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NOTE ON THE VARIANCE OF THE SUM OF GAUSSIAN FUNCTIONALS

Abstract. Let $(X_i, i = 1, 2, \dots)$ be a Gaussian sequence with $X_i \in N(0, 1)$ for each i and suppose its correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$ is the matrix of some linear operator $R: l_2 \rightarrow l_2$. Then for $f_i \in L^2(\mu)$, $i = 1, 2, \dots$, where μ is the standard normal distribution, we estimate the variation of the sum of the Gaussian functionals $f_i(X_i)$, $i = 1, 2, \dots$.

1. Introduction. Let (X, Y) be a Gaussian random vector such that $X, Y \in N(0, 1)$ and $E(XY) = \rho$, $(|\rho| < 1)$. We denote by μ the normalized one-dimensional Gaussian measure, i.e.

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx.$$

In $L^2(\mu)$ we have the scalar product

$$(f, g)_\mu = \int_{\mathbb{R}} f(x)g(x) \mu(dx).$$

Introducing a random variable $Z \in N(0, 1)$ such that Z, Y are independent, we find that the Gaussian vectors (X, Y) and (U, Y) with $U = \rho Y + \sqrt{1 - \rho^2} Z$ have the same joint distribution. Thus, for $f, g \in L^2(\mu)$ we have

$$(1.1) \quad E(f(X)g(Y)) = E(f(U)g(Y)) = E(P_\rho(Y)g(Y)),$$

where

$$P_\rho f(y) = E(f(U) | Y = y) = \int_{\mathbb{R}} f(\rho y + \sqrt{1 - \rho^2} z) d\mu(z), \quad y \in \mathbb{R},$$

is called the *Ornstein–Uhlenbeck operator*. The Ornstein–Uhlenbeck operator has a representation in terms of orthonormal Hermite polynomials

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$\{h_n\}_{n \geq 0} \subset L^2(\mu)$, namely

$$(1.2) \quad P_\rho f = \sum_{n=0}^{\infty} \rho^n (f, h_n)_\mu h_n, \quad f \in L^2(\mu).$$

In particular,

$$P_\rho h_n = \rho^n h_n, \quad n \geq 0.$$

From (1.2) we obtain Gebelein’s inequality (see [G] and [DK]):

PROPOSITION 1.1. *If $f \in L^2$ and $(f, 1)_\mu = 0$, then*

$$(1.3) \quad \|P_\rho f\|_2 \leq |\rho| \cdot \|f\|_2,$$

or equivalently for any $g \in L^2$ and f as above,

$$|(P_\rho f, g)_\mu| \leq |\rho| \cdot \|f\|_2 \cdot \|g\|_2.$$

In both inequalities we have equality if and only if $f(x) = cx$.

Consider a Gaussian sequence $(X_i, i = 1, 2, \dots)$ of random variables with $X_i \in N(0, 1)$ for each i . It is assumed that the correlation matrix $R = (\rho_{ij})_{i, j \geq 1}$, where $\rho_{ij} = E(X_i X_j)$, $i, j = 1, 2, \dots$, satisfies

$$(1.4) \quad C = \sup_{i \geq 1} \sum_{j \geq 1} |\rho_{ij}| < \infty.$$

It is evident that $C \geq 1$. The Frobenius Theorem (see [HLP]) implies that R is the matrix (in the standard basis) of a continuous linear operator (which we denote by the same letter) $R : l_p \rightarrow l_p$ for $1 \leq p \leq \infty$ with $\|R\| \leq C$. Hence, it is easily seen that for $C < 2$ the linear operator R is invertible. Using Gebelein’s inequality (1.3), one can prove (see [BC1], [BC2], [V])

LEMMA 1.1. *Let the Gaussian sequence $(X_i, i = 1, 2, \dots)$ with $X_i \in N(0, 1)$ for each i satisfy the hypothesis (1.4) and let $(f_i, i = 1, 2, \dots) \subset L^2(\mu)$. Then for each $n \geq 1$ we have*

$$(1.5) \quad (2 - C) \sum_{i=1}^n \text{Var}(f_i(X_i)) \leq \text{Var}\left(\sum_{i=1}^n f_i(X_i)\right) \leq C \sum_{i=1}^n \text{Var}(f_i(X_i)).$$

For $C \geq 2$ the left inequality in (1.5) holds trivially. In fact, we can say more: an inequality of the form

$$(1.6) \quad M \sum_{i=1}^n \text{Var}(f_i(X_i)) \leq \text{Var}\left(\sum_{i=1}^n f_i(X_i)\right),$$

where M is a positive constant, is not satisfied in general when $C \geq 2$.

Consider the following simple example: Let $(Y_i, i = 1, 2, \dots) \subset N(0, 1)$ be a sequence of independent Gaussian random variables. Let $a, b \in \mathbb{R}$ be such that $a^2 + b^2 = 1$ and define

$$X_{3k-2} = -Y_{2k}, \quad X_{3k-1} = a Y_{2k-1} - b Y_{2k}, \quad X_{3k} = a Y_{2k-1} + b Y_{2k}, \quad k \geq 1.$$

Moreover, we put

$$f_{3k-2}(t) = 2bt, \quad f_{3k-1}(t) = -t, \quad f_{3k}(t) = t, \quad t \in \mathbb{R}, k \geq 1.$$

It is easy to check that

$$C = \sup_{i \geq 1} \sum_{j \geq 1} |\rho_{ij}| = 1 + |b| + \max\{|b|, |1 - 2b^2|\} \geq 2$$

and

$$\text{Var}\left(\sum_{i=1}^{3n} f_i(X_i)\right) = 0 \quad \text{and} \quad \sum_{i=1}^{3n} \text{Var}(f_i(X_i)) > 0, \quad n \geq 1.$$

2. Main result. In this section we are going to prove the inequality (1.5) under a slightly weaker condition than (1.4). First let us introduce some notations. For a given correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$, we put

$$R_n^{(m)} = (\rho_{ij}^{(m)})_{1 \leq i,j \leq n}, \quad m, n \geq 1,$$

and let $\lambda_{n,1}^{(m)}$ and $\lambda_{n,n}^{(m)}$ denote the least and the greatest of the eigenvalues of the matrix $R_n^{(m)}$. By the Schur lemma (see [B]) the matrix $R_n^{(m)}$ is nonnegative definite. Hence, the eigenvalues $\lambda_{n,1}^{(m)}$ are nonnegative. For the matrix $R_n = R_n^{(1)}$ we use the well known decomposition

$$R_n = U_n D_n U_n^T,$$

where

$$D_n = \begin{pmatrix} \lambda_{n,1}^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{n,n}^{(1)} \end{pmatrix}$$

is a diagonal matrix with eigenvalues $\lambda_{n,i}^{(1)}$, $i = 1, \dots, n$, of R_n on the main diagonal. The matrix $U_n = (u_{n,ij})_{1 \leq i,j \leq n}$ is an orthogonal matrix and T stands for transposition. It follows that

$$(2.1) \quad \rho_{ij} = \sum_{k=1}^n \lambda_{n,k}^{(1)} u_{n,ik} u_{n,jk}, \quad 1 \leq i, j \leq n.$$

Now we can state the following simple result.

LEMMA 2.1. *Fix $n \geq 1$. Then the least and the greatest eigenvalues of the matrix $R_n^{(m)}$ are monotonic with respect to m , i.e.*

$$(2.2) \quad \lambda_{n,1}^{(m+1)} \geq \lambda_{n,1}^{(m)} \quad \text{and} \quad \lambda_{n,n}^{(m+1)} \leq \lambda_{n,n}^{(m)}, \quad \text{for } m = 1, 2, \dots$$

Proof. Since the matrix $R_n^{(m+1)}$ is symmetric, we have

$$(2.3) \quad \lambda_{n,1}^{(m+1)} = \inf_{\|c\|=1} (R_n^{(m+1)}c, c) = \inf_{\|c\|=1} \sum_{i,j=1}^n \rho_{ij}^{m+1} c_i c_j,$$

where $c = (c_1, \dots, c_n) \in l_2^n$ and l_2^n is the n -dimensional Euclidean space with the scalar product denoted here by (\cdot, \cdot) . From (2.3) and (2.1) we conclude that for every $c = (c_1, \dots, c_n) \in l_2^n$ with $\|c\| = 1$ we have

$$(2.4) \quad \begin{aligned} \sum_{i,j=1}^n \rho_{ij}^{m+1} c_i c_j &= \sum_{i,j=1}^n \rho_{ij}^m \rho_{ij} c_i c_j \\ &= \sum_{i,j=1}^n \rho_{ij}^m \sum_{k=1}^n \lambda_{n,k}^{(1)} u_{n,ik} u_{n,jk} c_i c_j = \sum_{k=1}^n \lambda_{n,k}^{(1)} \left(\sum_{i,j=1}^n \rho_{ij}^m c_i u_{n,ik} c_j u_{n,jk} \right) \\ &\geq \sum_{k=1}^n \lambda_{n,k}^{(1)} \sum_{i=1}^n c_i^2 u_{n,ik}^2 \inf_{\|b\|=1} (R_n^{(m)}b, b) = \lambda_{n,1}^{(m)}, \end{aligned}$$

since

$$\sum_{k=1}^n \lambda_{n,k}^{(1)} \sum_{i=1}^n c_i^2 u_{n,ik}^2 = \sum_{i=1}^n c_i^2 \sum_{k=1}^n \lambda_{n,k}^{(1)} u_{n,ik}^2 = 1$$

by (2.1). Taking the infimum in (2.4) over all $c = (c_1, \dots, c_n) \in l_2^n$ with $\|c\| = 1$ we obtain the first inequality of (2.2). The proof of the second one runs similarly. ■

We can now formulate our main result.

THEOREM 2.1. *Let $(X_i, i = 1, 2, \dots)$ be a Gaussian sequence with $X_i \in N(0, 1)$ for each i and suppose its correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$ is the matrix of some operator $R : l_2 \rightarrow l_2$. Then for $f_i \in L^2(\mu)$, $i = 1, 2, \dots$, and for every $n \geq 1$ we have*

$$(2.5) \quad \lambda_{\min} \sum_{i=1}^n \text{Var}(f_i(X_i)) \leq \text{Var}\left(\sum_{i=1}^n f_i(X_i)\right) \leq \lambda_{\max} \sum_{i=1}^n \text{Var}(f_i(X_i)),$$

where

$$\lambda_{\min} = \inf_{\|x\|=1} (Rx, x), \quad \lambda_{\max} = \sup_{\|x\|=1} (Rx, x).$$

REMARK. Let us point out that the assumption concerning the correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$ of the sequence $(X_i, i = 1, 2, \dots)$ is slightly weaker than the hypothesis (1.4). To see this, consider the following example: Let $(Y_i, i = 1, 2, \dots) \subset N(0, 1)$ be a sequence of independent Gaussian random

variables and define

$$X_1 = aY_1 + \sum_{j=2}^{\infty} Y_j/j, \quad \text{where } a = \sqrt{2 - \pi^2/6},$$

$$X_i = Y_i \quad \text{for } i \geq 2.$$

It follows immediately that the correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$ of the sequence $(X_i, i = 1, 2, \dots)$ is the matrix of some linear operator $R : l_2 \rightarrow l_2$ and the hypothesis (1.4) is not satisfied.

Proof of Theorem 1.1. First we prove the left inequality of (2.5). Without loss of generality we assume that $E(f_i(X_i)) = 0, i = 1, 2, \dots$. If $\lambda_{\min} = 0$ then the inequality holds trivially. Assume that $\lambda_{\min} \neq 0$. Expanding each $f_i, i \geq 1$, with respect to the Hermite basis in $L^2(\mu)$ we obtain

$$(2.6) \quad f_i = \sum_{k=1}^{\infty} c_{ik}h_k, \quad \|f_i\|_{\mu}^2 = \sum_{k=1}^{\infty} c_{ik}^2, \quad i = 1, 2, \dots$$

From (1.1) and (1.2) and the orthonormality of Hermite polynomials $\{h_n\}_{n \geq 1} \subset L^2(\mu)$ it follows that

$$(2.7) \quad E[h_n(X_i)h_m(X_j)] = \rho_{ij}^n \delta_m^n, \quad n, m, i, j = 1, 2, \dots,$$

where δ_m^n is the Kronecker delta. Combining (2.6) with (2.7) and using Lemma 2.1 we get

$$\begin{aligned} \text{Var}\left(\sum_{i=1}^n f_i(X_i)\right) &= E\left(\sum_{i=1}^n f_i(X_i)\right)^2 \\ &= \lim_{N \rightarrow \infty} E\left(\sum_{i=1}^n \sum_{k=1}^N c_{ik}h_k(X_i)\right)^2 = \lim_{N \rightarrow \infty} E\left(\sum_{k=1}^N \sum_{i=1}^n c_{ik}h_k(X_i)\right)^2 \\ &= \lim_{N \rightarrow \infty} \sum_{k,l=1}^N E\left[\left(\sum_{i=1}^n c_{ik}h_k(X_i)\right)\left(\sum_{j=1}^n c_{jl}h_l(X_j)\right)\right] \\ &= \lim_{N \rightarrow \infty} \sum_{k=1}^N E\left[\sum_{i=1}^n c_{ik}h_k(X_i)\right]^2 = \lim_{N \rightarrow \infty} \sum_{k=1}^N \sum_{i,j=1}^n \rho_{ij}^k c_{ik}c_{jk} \\ &\geq \lim_{N \rightarrow \infty} \sum_{k=1}^N \lambda_{n,1}^{(k)} \sum_{i=1}^n c_{ik}^2 \geq \lim_{N \rightarrow \infty} \sum_{k=1}^N \lambda_{n,1}^{(1)} \sum_{i=1}^n c_{ik}^2 \\ &\geq \lambda_{\min} \sum_{i=1}^n \sum_{k=1}^{\infty} c_{ik}^2 = \lambda_{\min} \sum_{i=1}^n E[f_i(X_i)]^2 = \lambda_{\min} \sum_{i=1}^n \text{Var}(f_i(X_i)). \end{aligned}$$

This proves the left inequality of (2.5). The proof of the right one is similar. ■

REMARK. Let us point out that under the assumptions of Theorem 2.1 the inequality (1.6) holds for all $f_i \in L^2(\mu)$, $i = 1, 2, \dots$, with a positive constant M if and only if the operator $R : l_2 \rightarrow l_2$ is invertible. ■

Adapting now the methods from [BC1] and [BC2] we can write the following two statements:

LEMMA 2.2 (Borel–Cantelli Lemma). *Let $(X_i, i = 1, 2, \dots)$ be a Gaussian sequence with $X_i \in N(0, 1)$ for $i \geq 1$ and suppose its correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$ is the matrix of some linear operator $R : l_2 \rightarrow l_2$. Then for every sequence of Borel sets $(A_i, i = 1, 2, \dots)$ such that $\sum_{i=1}^{\infty} P\{X_i \in A_i\} = \infty$ we have $P\{X_i \in A_i \text{ i.o.}\} = 1$. ■*

THEOREM 2.2 (Strong Law of Large Numbers). *Let $(X_i, i = 1, 2, \dots)$ be a Gaussian sequence with $X_i \in N(0, 1)$ for $i \geq 1$ and suppose its correlation matrix $R = (\rho_{ij})_{i,j \geq 1}$ is the matrix of some linear operator $R : l_2 \rightarrow l_2$. Then for $f \in L^1(\mu)$ we have*

$$\frac{1}{n} \sum_{i=1}^n f(X_i) \xrightarrow[n \rightarrow \infty]{} Ef(X_1) \quad \text{a.s.} \quad \blacksquare$$

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