exchange of  $T$  to  $T + e - f$  is called an *adjacent edge-exchange* if  $e$  and  $f$  are adjacent edges of  $G$ . If  $e$  and  $f$  are incident with a common end vertex of  $T$  (and then also of  $T + e - f$ ), then the transformation of  $T$  to  $T + e - f$  is called an *end edge-exchange*.

An integer-valued graph function  $\pi$  *interpolates over a connected graph*  $G$  if the set  $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$  consists of consecutive integers. The function  $\pi$  is called an *interpolating function* if  $\pi$  interpolates over each connected graph.

Now we present theorems and corollaries which are basic tools in our results.

**Theorem 1** [6] *If  $G$  is a 2-connected graph, then any  $T \in \mathcal{T}(G)$  can be transformed into any other  $T' \in \mathcal{T}(G)$  by a sequence of end edge-exchanges.*

**Theorem 2** [14] *An integer-valued graph function  $\pi$  is an interpolating function if and only if  $\pi$  interpolates over every unicyclic graph.*

**Corollary 3** [7] *An integer-valued graph function  $\pi$  is an interpolating function if one of the conditions holds:*

- (1) *for every unicyclic graph  $H$  and every edge  $uv$  belonging to the unique cycle of  $H$ ,  $\pi(H) \leq \pi(H - uv) \leq \pi(H) + 1$ ;*
- (2) *for every unicyclic graph  $H$  and every edge  $uv$  belonging to the unique cycle of  $H$ ,  $\pi(H) - 1 \leq \pi(H - uv) \leq \pi(H)$ .*

## 2 Results

This section is devoted to establishing the interpolating character of some variants of domination related parameters. We begin with a connected domination number of a graph introduced in [12]. Let  $G$  be a connected graph. A set  $D$  of vertices of  $G$  is a *connected dominating set* of  $G$  if  $D$  is dominating in  $G$  and a subgraph induced by  $D$  in  $G$ , denoted by  $\langle D \rangle_G$ , is connected. The cardinality of a minimum connected dominating set of  $G$  is called the *connected domination number* of  $G$  and is denoted by  $\gamma_c(G)$ . We now prove that the connected domination number is an interpolating function.

**Observation 1** *In a unicyclic graph  $G$  a connected dominating set  $S \subseteq V(G)$  contains all but at most two vertices from the unique cycle of  $G$ . A vertex on the cycle, but not in  $S$ , has valency two and two vertices on the cycle, but not in  $S$ , are adjacent.*



**Theorem 4** *The connected domination number  $\gamma_c$  is an interpolating function.*

**Proof.** Let  $G$  be a unicyclic graph and let  $C$  be the unique cycle of  $G$ . Let  $S$  be a connected dominating set of  $G$  such that  $|S| = a = \gamma_c(G)$ . We obtain all spanning trees of  $G$  by successively removing each edge of  $C$ ,  $\mathcal{T}(G) = \{T = G - uv : uv \in E(C)\}$ . On the basis of Observation 1 we consider three cases.

*Case 1.*  $V(C) - S = \emptyset$ . If we remove any edge  $uv \in E(C)$ , then  $S$  is also a dominating set of  $G - uv$  and  $\langle S \rangle_{G-uv}$  is connected, so  $S$  is a minimum connected dominating set of  $G - uv$ . Thus,  $\gamma_c(T) = a$  for all spanning trees  $T$  of  $G$ .

*Case 2.*  $|V(C) - S| = 1$ . Assume  $V(C) - S = \{x\}$ . Observe that  $d_G(x) = 2$ . If  $u = x$  or  $v = x$ , then  $S$  is a minimum connected dominating set of  $G - uv$ . Thus assume that  $u \neq x$  and  $v \neq x$ . Then one of the sets  $(S - \{v\}) \cup \{x\}$ ,  $(S - \{u\}) \cup \{x\}$  or  $S \cup \{x\}$  is a minimum connected dominating set of  $G - uv$ . Thus,  $\gamma_c(T(G)) = \{a\}$  or  $\gamma_c(T(G)) = \{a, a + 1\}$ .

*Case 3.*  $|V(C) - S| = 2$ . Assume  $V(C) - S = \{x, y\}$ . In this case  $d_G(x) = d_G(y) = 2$  and by Observation 1, we know that  $x$  and  $y$  are adjacent vertices of  $G$ . If  $|\{u, v\} \cap \{x, y\}| = 2$ , then  $S$  is a minimum connected dominating set of  $G - uv$ . Thus assume that  $|\{u, v\} \cap \{x, y\}| = 1$ , say  $u = x$ . Then  $(S - \{v\}) \cup \{y\}$  or  $S \cup \{y\}$  is a minimum connected dominating set of  $G - uv$ . Finally, assume that  $\{u, v\} \cap \{x, y\} = \emptyset$ . Then one of the sets  $(S - \{u, v\}) \cup \{x, y\}$ ,  $(S - \{u\}) \cup \{x, y\}$ ,  $(S - \{v\}) \cup \{x, y\}$  or  $S \cup \{x, y\}$  is a minimum connected dominating set of  $G - uv$ . Thus  $\gamma_c(G - uv) \in \{a, a + 1, a + 2\}$  and we note that if  $\gamma_c(G - uv) = \{a + 2\}$ , then there exists other edge  $wz \in E(C)$  such that  $d_G(w) = 2$  and  $d_G(z) > 2$  (because  $d_G(x) = d_G(y) = 2$ ,  $d_G(u) > 2$ ,  $d_G(v) > 2$  and  $xy, uv \in E(C)$ ). Hence  $(S - \{w\}) \cup \{x, y\}$  is a minimum connected dominating set of  $G - wz$  and  $\gamma_c(G - wz) = \{a + 1\}$ . Thus,  $\gamma_c(\mathcal{T}(G)) = \{a\}$  or  $\gamma_c(\mathcal{T}(G)) = \{a, a + 1\}$  or  $\gamma_c(\mathcal{T}(G)) = \{a, a + 1, a + 2\}$ .

Consequently, by Theorem 2, the number  $\gamma_c$  is an interpolating function. ■

The fact that  $\gamma_c$  is an interpolating function we may also prove in another way. We begin with an observation which is a consequence of the definition of the connected domination number.

**Observation 2** *Let  $T$  be a non-trivial tree of order  $n$ . Then  $\gamma_c(T) = n - n_1(T)$ , where  $n_1(T)$  is a number of end vertices of  $T$ .*

To prove that  $\gamma_c$  is an interpolating function, it is now enough to show the interpolating character of  $n_1$ . This fact was proved in [1, 2, 10, 13].



In [4], weakly connected domination was introduced. For a subset  $S$  of vertices of a connected graph  $G$ , the subgraph weakly induced by  $S$  in  $G$ , denoted by  $\langle S \rangle_G^w$ , is the graph  $(N_G[S], E')$ , where  $E'$  consists of the set of all edges of  $G$  having at least one vertex in  $S$ . A set  $S$  is a *weakly connected dominating set* of  $G$  if  $S$  is dominating in  $G$  and  $(N_G[S], E')$  is connected. The *weakly connected domination number* of  $G$ , denoted by  $\gamma_{wc}(G)$ , is the minimum cardinality of a weakly connected dominating set of  $G$ . We now consider relationships between a weakly connected dominating set and a connected dominating set.

**Lemma 5** *Let  $G$  be a connected graph and let  $D$  be a connected dominating set of  $G$ . Then  $D$  is a weakly connected dominating set of  $G$ .*

**Proof.** Since  $D$  is a connected dominating set of  $G$ , the subgraph  $\langle D \rangle_G$  is connected, i.e. any two vertices  $u, v \in D$  are linked by a path contained in  $D$ . Moreover,  $N_G(x) \cap D \neq \emptyset$  for each  $x \in V(G) - D = N_G[D] - D$ . Therefore  $D$  is also a weakly connected dominating set of  $G$ . ■

The converse implication of Lemma 5 is false. A weakly connected dominating set of graph does not have to be a connected dominating set of graph. The counterexample is a set  $D$  formed of two vertices adjacent to the end vertices of the path  $P_5$  (this path has five vertices). In spite of this, the weakly connected domination number is an interpolating function. We now prove this fact. First we cite a relationship between the number  $\gamma_{wc}$  and the independence number  $\alpha$ .

**Proposition 6** [4] *If  $T$  is a tree of order  $n \geq 2$ , then  $\gamma_{wc}(T) = n - \alpha(T)$ .*

**Theorem 7** *The weakly connected domination number  $\gamma_{wc}$  is an interpolating function.*

**Proof.** Let  $G$  be a unicyclic graph and let  $C$  be the unique cycle of  $G$ . Remove any edge  $uv \in E(G)$ , which belongs to  $C$ . Now we have a spanning tree  $G - uv$ . By Proposition 6, it suffices to show that the set  $\alpha(\mathcal{T}(G)) = \{\alpha(T) : T = G - uv, uv \in E(C)\}$  consists of consecutive integers, because  $n$  is a constant value for a given graph. This fact was proved in [8, 14]. ■

The fact that the weakly connected domination number  $\gamma_{wc}$  is an interpolating function is also an immediate consequence of the following lemma.

**Lemma 8** *If  $H$  is a unicyclic graph and  $uv$  is an edge belonging to the unique cycle  $C$  of  $H$ , then  $\gamma_{wc}(H) \leq \gamma_{wc}(H - uv) \leq \gamma_{wc}(H) + 1$ .*

**Proof.** Since every weakly connected dominating set of  $H - uv$  is also a weakly connected dominating set of  $H$ , so  $\gamma_{wc}(H) \leq \gamma_{wc}(H - uv)$ . On the other hand, let  $D$



be a minimum weakly connected dominating set of  $H$ . If  $D$  is also a weakly connected dominating set of  $H - uv$ , then  $\gamma_{wc}(H - uv) \leq |D| = \gamma_{wc}(H) \leq \gamma_{wc}(H) + 1$ . Assume now that  $D$  is not a weakly connected dominating set of  $H - uv$ . Then  $D \cap \{u, v\} \neq \emptyset$ . We consider two possible cases.

*Case 1.*  $|D \cap \{u, v\}| = 1$ , say  $v \in D$  and  $u \notin D$ . Then  $D$  is not dominating in  $H - uv$  or  $\langle D \rangle_{H-uv}^w$ , the subgraph weakly induced by  $D$  in  $H - uv$ , is not connected. If  $D$  is not dominating in  $H - uv$  and  $\langle D \rangle_{H-uv}^w$  is connected, then  $D \cup \{u\}$  is a weakly connected dominating set of  $H - uv$ . Thus assume that  $D$  is dominating in  $H - uv$  and  $\langle D \rangle_{H-uv}^w$  is not connected. Then there exists exactly one edge  $ab \in E(G)$  which belongs to  $C$  and  $D \cap \{a, b\} = \emptyset$ . Hence  $D \cup \{a\}$  or  $D \cup \{b\}$  is a weakly connected dominating set of  $H - uv$ . Finally, assume that  $D$  is not dominating in  $H - uv$  and  $\langle D \rangle_{H-uv}^w$  is not connected. Then there exists exactly one edge  $ab \in E(G)$  which belongs to  $C$  and  $D \cap \{a, b\} = \emptyset$  and  $a = u$ . Hence  $D \cup \{u\}$  or  $D \cup \{b\}$  is a weakly connected dominating set of  $H - uv$ . Thus,  $\gamma_{wc}(H - uv) \leq |D| = \gamma_{wc}(H) \leq \gamma_{wc}(H) + 1$ .

*Case 2.*  $|D \cap \{u, v\}| = 2$ . It is easy to observe that  $D$  is dominating in  $H - uv$  and  $\langle D \rangle_{H-uv}^w$ , the subgraph weakly induced by  $D$  in  $H - uv$  is not connected. Then there exists exactly one edge  $ab \in E(G)$  which belongs to  $C$  and  $D \cap \{a, b\} = \emptyset$ . Hence  $D \cup \{a\}$  or  $D \cup \{b\}$  is a weakly connected dominating set of  $H - uv$  and  $\gamma_{wc}(H - uv) \leq |D| + 1 = \gamma_{wc}(H) + 1$ .

■

A set  $D \subseteq V(G)$  is said to be a *double dominating set* of  $G$  if  $|N_G[v] \cap D| \geq 2$  for every vertex  $v \in V(G)$ . The *double domination number* of  $G$ , denoted by  $\gamma^{2d}(G)$ , is the minimum cardinality of a double dominating set of  $G$ . This parameter is only defined for graphs without an isolated vertex. It is worth observing that every end vertex of  $G$  and its neighbour always belong to every double dominating set of  $G$ . Double dominating sets were characterized in [5]. We now study the interpolating character of the double domination number. First we show that the removal of a non-end edge from a graph may increase its domination number by at most 2.

**Lemma 9** *Let  $G$  be a graph without an isolated vertex, and let  $uv$  be an edge of a graph  $G$ . If  $uv$  is not an end edge of  $G$ , then  $\gamma^{2d}(G) \leq \gamma^{2d}(G - uv) \leq \gamma^{2d}(G) + 2$ .*

**Proof.** Since every double dominating set of  $G - uv$  is also a double dominating set in  $G$ , the inequality  $\gamma^{2d}(G) \leq \gamma^{2d}(G - uv)$  is obvious. In order to prove the remaining inequality, let  $D$  be a minimum double dominating set of  $G$ . If  $D$  is also a double dominating set of  $G - uv$ , then  $\gamma^{2d}(G - uv) \leq |D| = \gamma^{2d}(G) \leq \gamma^{2d}(G) + 2$ . Assume now that  $D$  is not a double dominating set of  $G - uv$ . Then  $D \cap \{u, v\} \neq \emptyset$ . We consider two possible cases.



*Case 1.*  $|D \cap \{u, v\}| = 1$ , say  $v \in D$  and  $u \notin D$ . Now it is easy to observe that  $D \cup \{u\}$  is a double dominating set of  $G - uv$  and so  $\gamma^{2d}(G - uv) \leq |D| + 1 = \gamma^{2d}(G) + 1 \leq \gamma^{2d}(G) + 2$ .

*Case 2.*  $|D \cap \{u, v\}| = 2$ . Since  $D$  is not a double dominating set of  $G - uv$ , then at least one vertex  $v$  or  $u$  does not have a neighbour in  $D$ . Assume that  $N_{G-uv}(v) \cap D = \emptyset$  and  $N_{G-uv}(u) \cap D \neq \emptyset$ . Then  $D \cup \{x\}$  is a double dominating set of  $G - uv$  for a vertex  $x \in N_{G-uv}(v)$  and so  $\gamma^{2d}(G - uv) \leq |D| + 1 = \gamma^{2d}(G) + 1 \leq \gamma^{2d}(G) + 2$ . Finally, if  $N_{G-uv}(v) \cap D = \emptyset$  and  $N_{G-uv}(u) \cap D = \emptyset$ , then it is easy to observe that  $D \cup \{x, y\}$  is a double dominating set of  $G - uv$  for a vertex  $x \in N_{G-uv}(v)$  and a vertex  $y \in N_{G-uv}(u)$  and therefore  $\gamma^{2d}(G - uv) \leq |D| + 2 = \gamma^{2d}(G) + 2$ .

■

**Corollary 10** *Let  $G$  be a unicyclic graph with  $\gamma^{2d}(G) = a$ , then  $\gamma^{2d}(\mathcal{T}(G)) \subseteq \{a, a + 1, a + 2\}$ .*

Although Lemma 9 is true for unicyclic graphs,  $\gamma^{2d}$  is not an interpolating function. It follows from the counterexample shown in Fig. 1, in which the unicyclic graph  $G$  has only three nonisomorphic spanning trees  $T_1, T_2$  and  $T_3$  with  $\gamma^{2d}(T_1) = \gamma^{2d}(T_2) = 10$  and  $\gamma^{2d}(T_3) = 12$ . The marked vertices of each tree indicate a minimum double dominating set in this tree.

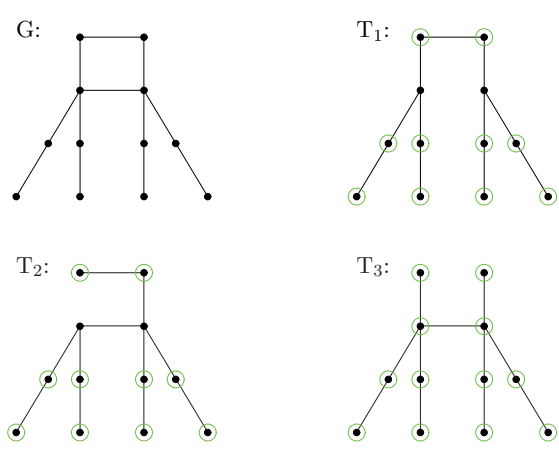


Fig. 1. A graph  $G$  with its spanning trees and  $\gamma^{2d}(\mathcal{T}(G)) = \{10, 12\}$

We now study the interpolating character of two opposite variants of domination parameters—the weak and strong domination numbers of a graph introduced in [11].



We say that a vertex  $v$  *weakly dominates* a vertex  $u$  in a graph  $G$  if  $u \in N_G[v]$  and  $\deg(v) \leq \deg(u)$ . Similarly, we say that a vertex  $u$  *strongly dominates* a vertex  $v$  in  $G$  if  $v \in N_G[u]$  and  $\deg(u) \geq \deg(v)$ . A set  $D$  of vertices of a graph  $G$  is a *weak dominating set* of  $G$  if every vertex in  $V(G) - D$  is weakly dominated by at least one vertex in  $D$ . Similarly,  $D \subseteq V(G)$  is said to be a *strongly dominating set* of  $G$  if every vertex in  $V(G) - D$  is strongly dominated by at least one vertex in  $D$ . The *weak (strong) domination number* of  $G$ , denoted by  $\gamma_w(G)$  ( $\gamma_s(G)$ ), is the minimum cardinality of a weak (strong) dominating set of  $G$ .

The weak domination number  $\gamma_w$  is not an interpolating function. This follows from the counterexample shown in Fig. 2, in which the unicyclic graph  $G$  has only three nonisomorphic spanning trees  $T_1, T_2$  and  $T_3$ . The marked vertices of each tree indicate a minimum weak dominating set in this tree. Thus,  $\gamma_w(\mathcal{T}(G)) = \{11, 12, 14\}$ .

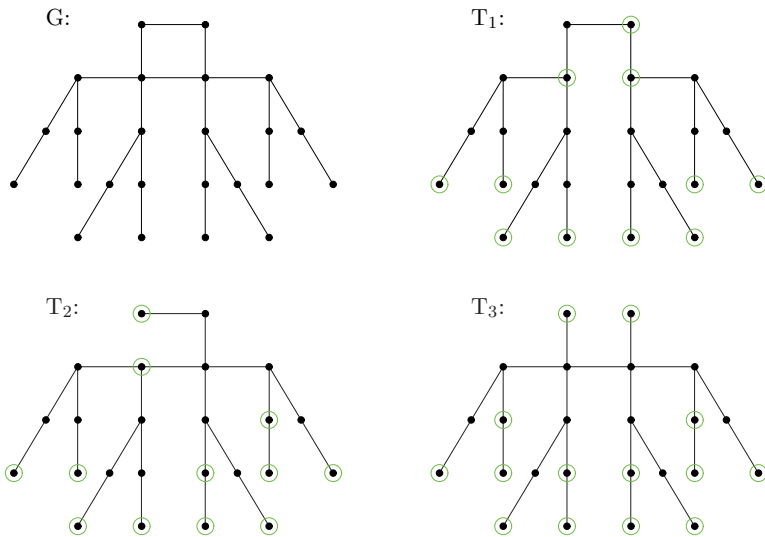


Fig. 2. A graph  $G$  with its spanning trees and  $\gamma_w(\mathcal{T}(G)) = \{11, 12, 14\}$

For 2-connected graphs we have the following theorem.

**Theorem 11** *The weak domination number  $\gamma_w$  interpolates over every 2-connected graph.*

**Proof.** Let  $G$  be a 2-connected graph, and let  $m$  and  $M$  be the smallest and the largest integer of  $\gamma_w(\mathcal{T}(G))$ , respectively. Let  $T_\circ, T^*$  be spanning trees of  $G$  such that  $\gamma_w(T_\circ) = m$  and  $\gamma_w(T^*) = M$ . Since  $G$  is a 2-connected graph then it follows from Theorem 1 that there exists a sequence of end edge-exchanges  $T_\circ, T_1, \dots, T_n = T^*$



transforming  $T_o$  into  $T^*$ . We now need only to show that each step of the end edge-exchange may increase the value of  $\gamma_w$  by at most one. Then  $\gamma_w(T_{l+1}) \leq \gamma_w(T_l) + 1$  for  $l = 0, \dots, n-1$  and it implies that the sequence  $(\gamma_w(T_o), \gamma_w(T_1), \dots, \gamma_w(T_n))$  contains  $(m, m+1, \dots, M)$  as a subsequence and consequently  $\gamma_w(\mathcal{T}(G)) = \{m, m+1, \dots, M\}$ . Let  $D$  be a minimum weak dominating set of  $T_l$  and assume that  $T_{l+1} = T_l - uv + uw$ , where  $u$  is an end vertex of  $T_l$  and of  $T_{l+1}$ . If  $D$  is also a weak dominating set of  $T_{l+1}$ , then  $\gamma_w(T_{l+1}) \leq \gamma_w(T_l) + 1$ . Assume now that  $D$  is not a weak dominating set of  $T_{l+1}$ . It follows from the definition of the weak dominating set that  $u \in D$ . Thus,  $v$  is the unique vertex in  $T_{l+1}$ , which is not weak dominated by any vertex of  $T_{l+1}$ . Therefore  $D \cup \{v\}$  is a weak dominating set of  $T_{l+1}$  and so  $\gamma_w(T_{l+1}) \leq |D| + 1 \leq \gamma_w(T_l) + 1$ . ■

We now prove that the strong domination number, like the weak domination number, interpolates over every 2-connected graph.

**Theorem 12** *The strong domination number  $\gamma_s$  interpolates over any 2-connected graph.*

**Proof.** Let  $G$  be a 2-connected graph. As in the proof of Theorem 11, it is enough to show that  $\gamma_s(T') \leq \gamma_s(T) + 1$  for every end edge-exchange of a spanning tree  $T$  into a spanning tree  $T' = T - uv + uw$  of  $G$ , where  $u$  is an end vertex of  $T$  and of  $T'$ . Let  $D$  be a minimum strong dominating set of  $T$ . If  $D$  is also a strong dominating set of  $T'$ , then  $\gamma_s(T') \leq \gamma_s(T) \leq \gamma_s(T) + 1$ . Assume now that  $D$  is not a strong dominating set of  $T'$ . We may assume that  $u \notin D$ . Otherwise it follows from the minimality of  $D$  that  $u \in D$  and  $v \notin D$  and then  $(D - \{u\}) \cup \{v\}$  is also a minimum strong dominating set of  $T$ . It is now easy to observe that  $D \cup \{w\}$  is a strong dominating set of  $T'$ , so  $\gamma_s(T') \leq |D| + 1 \leq \gamma_s(T) + 1$ . ■

In the end we prove that a very interesting both mathematical and historical variant of domination parameter is an interpolating function. This variant of the domination number, which is suggested by the article in *Scientific American* (1999) by Ian Stewart, entitled “Defend the Roman Empire!”, was first defined and characterized in [3].

For a graph  $G = (V, E)$ , let  $f : V \rightarrow \{0, 1, 2\}$  be a function, and let  $(V_0, V_1, V_2)$  be the ordered partition of  $V$  induced by  $f$ , where  $V_i = \{v \in V | f(v) = i\}$  and  $|V_i| = n_i$  for  $i = 0, 1, 2$ . Between the functions  $f$  and the ordered partitions  $(V_0, V_1, V_2)$  of  $V$  exists 1 – 1 correspondence and so we write  $f = (V_0, V_1, V_2)$ . A function  $f : V \rightarrow \{0, 1, 2\}$  is a *Roman dominating function* (RDF) of a graph  $G = (V, E)$  if every vertex  $u$  for which  $f(u) = 0$  is adjacent to at least one vertex  $v$  for which  $f(v) = 2$ . The weight of  $f$  is  $f(V) = \sum_{v \in V(G)} f(v) = 2n_2 + n_1$ . The *Roman domination number*, denoted by  $\gamma_R(G)$ , equals the minimum weight of an RDF of  $G$ . A function  $f = (V_0, V_1, V_2)$  is a  $\gamma_R$ -function if it is an RDF and  $f(V) = \gamma_R(G)$ .

**Theorem 13** *The Roman domination number  $\gamma_R$  is an interpolating function.*





**Proof.** By Corollary 3(1), it suffices to show that the inequality  $\gamma_R(G) \leq \gamma_R(G - uv) \leq \gamma_R(G) + 1$  is true for every unicyclic graph  $G$  and every edge of the unique cycle of  $G$ . If  $f = (V_0, V_1, V_2)$  is a  $\gamma_R$ -function on a graph  $G - uv$ , then  $f = (V_0, V_1, V_2)$  is also an RDF on a graph  $G$  and so  $\gamma_R(G) \leq |V_1| + 2|V_2| = \gamma_R(G - uv)$ . Let now  $g = (V_0, V_1, V_2)$  be a  $\gamma_R$ -function on a graph  $G$ , and let  $uv$  be an edge of the unique cycle of  $G$ . If one of the vertices  $v$  or  $u$  belongs to  $V_0$  and the other belongs to  $V_2$ , say  $v \in V_0$  and  $u \in V_2$ , then function  $g' = (V_0 - \{v\}, V_1 \cup \{v\}, V_2)$  is an RDF on  $G - uv$ . Thus,  $\gamma_R(G - uv) \leq g'(V(G)) = |V_1 \cup \{v\}| + 2|V_2| = |V_1| + 2|V_2| + 1 = \gamma_R(G) + 1$ . In other simpler cases  $g = (V_0, V_1, V_2)$  is also an RDF on a graph  $G - uv$  and therefore we have  $\gamma_R(G - uv) \leq \gamma_R(G) \leq \gamma_R(G) + 1$ . ■

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