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Efficiency of acoustic heating produced in the thermoviscous flow of a fluid with relaxation

Research Article

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Abstract: Instantaneous acoustic heating of a fluid with thermodynamic relaxation is the subject of investigation. Among others, viscoelastic biological media described by the Maxwell model of the viscous stress tensor, belong to this type of fluid. The governing equation of acoustic heating is derived by means of the special linear combination of conservation equations in differential form, allowing the reduction of all acoustic terms in the linear part of the final equation, but preserving terms belonging to the thermal mode responsible for heating. The procedure of decomposition is valid for weakly nonlinear flows, resulting in the nonlinear terms responsible for the modes interaction. Nonlinear acoustic terms form a source of acoustic heating in the case of dominative sound, which reflects the thermoviscous and dispersive properties of a fluid. The method of deriving the governing equations does not need averaging over the sound period, and the final governing dynamic equation of the thermal mode is instantaneous. Some examples of acoustic heating are illustrated and discussed, conclusions about efficiency of heating caused by different sound impulses are made.

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Keywords: acoustic heating • weak dispersion • relaxation

1. Introduction

It is well-known, that standard attenuation of fluids leads to a linear dissipation of sound. The acoustic heating is an increase of the ambient fluid temperature caused by *nonlinear* losses in acoustic energy. This increase in temperature is not an acoustic quantity but a value referred to as the entropy, or thermal mode. The increase in the ambient temperature should be distinguished from the excess temperature associated with the sound wave. The latter of which is a wave quantity, damped during sound propagation in a fluid with standard attenuation. The role of periodic sound in the origin of acoustic heating in standard thermoviscous fluid flows is well-studied theoretically and experimentally [1–3]. Interest in acoustic heating has grown over the last few years in connection with biomedical applications. Such applications require accurate estimation of heating during medical therapy, which applies sound of different kinds including impulses [3, 4].

A general description of fluid dynamics, covering a wide frequency range, needs to recognize that equilibrium and

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non-equilibrium thermodynamic deviations involve relaxation. Examples are vibrational relaxation of diatomic molecules (as in air at audible frequencies) and chemical relaxation in seawater below 500 kHz. In air, vibrational relaxation is the dominant attenuation mechanism at audible frequencies. Relaxation in general is an inalienable part of physical reality, and one of its manifestations is attenuation of sound [5]. It plays a significant role in the dynamics of liquids, especially those that are biological. This study is devoted to nonlinear dissipation of sound energy in a fluid where relaxation processes take place. The mathematical technique has been worked out and applied previously by one of the authors to some problems of nonlinear flow. It allows the separatation of the equations governing sound, vorticity and entropy modes [6–8]. The method and results based on its application are described in Secs. 3, 4. Some illustrations of acoustic heating caused by stationary or impulse sound are discussed in Sec. 5.

2. Dynamic equations in a fluid with dispersive properties

The continuity, momentum and energy equations for a thermoviscous fluid flow without external forces read:

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0\\
&\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\frac{1}{\rho}\left(-\vec{\nabla}p+Div\,\mathbf{P}\right),\\
&\frac{\partial e}{\partial t}+(\vec{v}\cdot\vec{\nabla})e=\frac{1}{\rho}\left(-p(\vec{\nabla}\cdot\vec{v})+\chi\Delta T\right.\\
&\qquad\qquad\left.+\mathbf{P}:Grad\,\vec{v}\right).
\end{aligned}\tag{1}$$

Here, $\vec{v}$ denotes the velocity of the fluid, ρ, p are density and pressure, e, T mark the energy per unit mass and temperature, correspondingly, χ is the thermal conductivity, and x_i, t spacial coordinates and time. The operators Div and $Grad$ denote the tensor divergence and dyad gradient respectively. $\mathbf{P}$ is the tensor of viscous stress. The equation connecting the viscous stress tensor and particle displacements $u_i(\vec{r},t)$ in the medium at a given point in space and time, for viscous liquids fits the Maxwell model, in two equivalent forms (τ_R is the relaxation time) [5, 9]:

$$\begin{aligned}
&\frac{\partial\mathbf{P}_{i,k}}{\partial t}+\frac{1}{\tau_R}\mathbf{P}_{i,k}=\mu\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right),\\
&\mathbf{P}_{i,k}=\mu\int_{-\infty}^{t}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)e^{-(t-t')/\tau_R}dt'.
\end{aligned}\tag{2}$$

Two thermodynamic functions $e(p,\rho)$, $T(p,\rho)$ complete the system (1). Their excess quantities may be written as a series of excess internal energy $e'=e-e_0$ and temperature $T'=T-T_0$ in powers of excess pressure and density $p'=p-p_0$, $\rho'=\rho-\rho_0$ (ambient quantities are marked by the index 0):

$$\begin{aligned}
e'&=\frac{E_1}{\rho_0}p'+\frac{E_2p_0}{\rho_0^2}\rho'+\frac{E_3}{p_0\rho_0}p'^2+\frac{E_4p_0}{\rho_0^3}\rho'^2+\frac{E_5}{\rho_0^2}\rho'p',\\
T'&=\frac{\Theta_1}{\rho_0C_v}p'+\frac{\Theta_2p_0}{\rho_0^2C_v}\rho'+\frac{\Theta_3}{p_0\rho_0C_v}p'^2\\
&\qquad+\frac{\Theta_4p_0}{\rho_0^3C_v}\rho'^2+\frac{\Theta_5}{\rho_0^2C_v}\rho'p',
\end{aligned}\tag{3}$$

where $E_1,\ldots\Theta_5$ are dimensionless coefficients, and C_v marks the heat capacity per unit mass under constant volume. The series (3) allows the consideration of a wide variety of fluids in the general form. A discrepancy in the thermodynamic properties of fluids is manifested namely by the coefficients different for different fluids. The expressions for coefficients E_1 and E_2 are as follows:

$$E_1=\frac{\rho_0C_v\kappa}{\beta},\qquad E_2=-\frac{C_p\rho_0}{\beta p_0}+1,\tag{4}$$

where C_p denotes the heat capacity per unit mass under constant pressure, κ and β are the compressibility and thermal expansion, correspondingly:

$$\begin{aligned}
\kappa&=-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T=\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T,\\
\beta&=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p.
\end{aligned}\tag{5}$$

The common practice in nonlinear acoustics is to focus on the equations of the second order of acoustic Mach number $M=v_0/c_0$, where v_0 is the magnitude of a particles velocity, and $c_0=\sqrt{\frac{(1-E_2)p_0}{E_1\rho_0}}$ is an infinitely small signal velocity. The present study is further constrained by considering nonlinearities of the second order, so that in the series (3) only terms up to the second order are kept. A small variation in entropy is a total differential, that gives the link of the first coefficient in the series of excess temperature (3):

$$\Theta_2=\frac{C_v\rho_0T_0}{E_1p_0}-\frac{(1-E_2)\Theta_1}{E_1}.\tag{6}$$

The next small parameter, responsible for relaxation $m=\mu/(\rho_0c_0^2)$ and the dimensionless thermal conductivity: $\delta=\frac{\chi T_0\omega}{c_0^4E_1^2\rho_0}$ (ω is the characteristic frequency of sound), should be of the same order. We choose to treat attenuation due to thermal conductivity δ and M of comparable smallness. We shall consider weakly nonlinear flows discarding $O(M^3)$ terms in all expansions and confining terms to be considered to those including m^0 and m^1. The resulting model accounts for the combined effects of nonlinearity, attenuation and weak dispersion of one-dimensional sound and thermal modes.

3. Definition of modes in the planar flow of infinitely small amplitude

We consider the one-dimensional flow along axis Ox. Based on the linearized version of Eq. (1), the roots of the dispersion equation can be obtained. They determine three independent modes of infinitely small-signal disturbances in an unbounded fluid. In one dimension, there exist the acoustic (two branches), and thermal (or entropy) modes. In general, every perturbation of the field variables contains contributions from each of the three modes, for example, $\rho' = \rho'_{a,1} + \rho'_{a,2} + \rho'_e$. This allows the separation of the governing equations into linear parts using the specific properties of respective modes. The method developed in [6, 7] provides the possibility of consequent decoupling of the initial system. All formulae that follow, including links of modes and governing equations, are written in the leading order.

It is convenient to rearrange formulae into the dimensionless quantities in the following way:

$$
\begin{aligned}
p^* &= \frac{p'}{c_0^2\cdot\rho_0}, \qquad \rho^* = \frac{\rho'}{\rho_0}, \qquad v^* = \frac{v}{c_0},\\
x^* &= \frac{\omega x}{c_0}, \qquad t^* = \omega t, \qquad \tau^* = \omega\tau_R.
\end{aligned}
\tag{7}
$$

Starting from Eq. (8), the upper indexes (asterisks) denoting dimensionless quantities will be omitted throughout the text. In the dimensionless quantities, accounting for Eqs. (2, 3), Eqs. (1) take the form:

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} + \frac{\partial v}{\partial x} &= -v\frac{\partial\rho}{\partial x} - \rho\frac{\partial v}{\partial x},\\
\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - 2\widehat{A}\frac{\partial^2 v}{\partial x^2} &= -v\frac{\partial v}{\partial x} + \rho\frac{\partial p}{\partial x} - 2\rho\widehat{A}\frac{\partial^2 v}{\partial x^2},\\
\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} - \delta_1\frac{\partial^2 p}{\partial x^2} - \delta_2\frac{\partial^2 \rho}{\partial x^2} &= -v\frac{\partial p}{\partial x} + (D_1 p + D_2\rho)\frac{\partial v}{\partial x} + \frac{2}{E_1}\frac{\partial v}{\partial x}\widehat{A}\frac{\partial v}{\partial x}\\
&\quad + \delta_3\frac{\partial^2 p^2}{\partial x^2} + \delta_4\frac{\partial^2\rho^2}{\partial x^2} + \delta_5\frac{\partial^2(\rho p)}{\partial x^2}.
\end{aligned}
\tag{8}
$$

where $m = \mu/(\rho_0 c_0^2) = c_\infty^2/c_0^2 - 1$ is the dimensionless dispersion (c_∞ is a frozen speed of sound, of infinitely large frequency), and $\widehat{A}$ denotes the dimensionless operator acting on a scalar function $\phi(x,t)$: $\widehat{A}\phi = m\int_{-\infty}^{t}\phi e^{-(t-t')/\tau}dt'$. The terms of order M^2 form the right-hand side of the set (8). The dynamic equations in the rearranged form include the following dimensionless quantities:

$$
\begin{aligned}
\delta_1 &= \frac{\chi\Theta_1\omega}{\rho_0 c_0^2 C_v E_1}, & \delta_2 &= \frac{\chi\Theta_2\omega}{\rho_0 c_0^2 C_v(1-E_2)},\\
\delta_3 &= \frac{\Theta_3\chi\omega}{E_1\rho_0 c_0^2 C_v}\frac{1-E_2}{E_1}, & \delta_4 &= \frac{\Theta_4\chi\omega}{(1-E_2)\rho_0 c_0^2\lambda C_v}, \qquad \delta_5 = \frac{\Theta_5\chi\omega}{E_1\rho_0 c_0^2 C_v},\\
D_1 &= \frac{1}{E_1}\left(-1 + 2\frac{1-E_2}{E_1}E_3 + E_5\right), & D_2 &= \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right).
\end{aligned}
\tag{9}
$$

The sum of the first two coefficients, is the coefficient of linear attenuation due to thermal conductivity, $\delta = \delta_1 + \delta_2$. The linearized version of Eq. (8) describes a flow of infinitely small amplitude, when $M \to 0$:

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} + \frac{\partial v}{\partial x} &= 0,\\
\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} - 2\widehat{A}\frac{\partial^2 v}{\partial x^2} &= 0,\\
\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} - \delta_1\frac{\partial^2 p}{\partial x^2} - \delta_2\frac{\partial^2\rho}{\partial x^2} &= 0.
\end{aligned}
\tag{10}
$$

The linear hydrodynamic field is represented by acoustic modes, propagating in the positive and negative directions of axis Ox and the entropy mode. Every type of motion is determined by one of the roots of the dispersion relation of the linear flow, $\omega(k)$ (k is the wave number) [1, 2, 10] and fixes links of perturbations, which are independent of time [6, 7]. The dispersion relations for acoustic modes propagating in the positive direction of axis Ox (marked by index 1), the negative direction of axis Ox (marked by index 2), and entropy modes (marked by index 3) are as follows [8]:

$$\omega_{a,1}=k+\frac{m(k\tau)^2}{1+(\tau k)^2}+\frac{ik^2}{2}\delta+i\frac{mk^2\tau}{1+(\tau k)^2},\qquad \omega_{a,2}=-k-\frac{m(\tau k)^2}{1+(\tau k)^2}+\frac{ik^2}{2}\delta+i\frac{mk^2\tau}{1+(\tau k)^2},\qquad \omega_e=-ik^2\delta_2. \quad (11)$$

They uniquely determine links of excess density inside every mode, which are valid at any time:

$$\psi_{a,1}=\begin{pmatrix}\rho_{a,1}\\ v_{a,1}\\ p_{a,1}\end{pmatrix}=\begin{pmatrix}1\\ 1-\left(\frac{1}{2}\delta+\widehat{A}\right)\frac{\partial}{\partial x}\\ 1-\delta\frac{\partial}{\partial x}\end{pmatrix}\rho_{a,1},\qquad \psi_{a,2}=\begin{pmatrix}1\\ -1-\left(\frac{1}{2}\delta+\widehat{A}\right)\frac{\partial}{\partial x}\\ 1+\delta\frac{\partial}{\partial x}\end{pmatrix}\rho_{a,2},\qquad \psi_e=\begin{pmatrix}1\\ \delta_2\frac{\partial}{\partial x}\\ 0\end{pmatrix}\rho_e. \quad (12)$$

The equations for every type of motion may be extracted from the system (8) in accordance with specific links inside every mode. This may be formally accomplished by means of projecting of the equations into specific sub-spaces [6–8]. Every equation includes a first-order derivative with respect to time.

The linear dynamic equations are obviously independent. The equation describing the acoustic excess density in a wave, propagating in the positive direction of axis Ox, is:

$$\frac{\partial\rho_{a,1}}{\partial t}+\frac{\partial\rho_{a,1}}{\partial x}-\left(\widehat{A}+\frac{\delta}{2}\right)\frac{\partial^2\rho_{a,1}}{\partial x^2}=0. \quad (13)$$

The density perturbation for entropy motion satisfies the diffusion equation:

$$\frac{\partial\rho_e}{\partial t}+\delta_2\frac{\partial^2\rho_e}{\partial x^2}=0. \quad (14)$$

4. Dynamic equations in a weakly nonlinear flow

4.1. Weakly nonlinear dynamic equation for sound

The nonlinear terms in every conservation equation from the right-hand side of system (8) include, in general, inputs of every mode in the weakly nonlinear flow. We fix links determining every mode in the linear flow and consider every excess quantity as a sum of the specific excess quantities of every mode. The consequent decomposing of the governing equations for sound and thermal modes may still be achieved by means of linear projection, for details refer to [8]. In simple terms, projecting is the linear combination of equations in such a way as to keep the terms of the chosen mode in the linear part, and reduce all other terms. Keeping only the terms corresponding to the acoustic rightwards progressive wave in the nonlinear part, and expressing all acoustic quantities in terms of excess acoustic density by use of links ($\psi_{a,1}$ from Eqs. (12)), one can easily obtain the equation supplementing the well-know Burgers' equation (accounting for standard attenuation exclusively) by the terms responsible for dispersion:

$$\frac{\partial\rho_{a,1}}{\partial x}+\frac{\partial\rho_{a,1}}{\partial t}-\widehat{A}\frac{\partial^2\rho_{a,1}}{\partial x^2}-\frac{\delta}{2}\frac{\partial^2\rho_{a,1}}{\partial x^2}=-\frac{1-D_1-D_2}{2}\rho_{a,1}\frac{\partial}{\partial x}\rho_{a,1}. \quad (15)$$

The nonlinear term in the right-hand side of Eq. (15) may be considered as a result of the self-action of sound, which corrects the dynamic equations by nonlinear terms.

4.2. Interaction of thermal mode with dominant sound. Acoustic heating.

The important property of projection is not only to decompose the specific perturbations in the linear part of equations, but to distribute nonlinear terms correctly between different dynamic equations. In the context of acoustic heating, the magnitudes of excess density specific to the entropy mode is small compared to that of the sound. It may be easily verified, that the modes with links (12) satisfy, in the leading order terms, (up to order δ^2) the equality below:

$$\begin{pmatrix} 1 & -\delta\frac{\partial}{\partial x} & -1 \end{pmatrix} \begin{pmatrix} \rho_{a,1}+\rho_{a,2}+\rho_e \\ v_{a,1}+v_{a,2}+v_e \\ p_{a,1}+p_{a,2}+p_e \end{pmatrix} = \rho_e, \tag{16}$$

which suggests a way of combining of the set of Eqs. (8). The links inside dominant sound should be completed by nonlinear quadratic terms making sound isentropic in the leading order. These corrections are similar to those specific to the Riemann wave in the ideal gas [11]:

$$v_{a,1} = \rho_{a,1} - \widehat{A}\frac{\partial}{\partial x}\rho_{a,1} - \frac{\delta}{2}\frac{\partial}{\partial x}\rho_{a,1} - \frac{1}{4}(3+D_1+D_2)\rho_{a,1}^2,$$
$$p_{a,1} = \rho_{a,1} - \delta\frac{\partial}{\partial x}\rho_{a,1} - \frac{1}{2}(1+D_1+D_2)\rho_{a,1}^2. \tag{17}$$

The nonlinear corrections of second and higher order terms depend on the equation of state, and in the case of an ideal gas they coincide with the well-known links in the Riemann wave with $D_1 = -\gamma, D_2 = 0$.

For simplicity, let sound be associated only with waves propagating in the positive direction of axis Ox: $p_a = p_{a,1}$, $\rho_a = \rho_{a,1}$, $v_a = v_{a,1}$. The linear combination of the left-hand sides of the equations of (8) in accordance to (16) results in:

$$\frac{\partial}{\partial t}(\rho - p - \delta\frac{\partial}{\partial x}v) - \delta\frac{\partial^2}{\partial x^2}p + \delta_1\frac{\partial^2 p}{\partial x^2} + \delta_2\frac{\partial^2 \rho}{\partial x^2}$$
$$\approx \frac{\partial}{\partial t}\rho_e + \delta_2\frac{\partial^2\rho_e}{\partial x^2} - \frac{\delta}{4}(3+D_1+D_2)\frac{\partial^2}{\partial x^2}\rho_a^2 + (1+D_1+D_2)\left(-\rho_a\frac{\partial\rho_a}{\partial x} + \frac{\delta_2}{2}\frac{\partial^2}{\partial x^2}\rho_a^2\right). \tag{18}$$

In the simple evaluations above, the corrected links (17) are used, as well as the dynamic Eq. (13) to exclude the partial time derivative in the nonlinear terms. In the context of acoustic heating, the sound is dominant, so that only acoustic quadratic terms are kept. Combining in a similar way the right-hand sides of the equations from the set (8), and comparing the result with Eq. (18), one obtains the dynamic equation for acoustic heating:

$$\frac{\partial}{\partial t}\rho_e + \delta_2\frac{\partial^2\rho_e}{\partial x^2} - \frac{\delta}{4}(3+D_1+D_2)\frac{\partial^2\rho_a^2}{\partial x^2} + (1+D_1+D_2)\left(-\rho_a\frac{\partial\rho_a}{\partial x} + \frac{\delta_2}{2}\frac{\partial^2\rho_a^2}{\partial x^2}\right) =$$
$$-(1+D_1+D_2)(\rho_a\frac{\partial\rho_a}{\partial x} - \frac{2}{E_1}\frac{\partial}{\partial x}\rho_a\widehat{A}\frac{\partial}{\partial x}\rho_a + \delta\left(D_1\left(\frac{\partial\rho_a}{\partial x}\right)^2 - \rho_a\frac{\partial^2\rho_a}{\partial x^2}\right) - (\delta_3+\delta_4+\delta_5)\left(\frac{\partial^2\rho_a^2}{\partial x^2}\right), \tag{19}$$

which becomes simpler after ordering:

$$\frac{\partial}{\partial t}\rho_e + \delta_2\frac{\partial^2\rho_e}{\partial x^2} = -\frac{2}{E_1}\frac{\partial}{\partial x}\rho_a\widehat{A}\frac{\partial}{\partial x}\rho_a + \left((\frac{\delta}{2} - \delta_2)(1+D_1+D_2) - 2(\delta_3+\delta_4+\delta_5)\right)\rho_a\frac{\partial^2\rho_a}{\partial x^2}$$
$$+\left(\frac{\delta}{2}(3D_1+D_2+3) - \delta_2(1+D_1+D_2) - 2(\delta_3+\delta_4+\delta_5)\right)\left(\frac{\partial\rho_a}{\partial x}\right)^2. \tag{20}$$

It is remarkable, that the dynamic equation for acoustic heating is a result of the combining of the energy and continuity equations in the absence of thermal conduction. When there is thermal conduction it is the result of the combining of momentum, energy and continuity equations in accordance with Eq. (16). The acoustic terms of the leftwards propagating sound are also completely reduced in the linear part of final equation.

This section is restricted to the consideration of the acoustic field represented by rightwards propagating sound, thought it may be easily expanded to include leftwards propagating waves or any mixture of the two acoustic branches.

5. Numerical examples

The solution of Eq. (20) governing the decrease in the ambient density ρ_e, is a fairly complex problem considering that the excess acoustic density itself should satisfy Eq. (15), which itself is nonlinear and accounts for attenuation due to thermal conduction and dispersion. It should be noted that ρ_e is not an acoustic quantity. The equation governing its dynamics includes nonlinear acoustic terms proportional to dissipative coefficients. They play the role of a nonlinear source of heating and reflect the fact that the origin of the phenomenon are nonlinearity and viscosity, which follow from dispersion and thermal conductivity. The diffusion equation Eq. (20) is instantaneous, it describes the dynamics of the thermal mode at any time, and does not require the periodicity of the sound in the role of a source. Let us consider only terms originating from relaxation, both in the governing equations for sound and entropy excess density (Eqs. (15, 20)).
In terms of dimensional temperature T_e, and accounting for Eqs. (3), (20), the governing equation of acoustic heating becomes:

$$\frac{\partial \mathrm{T}_e}{\partial t}=\frac{\Theta_2 p_0}{\rho_0 C_v}\frac{\partial}{\partial t}\rho_e \\ =-\frac{2\Theta_2 p_0 m}{\rho_0 C_v E_1}\frac{\partial \rho_a}{\partial x}\int_{-\infty}^{t}\frac{\partial \rho_a}{\partial x}exp(-(t-t')/\tau)dt'. \quad (21)$$

Assuming, that

$$\Theta_2=\frac{\rho_0^2 C_v}{p_0}\left(\frac{\partial T}{\partial \rho}\right)_p=-\frac{\rho_0 C_v}{p_0\beta}, \quad (22)$$

Eq. (21) rearranges into

$$\frac{\partial \mathrm{T}_e}{\partial t}=\frac{2m}{\beta E_1}\frac{\partial \rho_a}{\partial x}\int_{-\infty}^{t}\frac{\partial \rho_a}{\partial x}exp(-(t-t')/\tau)dt' \\ =\frac{2m}{\rho_0 C_v \kappa}\frac{\partial \rho_a}{\partial x}\int_{-\infty}^{t}\frac{\partial \rho_a}{\partial x}exp(-(t-t')/\tau)dt'. \quad (23)$$

Eq. (23), along with Eq. (20), is the main result of this study.

5.1. Acoustic heating caused by stationary sound

The stationary solution of nonlinear Eq. (15) with $\delta=0$, has the form [13]

$$\rho_a(\eta)=M\tanh\left(\eta/2G\tau\right), \quad (24)$$

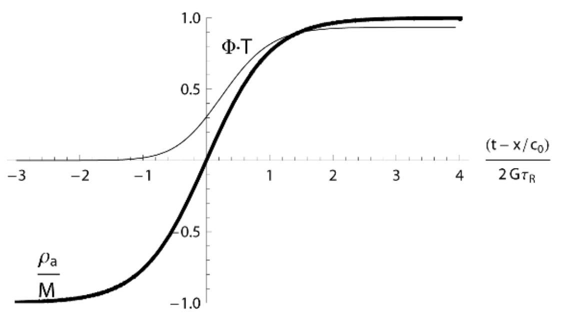

Figure 1. Stationary waveform (bold line) and associated increase in temperature (normal line).

where $\eta=t-x$ is the retarded time, and $G=\frac{2m}{(1-D_1-D_2)M}$ measures the ratio of relaxation effects to nonlinear effects. It is valid in the limit of weak nonlinearity, $G\gg 1$. The stationary waveform and dimensional temperature $\Phi\mathbf{T_e}$ ($\Phi=\frac{\rho_0 C_v k}{2M^2 m}=\frac{\beta E_1}{2M^2 m}$ is measured in $1/K^o$), are calculated with the help of *Mathematica* as functions of $(\eta/2G\tau)$, which equals $(t-x/c_0)/2G\tau_R$ in dimensional t, x, τ_R, are shown by Fig. 1.
The constant asymptotic value of temperature at infinitely large η, is the trace which sound gives up after its passing. It is positive due to the nonlinear transform of acoustic energy into that of the thermal mode.

5.2. Efficiency of acoustic heating caused by pulses

Let us consider an excess dimensionless acoustic density in a form of three traveling single waves, the first, second, and third, correspondingly:

$$\rho_a(\eta)=\sqrt{2}M\exp\left(-\eta^2\right), \\ \rho_a(\eta)=M\exp\left(-\eta^2/4\right), \\ \rho_a(\eta)=2\sqrt{2}M\eta\exp\left(-\eta^2\right). \quad (25)$$

They are solutions of the linear wave Eq. (13). The energy of all waveforms, proportional to $\int_{-\infty}^{\infty}\rho_a(\eta)^2 d\eta$, is equal for all three impulses. Eq. (23) may be numerically integrated. The results for different relaxation times are shown by Figs. 2b, 2c, 2d. The initial waveforms are shown in Fig. 2a. Fig. 2 shows dimensional time and temperature, ω represents the characteristic inverse duration of the initial waveform. Despite the fact that all waveforms from Eq. (25) have equal energy, the third waveform causes the most effective heating, producing the largest excess temperature after the pulse amplitude decreases to zero. The difference obviously is much greater in the vicinity of

$\omega\tau_R = 1$, where the attenuation of sound itself achieves a maximum.
The next series of impulses compared are the following (the first, second and third, correspondingly):

$$\begin{aligned}\rho_a(\eta) =& 1.3511M\exp(-n_1\cdot\eta^2)\sin(\eta),\\ \rho_a(\eta) =& M\exp(-n_2\cdot\eta^2)\sin(\eta),\\ \rho_a(\eta) =& 0.6815M\exp(-n_3\cdot\eta^2)\sin(\eta),\end{aligned} \qquad (26)$$

where $n_1 = 0.01$, $n_2 = 0.003$, $n_3 = 0.0005$ to keep the energy of all waveforms equal. The initial waveforms are shown by Figs. 3a, 3c, 3e, the dimensionless temperature $\Phi\mathbf{T_e}$ for different relaxation times are shown by Figs. 3b, 3d, 3f. The acoustic heating increases with decreasing of values n. The concrete quantities of excess temperature depend on the Mach number M, the dispersion parameter m, the characteristic inverse duration of an impulse ω and thermodynamic properties of a fluid. For an ideal gas, $\beta E_1 = 1/(T_0(\gamma-1))$, but for liquids (with the exception of the metallic ones) it is a small quantity, much smaller than that of gases. The coefficient $\Phi\cdot M^2 = \frac{\rho_0 C_v k}{2m}$ depends exclusively on the molecular properties of a fluid. The viscoelastic fluid glycerin, which can be described by the Maxwell stress tensor $\mathbf{P}$, has a $\Phi\cdot M^2$ of $22207\frac{1}{K^o}$. The authors have computed it based on experimental results [14–16]. The acoustic Mach number is typically in the domain between 10^{-3} and 10^{-2}. A single pulse is hardly expected to produce a large increase in temperature. Pulse polarity is not of importance because of the quadratic form of the acoustic source. Its curvature however, plays a significant role.

6. Conclusions

The equation governing acoustic heating, Eq. (20) is the result of the decomposition of the weakly nonlinear equations for acoustic and non-acoustic motion. The method may be applied to a wide variety of flows with different mechanisms of dissipation and dispersion. It leads to instantaneous equations and does not need temporal averaging of the conservative equations with respect to the period of sound. This distinguishes it from the traditional decomposition of equations for acoustic and non-acoustic motion based on averaging of conservation equations over the period of sound [1, 2]. The main result of this study, besides Eq. (20), is Eq. (23) describing the excess temperature of the entropy mode of the relaxing fluid.
The acoustic heating grows with increase of acoustic Mach number M and dispersive parameter m. It also increases with increasing $\omega\tau_R$, which measures the ratio of the characteristic duration of sound and the relaxation time.

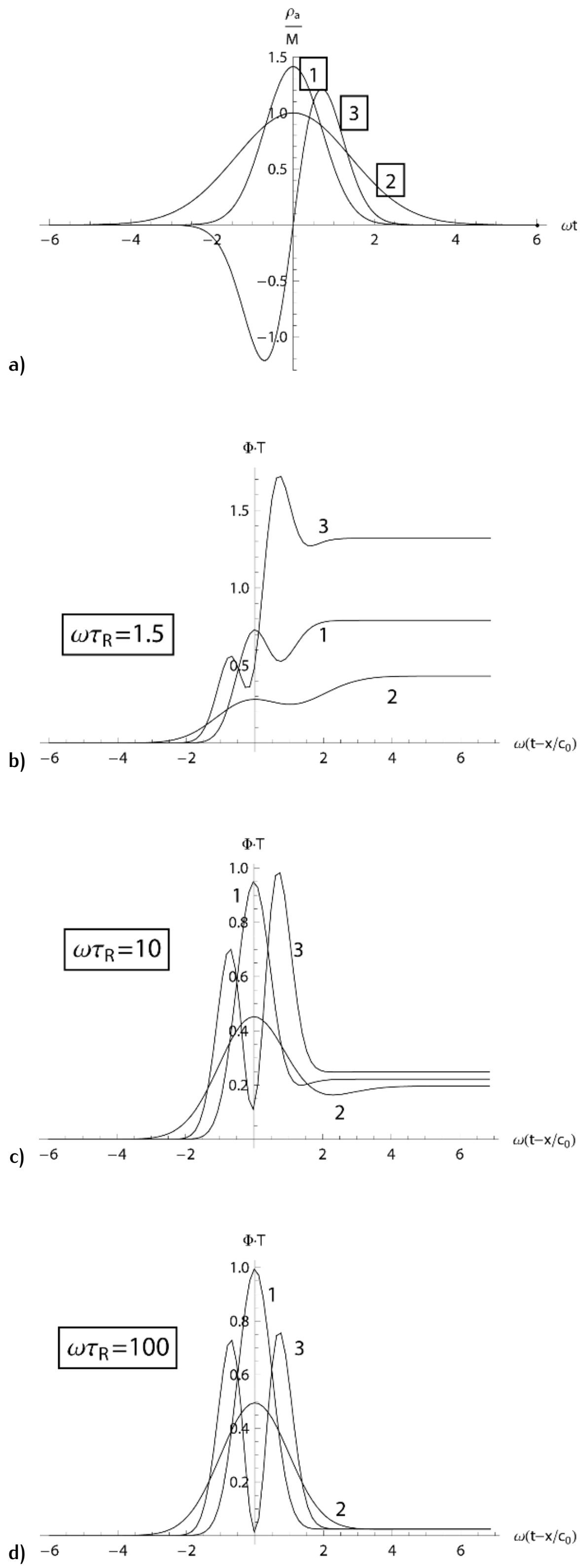

Figure 2. Initial waveforms (2a) (Eqs. (25)) and excess temperatures caused by them for different $\tau = \omega\tau_R$ (2b, 2c, 2d).

Smaller $\omega\tau_R$ results in more efficient heating. In view of the mathematical difficulties, the effects of shear viscosity and thermal conductivity are not considered in the numerical examples. The inclusion of shear viscosity and thermal conductivity would result in the larger attenuation of the sound itself, and the thermal conductivity would further result in the attenuation of the entropy mode.
These general peculiarities may be concluded a priori. The question about efficiency of heating caused by different impulses of equal energy, is of importance in many medical and technical applications of ultrasound. Investigations have to be based on instantaneous equations governing acoustic heating, Eq. (23). Preliminary numerical evaluations reveal the comparative efficiency of heating caused by some types of sound. They may be repeated for any other waveform as the origin of acoustic heating.

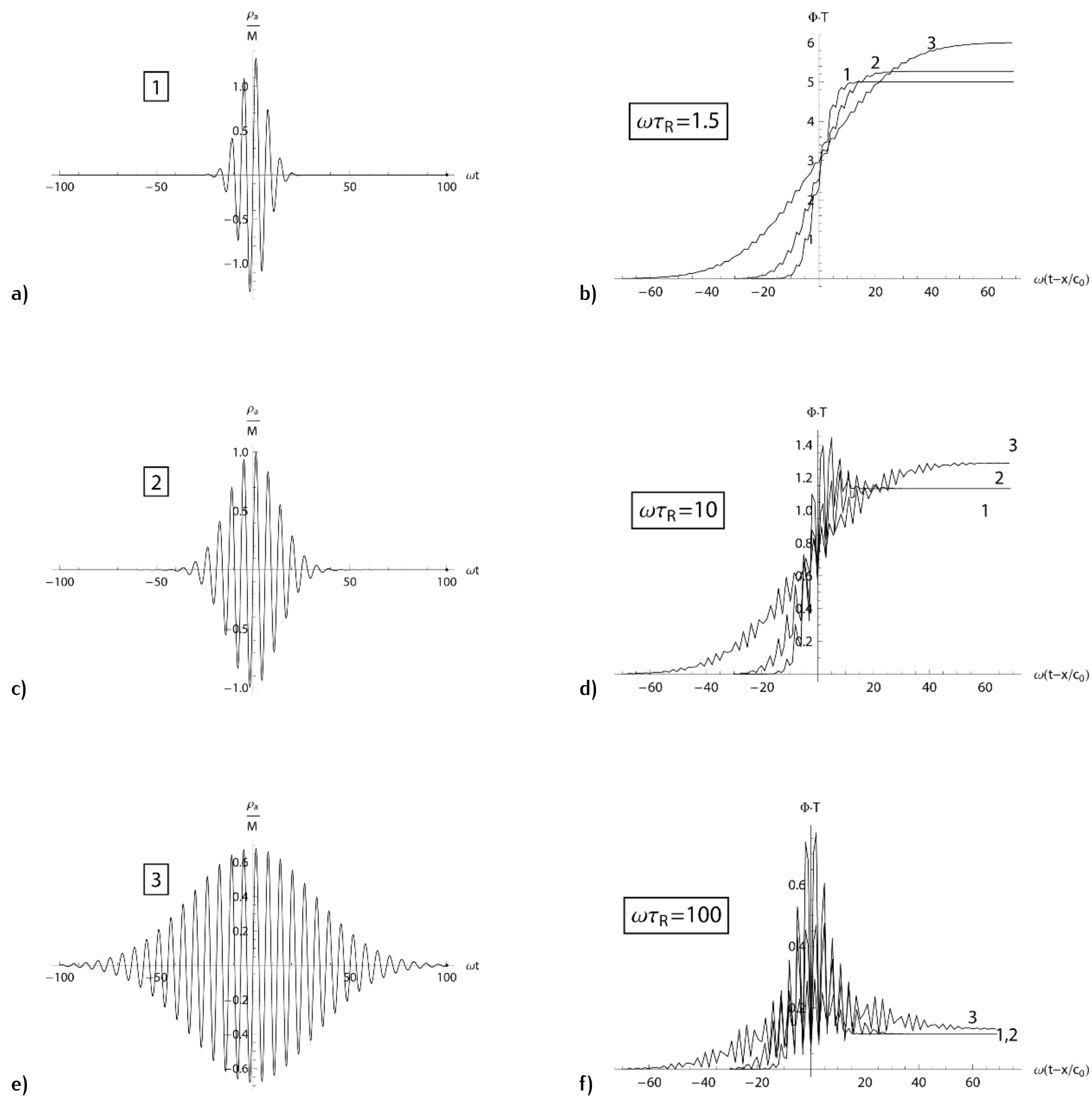

Figure 3. Initial waveforms (3a, 3c, 3e) (Eqs. (26)) and excess temperatures caused by them for different $\tau = \omega\tau_R$: $\omega\tau_R = 1.5$ (3b), $\omega\tau_R = 10$ (3d), $\omega\tau_R = 100$ (3f).

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