

## Almost homoclinic solutions for a certain class of mixed type functional differential equations

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**Abstract.** We shall be concerned with the existence of almost homoclinic solutions for a class of second order functional differential equations of mixed type:  $\ddot{q}(t) + V_q(t, q(t)) + u(t, q(t), q(t - T), q(t + T)) = f(t)$ , where  $t \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$  and  $T > 0$  is a fixed positive number.

By an almost homoclinic solution (to 0) we mean one that joins 0 to itself and  $q \equiv 0$  may not be a stationary point. We assume that  $V$  and  $u$  are  $T$ -periodic with respect to the time variable,  $V$  is  $C^1$ -smooth and  $u$  is continuous. Moreover,  $f$  is non-zero, bounded, continuous and square-integrable. The main result provides a certain approximative scheme of finding an almost homoclinic solution.

**1. Introduction.** In this work, we shall be concerned with the existence of almost homoclinic solutions for second order functional differential equations of mixed type (sometimes known as forward-backward differential equations) of the form

$$(1.1) \quad \ddot{q}(t) + V_q(t, q(t)) + u(t, q(t), q(t - T), q(t + T)) = f(t),$$

where  $t \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$ ,  $T > 0$  is a fixed positive number, under the following assumptions:

- (H1)  $V: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^1$ -smooth,  $T$ -periodic with respect to  $t$ ,
- (H2)  $u: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous,  $T$ -periodic in  $t$ ,
- (H3)  $f: \mathbb{R} \rightarrow \mathbb{R}^n$  is non-zero, continuous, bounded and square-integrable.

Here and subsequently,  $V_q$  denotes the gradient of  $V$  with respect to  $q$ .

**DEFINITION 1.1.** We will say that a solution  $q: \mathbb{R} \rightarrow \mathbb{R}^n$  of (1.1) is *almost homoclinic* (to 0) if  $q(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ .

Let us remark that  $q \equiv 0$  may not satisfy (1.1). That is why we have decided to call solutions joining 0 to itself almost homoclinic.

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From now on,  $(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  stands for the standard inner product in  $\mathbb{R}^n$  and  $|\cdot|: \mathbb{R}^n \rightarrow [0, \infty)$  denotes the induced norm. Let  $E = W^{1,2}(\mathbb{R}, \mathbb{R}^n)$  be the Hilbert space of functions from  $\mathbb{R}$  into  $\mathbb{R}^n$  under the usual norm

$$\|q\|_E^2 = \int_{-\infty}^{\infty} (|q(t)|^2 + |\dot{q}(t)|^2) dt.$$

For each  $k \in \mathbb{N}$ , let  $f_k: \mathbb{R} \rightarrow \mathbb{R}^n$  denote a  $2kT$ -periodic extension of  $f|_{[-kT, kT]}$  over  $\mathbb{R}$ . Remark that  $f_k$  may not be continuous at  $kT \pm 2kTj$ ,  $j \in \mathbb{Z}$ . Let us consider a sequence of functional differential equations

$$(1.2) \quad \ddot{q}(t) + V_q(t, q(t)) + u(t, q(t), q(t-T), q(t+T)) = f_k(t).$$

Let  $E_k = W_{2kT}^{1,2}(\mathbb{R}, \mathbb{R}^n)$  be the Hilbert space of  $2kT$ -periodic functions from  $\mathbb{R}$  into  $\mathbb{R}^n$  with the standard norm

$$\|q\|_{E_k}^2 = \int_{-kT}^{kT} (|q(t)|^2 + |\dot{q}(t)|^2) dt.$$

Let us denote by  $C_{\text{loc}}^l(\mathbb{R}, \mathbb{R}^n)$ ,  $l \in \mathbb{N}$ , the space of functions on  $\mathbb{R}$  with values in  $\mathbb{R}^n$  under the topology of almost uniform convergence of functions and all derivatives up to order  $l$ .

We will prove the following theorem.

**THEOREM 1.2.** *Let  $V$ ,  $u$  and  $f$  satisfy (H1)–(H3). Assume that for each  $k \in \mathbb{N}$  there is a solution  $q_k \in E_k$  of (1.2). If the sequence  $\{\|q_k\|_{E_k}\}_{k \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  then there exist a subsequence  $\{q_{k_j}\}_{j \in \mathbb{N}}$  and a function  $q_0 \in E$  such that*

$$q_{k_j} \rightarrow q_0 \quad \text{as } j \rightarrow \infty$$

*in the topology of  $C_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^n)$  and  $q_0$  is an almost homoclinic solution of (1.1).*

Recently, differential equations involving both advanced and retarded arguments have appeared in an increasing number of models originating from a wide variety of scientific disciplines (see [HVL1, HVL2] and references given there).

Theorem 1.2 yields a certain approximative scheme of finding an almost homoclinic solution for (1.1). This method generalizes that of [J] for the Newtonian systems  $\ddot{q} + V_q(t, q) = f(t)$ , where  $V$  and  $f$  satisfy (H1) and (H3), respectively (see also [IJ1, IJ2]). Similar results for a class of Newtonian systems (with  $f \equiv 0$ ) have been obtained by Rabinowitz in [R] and for a family of first order Hamiltonian systems by Tanaka in [T].

The paper is organized as follows. Theorem 1.2 will be proved in Section 2 by means of the Ascoli–Arzelà lemma. In Section 3, some applications of this theorem will be given for certain problems of variational nature.



**2. Proof of Theorem 1.2.** For each  $k \in \mathbb{N}$ , let  $L_{2kT}^\infty(\mathbb{R}, \mathbb{R}^n)$  denote the space of  $2kT$ -periodic essentially bounded measurable functions from  $\mathbb{R}$  into  $\mathbb{R}^n$  with the norm

$$\|q\|_{L_{2kT}^\infty} = \text{ess sup}\{|q(t)| : t \in [-kT, kT]\}.$$

FACT 2.1. *There is  $C > 0$ , independent of  $k \in \mathbb{N}$ , such that*

$$(2.1) \quad \|q\|_{L_{2kT}^\infty} \leq C\|q\|_{E_k} \quad \text{for all } q \in E_k.$$

The proof of this fact may be found in [IJ1].

LEMMA 2.2. *Let  $V, u$  and  $f$  satisfy (H1)–(H3). Assume that for each  $k \in \mathbb{N}$  there is a solution  $q_k \in E_k$  of (1.2). If the sequence  $\{\|q_k\|_{E_k}\}_{k \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  then there exist a subsequence  $\{q_{k_j}\}_{j \in \mathbb{N}}$  and a function  $q_0 \in E$  such that  $q_{k_j} \rightarrow q_0$  as  $j \rightarrow \infty$  in the topology of  $C_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^n)$ .*

*Proof.* There is  $M > 0$  such that for each  $k \in \mathbb{N}$ ,

$$(2.2) \quad \|q_k\|_{E_k} \leq M.$$

Combining (2.2) with (2.1), we get

$$(2.3) \quad \|q_k\|_{L_{2kT}^\infty} \leq CM.$$

Since  $q_k$  is a solution of (1.2), from (H1)–(H3) and (2.3) it follows that there is  $M_1 > 0$ , independent of  $k \in \mathbb{N}$ , such that

$$(2.4) \quad \|\dot{q}_k\|_{L_{2kT}^\infty} \leq M_1.$$

Fix  $k \in \mathbb{N}$  and  $t \in \mathbb{R}$ . If  $n = 1$  (i.e.  $q_k : \mathbb{R} \rightarrow \mathbb{R}$ ) then there is  $s_k \in (t - 1, t)$  such that

$$\dot{q}_k(s_k) = \int_{t-1}^t \ddot{q}_k(s) ds = q_k(t) - q_k(t - 1)$$

and

$$\dot{q}_k(t) = \int_{s_k}^t \ddot{q}_k(s) ds + \dot{q}_k(s_k).$$

Consequently,

$$|\dot{q}_k(t)| \leq \int_{t-1}^t |\ddot{q}_k(s)| ds + |q_k(t) - q_k(t - 1)| \leq M_1 + 2CM \equiv M_2.$$

Hence, if  $n \geq 1$ , we have

$$(2.5) \quad \|\dot{q}_k\|_{L_{2kT}^\infty} \leq \sqrt{n}M_2.$$

To finish the proof, it suffices to show that  $\{q_k\}_{k \in \mathbb{N}}$  and  $\{\dot{q}_k\}_{k \in \mathbb{N}}$  are equicontinuous. For each  $k \in \mathbb{N}$  and  $t, s \in \mathbb{R}$ , we have

$$|q_k(t) - q_k(s)| = \left| \int_s^t \dot{q}_k(\tau) d\tau \right| \leq \sqrt{n}M_2|t - s|,$$

by (2.5), and

$$|\dot{q}_k(t) - \dot{q}_k(s)| \leq M_1|t - s|,$$

by (2.4). Applying now the Ascoli–Arzelà lemma, we get the claim. ■

FACT 2.3. *Let  $q: \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous mapping. If a weak derivative  $\dot{q}: \mathbb{R} \rightarrow \mathbb{R}^n$  is continuous at  $t_0 \in \mathbb{R}$ , then  $q$  is differentiable at  $t_0$  and*

$$\lim_{t \rightarrow t_0} \frac{q(t) - q(t_0)}{t - t_0} = \dot{q}(t_0).$$

Let  $L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^n)$  be the space of functions from  $\mathbb{R}$  into  $\mathbb{R}^n$  that are locally square-integrable.

FACT 2.4. *Let  $q: \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous map such that  $\dot{q} \in L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^n)$ . Then for each  $t \in \mathbb{R}$ ,*

$$(2.6) \quad |q(t)| \leq \sqrt{2} \left( \int_{t-1/2}^{t+1/2} (|q(s)|^2 + |\dot{q}(s)|^2) ds \right)^{1/2}.$$

The proofs of Facts 2.3 and 2.4 can be found in [IJ1].

LEMMA 2.5. *Let  $V$ ,  $u$  and  $f$  satisfy (H1)–(H3). If  $\{q_{k_j}\}_{j \in \mathbb{N}}$  and  $q_0$  are given by Lemma 2.2 then  $q_0$  is an almost homoclinic solution of (1.1) and  $q_{k_j} \rightarrow q_0$  as  $j \rightarrow \infty$  in the topology of  $C_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^n)$ .*

*Proof.* Fix  $a, b \in \mathbb{R}$  such that  $a < b$ . There is  $j_0 \in \mathbb{N}$  such that  $[a, b] \subset [-k_j T, k_j T]$  for all  $j > j_0$ . Thus

$$\ddot{q}_{k_j}(t) = f(t) - V_q(t, q_{k_j}(t)) - u(t, q_{k_j}(t), q_{k_j}(t - T), q_{k_j}(t + T))$$

for all  $t \in [a, b]$  and  $j > j_0$ . Hence, if  $j > j_0$  then the restriction of  $\ddot{q}_{k_j}$  onto  $[a, b]$  is continuous. From Fact 2.3 it follows that  $\ddot{q}_{k_j}$  is a derivative of  $\dot{q}_{k_j}$  in  $(a, b)$  for all  $j > j_0$ . Since  $q_{k_j} \rightarrow q_0$  and  $\dot{q}_{k_j} \rightarrow \dot{q}_0$  almost uniformly on  $\mathbb{R}$ ,

$$\ddot{q}_{k_j}(t) \rightarrow f(t) - V_q(t, q_0(t)) - u(t, q_0(t), q_0(t - T), q_0(t + T))$$

uniformly on  $[a, b]$ . Consequently,

$$\ddot{q}_0(t) = f(t) - V_q(t, q_0(t)) - u(t, q_0(t), q_0(t - T), q_0(t + T))$$

in  $(a, b)$ . By the above, we conclude that  $q_0$  is a solution of (1.1) and  $q_{k_j} \rightarrow q_0$  as  $j \rightarrow \infty$  in the topology of  $C_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^n)$ .

To finish the proof, we have to show that  $q_0$  is an almost homoclinic solution. Take  $l \in \mathbb{N}$ . There is  $j_0 \in \mathbb{N}$  such that  $[-lT, lT] \subset [-k_j T, k_j T]$  for all  $j > j_0$ . By (2.2), for all  $j > j_0$  we get

$$\int_{-lT}^{lT} (|q_{k_j}(t)|^2 + |\dot{q}_{k_j}(t)|^2) dt \leq M^2.$$



Letting  $j \rightarrow \infty$ , we obtain

$$\int_{-lT}^{lT} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \leq M^2.$$

Finally, since  $l$  is an arbitrary positive integer,

$$\int_{-\infty}^{\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \leq M^2,$$

which implies

$$(2.7) \quad \int_{|t|>r} (|q_0(t)|^2 + |\dot{q}_0(t)|^2) dt \rightarrow 0$$

as  $r \rightarrow \infty$ . Combining (2.7) with (2.6), we get  $q_0(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ . ■

**3. Applications.** In this section some applications of Theorem 1.2 are indicated. Let assumptions (H1)–(H3) hold. From now on, we will also assume that

(H4) there is a  $C^1$ -map  $U: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  of variables  $(t, x, y)$ ,  $T$ -periodic in  $t \in \mathbb{R}$  such that

$$u(t, q(t), q(t-T), q(t+T)) = U_x(t, q(t), q(t-T)) + U_y(t, q(t+T), q(t)),$$

where  $U_x$  and  $U_y$  denote the gradients of  $U$  with respect to  $x$  and  $y$ , respectively.

For each  $k \in \mathbb{N}$ , let  $I_k: E_k \rightarrow \mathbb{R}$  be given by

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT} \left( \frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) - U(t, q(t), q(t-T)) \right) dt \\ &\quad + \int_{-kT}^{kT} (f_k(t), q(t)) dt. \end{aligned}$$

The functional  $I_k$  is differentiable on  $E_k$  and it is easy to check that

$$\begin{aligned} I'_k(q)p &= \int_{-kT}^{kT} [(\dot{q}(t), \dot{p}(t)) - (V_q(t, q(t)), p(t))] dt \\ &\quad - \int_{-kT}^{kT} [(U_x(t, q(t), q(t-T)), p(t)) + (U_y(t, q(t+T), q(t)), p(t))] dt \\ &\quad + \int_{-kT}^{kT} (f_k(t), p(t)) dt, \end{aligned}$$



and by (H4),

$$I'_k(q)p = \int_{-kT}^{kT} [(\dot{q}(t), \dot{p}(t)) - (V_q(t, q(t)), p(t))] dt - \int_{-kT}^{kT} (u(t, q(t), q(t - T), q(t + T)), p(t)) dt + \int_{-kT}^{kT} (f_k(t), p(t)) dt.$$

Moreover, by the Fundamental Lemma (see [MW]), for a fixed  $k \in \mathbb{N}$  the critical points of  $I_k$  are  $2kT$ -periodic solutions of (1.2).

EXAMPLE 1. Let us consider second order functional differential equations of mixed type of the form

$$(3.1) \quad \ddot{q}(t) + u(t, q(t), q(t - T), q(t + T)) = f(t),$$

where  $t \in \mathbb{R}$ ,  $q \in \mathbb{R}^n$  and  $T > 0$  is a fixed positive number. Suppose that (H2)–(H4) hold, and furthermore,

$$(H5) \quad \text{there are } a, b > 0 \text{ such that for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^n, \\ -U(t, x, y) \geq -U(t, 0, 0) + a|x|^2 + b|y|^2,$$

$$(H6) \quad \int_0^T U(t, 0, 0) dt = 0.$$

THEOREM 3.1. Under assumptions (H2)–(H6), there is an almost homoclinic solution  $q_0: \mathbb{R} \rightarrow \mathbb{R}^n$  of (3.1) such that  $\dot{q}_0(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ .

An approximative sequence of functional differential equations for (3.1) is as follows:

$$(3.2) \quad \ddot{q}(t) + u(t, q(t), q(t - T), q(t + T)) = f_k(t),$$

where  $f_k: \mathbb{R} \rightarrow \mathbb{R}^n$  is a  $2kT$ -periodic extension of  $f|_{[-kT, kT]}$  onto  $\mathbb{R}$ . In this case,  $I_k: E_k \rightarrow \mathbb{R}$  is given by

$$(3.3) \quad I_k(q) = \int_{-kT}^{kT} \left( \frac{1}{2} |\dot{q}(t)|^2 - U(t, q(t), q(t - T)) + (f_k(t), q(t)) \right) dt$$

and

$$(3.4) \quad I'_k(q)p = \int_{-kT}^{kT} (\dot{q}(t), \dot{p}(t)) dt - \int_{-kT}^{kT} (U_x(t, q(t), q(t - T)), p(t)) dt - \int_{-kT}^{kT} (U_y(t, q(t + T), q(t)), p(t)) dt + \int_{-kT}^{kT} (f_k(t), p(t)) dt.$$

For each  $k \in \mathbb{N}$ , let  $L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$  denote the Hilbert space of  $2kT$ -periodic functions on  $\mathbb{R}$  with values in  $\mathbb{R}^n$  under the norm



$$\|q\|_{L^2_{2kT}}^2 = \int_{-kT}^{kT} |q(t)|^2 dt.$$

Applying (H3), (H5) and (H6), we get

$$\begin{aligned} (3.5) \quad I_k(q) &\geq \int_{-kT}^{kT} \left( \frac{1}{2} |\dot{q}(t)|^2 + a|q(t)|^2 + (f_k(t), q(t)) \right) dt \\ &\geq \min\{1/2, a\} \int_{-kT}^{kT} (|\dot{q}(t)|^2 + |q(t)|^2) dt - \|f_k\|_{L^2_{2kT}} \|q\|_{E_k} \\ &\geq A\|q\|_{E_k}^2 - B\|q\|_{E_k}, \end{aligned}$$

where  $A = \min\{1/2, a\}$  and  $B = \|f\|_{L^2(\mathbb{R}, \mathbb{R}^n)}$ .

LEMMA 3.2. Under (H2)–(H6),  $I_k$  defined by (3.3) satisfies the Palais–Smale condition.

*Proof.* Let  $\{p_j\}_{j \in \mathbb{N}} \subset E_k$  be a sequence such that  $\{I_k(p_j)\}_{j \in \mathbb{N}}$  is bounded and  $I'_k(p_j) \rightarrow 0$  as  $j \rightarrow \infty$ . We have to show that  $\{p_j\}_{j \in \mathbb{N}}$  has a convergent subsequence. From (3.5) it follows that  $\{p_j\}_{j \in \mathbb{N}}$  is bounded in the Hilbert space  $E_k$ . Therefore, along a subsequence,  $\{p_j\}_{j \in \mathbb{N}}$  converges weakly in  $E_k$  and strongly in  $L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$  to  $p_0 \in E_k$ . Hence, passing to a subsequence if necessary, we have  $p_j \rightarrow p_0$  in  $L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$ ,  $(I'_k(p_j) - I'_k(p_0))(p_j - p_0) \rightarrow 0$ ,

$$\int_{-kT}^{kT} (U_x(t, p_j(t), p_j(t-T)) - U_x(t, p_0(t), p_0(t-T)), p_j(t) - p_0(t)) dt \rightarrow 0$$

and

$$\int_{-kT}^{kT} (U_y(t, p_j(t+T), p_j(t)) - U_y(t, p_0(t+T), p_0(t)), p_j(t) - p_0(t)) dt \rightarrow 0$$

as  $j \rightarrow \infty$ . By (3.4), we get

$$\begin{aligned} &\int_{-kT}^{kT} |\dot{p}_j(t) - \dot{p}_0(t)|^2 dt \\ &= \int_{-kT}^{kT} (U_x(t, p_j(t), p_j(t-T)) - U_x(t, p_0(t), p_0(t-T)), p_j(t) - p_0(t)) dt \\ &\quad + \int_{-kT}^{kT} (U_y(t, p_j(t+T), p_j(t)) - U_y(t, p_0(t+T), p_0(t)), p_j(t) - p_0(t)) dt \\ &\quad + (I'_k(p_j) - I'_k(p_0))(p_j - p_0), \end{aligned}$$

which implies  $\int_{-kT}^{kT} |\dot{p}_j(t) - \dot{p}_0(t)|^2 dt \rightarrow 0$  as  $j \rightarrow \infty$ . Consequently,  $p_j \rightarrow p_0$  in  $E_k$  as  $j \rightarrow \infty$ . ■

*Proof of Theorem 3.1.* By Ekeland's Variational Principle (see Theorems 4.1 and 4.2 in [MW]) and Lemma 3.2, for each  $k \in \mathbb{N}$  there is  $q_k \in E_k$  such that  $I_k(q_k) = \inf_{q \in E_k} I_k(q)$  and  $I'_k(q_k) = 0$ . From (3.3) and (H6) it follows that  $I_k(q_k) \leq 0 = I_k(0)$ .

Set

$$\varrho = \frac{B + \sqrt{B^2 + 4A}}{2A}.$$

(3.5) implies that if  $\|q\|_{E_k} \geq \varrho$  then  $I_k(q) \geq 1$ . Hence  $\|q_k\|_{E_k} < \varrho$  for each  $k \in \mathbb{N}$ . By Theorem 1.2, we conclude that (3.1) has an almost homoclinic solution  $q_0 \in E$ .

To finish the proof of Theorem 3.1, we have to show that  $\dot{q}_0(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ . From Fact 2.4 it follows that for each  $t \in \mathbb{R}$ ,

$$|\dot{q}_0(t)|^2 \leq 2 \int_{t-1/2}^{t+1/2} (|\dot{q}_0(s)|^2 + |\ddot{q}_0(s)|^2) ds.$$

By (2.7), it suffices to prove that

$$\int_r^{r+1} |\ddot{q}_0(s)|^2 ds \rightarrow 0$$

as  $r \rightarrow \pm\infty$ . Since  $q_0$  satisfies (3.1), we get

$$\begin{aligned} \int_r^{r+1} |\ddot{q}_0(s)|^2 ds &\leq 2 \int_r^{r+1} |f(s)|^2 ds \\ &\quad + 2 \int_r^{r+1} |u(s, q_0(s), q_0(s-T), q_0(s+T))|^2 ds. \end{aligned}$$

From (H4) and (H5) we have  $u(t, 0, 0, 0) = 0$  for each  $t \in \mathbb{R}$ . From this and (H2) we deduce that

$$\int_r^{r+1} |u(s, q_0(s), q_0(s-T), q_0(s+T))|^2 ds \rightarrow 0 \quad \text{as } r \rightarrow \pm\infty.$$

Finally, by (H3),

$$\int_r^{r+1} |f(s)|^2 ds \rightarrow 0 \quad \text{as } r \rightarrow \pm\infty. \quad \blacksquare$$

EXAMPLE 2. Consider now second order functional differential equations of mixed type

$$(3.6) \quad \ddot{q}(t) - V_q(t, q(t)) + u(t, q(t), q(t-T), q(t+T)) = f(t),$$



where  $V: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $u: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $f: \mathbb{R} \rightarrow \mathbb{R}^n$  satisfy assumptions (H1)–(H4), and moreover,

(H7) there exist constants  $b_1, b_2 > 0$  such that for all  $t \in \mathbb{R}$  and  $q \in \mathbb{R}^n$ ,

$$b_1|q|^2 \leq V(t, q) \leq b_2|q|^2,$$

(H8) for all  $t \in \mathbb{R}$  and  $q \in \mathbb{R}^n$ ,

$$V(t, q) \leq (q, V_q(t, q)) \leq 2V(t, q),$$

(H9) there is  $\mu > 2$  such that for all  $t \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ ,

$$0 < \mu U(t, x, y) \leq (U_x(t, x, y), x) + (U_y(t, x, y), y),$$

(H10) the gradient of  $U$  with respect to  $(x, y)$  is equal to  $o(\sqrt{|x|^2 + |y|^2})$  as  $|x|^2 + |y|^2 \rightarrow 0$  uniformly in  $t$ .

Let us remark that (H9)–(H10) imply that  $U(t, x, y) = o(|x|^2 + |y|^2)$  as  $|x|^2 + |y|^2 \rightarrow 0$  uniformly in  $t$ .

Set

$$\bar{U} = \sup\{U(t, x, y) : |x|^2 + |y|^2 = 1, t \in [0, T]\}, \quad \bar{b}_1 = \min\{2b_1, 1\}$$

and suppose that

(H11)  $\bar{b}_1 > 4\bar{U}$  and  $\|f\|_{L^2(\mathbb{R}, \mathbb{R}^n)} < \frac{\sqrt{2}}{4C}(\bar{b}_1 - 4\bar{U})$ , where  $C$  is the positive constant given by (2.1).

**THEOREM 3.3.** *If (H1)–(H4) and (H7)–(H11) are satisfied, then (3.6) has an almost homoclinic solution  $q_0: \mathbb{R} \rightarrow \mathbb{R}^n$ . Moreover,  $\dot{q}_0(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ .*

We will study an approximative sequence of functional differential equations for (3.6) given by

$$(3.7) \quad \ddot{q}(t) - V_q(t, q(t)) + u(t, q(t), q(t - T), q(t + T)) = f_k(t),$$

where  $f_k: \mathbb{R} \rightarrow \mathbb{R}^n$  is a  $2kT$ -periodic extension of  $f|_{[-kT, kT]}$  onto  $\mathbb{R}$ . Now,  $I_k: E_k \rightarrow \mathbb{R}$  is defined by

$$(3.8) \quad I_k(q) = \int_{-kT}^{kT} \left( \frac{1}{2} |\dot{q}(t)|^2 + V(t, q(t)) - U(t, q(t), q(t - T)) \right) dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt$$



and

$$\begin{aligned}
 (3.9) \quad I'_k(q)p &= \int_{-kT}^{kT} [(\dot{q}(t), \dot{p}(t)) + (V_q(t, q(t)), p(t))] dt \\
 &\quad - \int_{-kT}^{kT} (U_x(t, q(t), q(t-T)), p(t)) dt \\
 &\quad - \int_{-kT}^{kT} (U_y(t, q(t+T), q(t)), p(t)) dt \\
 &\quad + \int_{-kT}^{kT} (f_k(t), p(t)) dt.
 \end{aligned}$$

LEMMA 3.4. *Let assumptions (H1)–(H4) and (H7)–(H11) hold. Then  $I_k$  defined by (3.8) satisfies the Palais–Smale condition.*

*Proof.* Let  $\{p_j\}_{j \in \mathbb{N}} \subset E_k$  be a sequence such that  $\{I_k(p_j)\}_{j \in \mathbb{N}}$  is bounded and  $I'_k(p_j) \rightarrow 0$  as  $j \rightarrow \infty$ . We have to show that  $\{p_j\}_{j \in \mathbb{N}}$  has a convergent subsequence.

There is  $C_k > 0$  such that  $|I_k(p_j)| \leq C_k$  for each  $j \in \mathbb{N}$ . Moreover, there is  $j_0 \in \mathbb{N}$  such that  $\|I'_k(p_j)\|_{E_k^*} < \mu$  for all  $j > j_0$ . Using (3.8), (3.9) and (H7)–(H9), we immediately check that

$$2I_k(p_j) - \frac{2}{\mu} I'_k(p_j)p_j \geq \left(1 - \frac{2}{\mu}\right) \bar{b}_1 \|p_j\|_{E_k}^2 + \left(2 - \frac{2}{\mu}\right) \int_{-kT}^{kT} (f_k(t), p_j(t)) dt.$$

Hence, for all  $j > j_0$ ,

$$\left(1 - \frac{2}{\mu}\right) \bar{b}_1 \|p_j\|_{E_k}^2 - \left(2 - \frac{2}{\mu}\right) \|f\|_{L^2(\mathbb{R}, \mathbb{R}^n)} \|p_j\|_{E_k} - 2\|p_j\|_{E_k} - 2C_k \leq 0.$$

Since  $\mu > 2$ , we conclude that  $\{p_j\}_{j \in \mathbb{N}}$  is bounded in  $E_k$ .

Arguments similar to that in the proof of Lemma 3.2 show that  $\{p_j\}_{j \in \mathbb{N}}$  has a convergent subsequence. ■

FACT 3.5. *For each  $t \in [0, T]$  and  $x, y \in \mathbb{R}^n$ , if  $|x|^2 + |y|^2 \leq 1$  then*

$$(3.10) \quad U(t, x, y) \leq \bar{U} \cdot (\sqrt{|x|^2 + |y|^2})^\mu.$$

To prove this fact, it is sufficient to notice that a real-valued function

$$(0, \infty) \ni \zeta \mapsto U(t, \zeta^{-1}x, \zeta^{-1}y)\zeta^\mu$$

is non-increasing for  $t \in [0, T]$  and  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$ . This is a trivial consequence of (H9).



*Proof of Theorem 3.3.* Fix  $k \in \mathbb{N}$ . Set  $\varrho = \sqrt{2}/(2C)$ , where  $C$  is given by (2.1). Assume  $0 < \|q\|_{E_k} \leq \varrho$ . From (2.1) it follows that  $0 < \|q\|_{L^\infty_{2kT}} \leq \sqrt{2}/2$ . By (3.10), we have

$$(3.11) \quad \int_{-kT}^{kT} U(t, q(t), q(t-T)) dt \leq \bar{U} \int_{-kT}^{kT} (|q(t)|^2 + |q(t-T)|^2)^{\mu/2} dt \\ \leq 2\bar{U} \int_{-kT}^{kT} |q(t)|^2 dt \leq 2\bar{U} \|q\|_{E_k}^2.$$

Using (3.8), (H7) and (3.11), we get

$$(3.12) \quad I_k(q) \geq \frac{1}{2} \bar{b}_1 \|q\|_{E_k}^2 - 2\bar{U} \|q\|_{E_k}^2 - \|f_k\|_{L^2_{2kT}} \|q\|_{E_k} \\ \geq \frac{\bar{b}_1 - 4\bar{U}}{2} \|q\|_{E_k}^2 - \|f\|_{L^2(\mathbb{R}, \mathbb{R}^n)} \|q\|_{E_k}.$$

Moreover, if  $\|q\|_{E_k} = \varrho$ , then

$$(3.13) \quad I_k(q) \geq \frac{\bar{b}_1 - 4\bar{U}}{4C^2} - \frac{\sqrt{2}}{2C} \|f\|_{L^2(\mathbb{R}, \mathbb{R}^n)} \equiv \alpha > 0 = I_k(0),$$

by (H11). Note that both  $\varrho$  and  $\alpha$  are independent of  $k$ . Applying Ekeland's Variational Principle and Lemma 3.4, we obtain a sequence  $\{q_k\}_{k \in \mathbb{N}}$  such that  $I_k(q_k) = \inf_{\|q\|_{E_k} \leq \varrho} I_k(q)$  and  $I'_k(q_k) = 0$ . By Theorem 1.2, we conclude that (3.6) has an almost homoclinic solution  $q_0 \in E$ . Analysis similar to that in the proof of Theorem 3.1 shows that  $\dot{q}_0(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ . ■

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