Are quantum correlations symmetric ?

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To our knowledge, all known bipartite entanglement measures are symmetric under exchange of subsystems. We ask if an entanglement measure that is not symmetric can exist. A related question is if there is a state that cannot be swapped by means of LOCC. We show, that in general one cannot swap states by LOCC. This allows to construct nonsymmetric measure of entanglement, and a parameter that reports asymmetry of entanglement contents of quantum state. We propose asymptotic measure of asymmetry of entanglement, and show that states for which it is nonzero, contain necessarily bound entanglement.

To our knowledge all known bipartite entanglement measures (EM) are symmetric under exchange of subsystems. To see it, it is enough to to check whether objects entering definitions of a given measure are symmetric. For example, operational EMs like distillable entanglement E_D [1], and distillable key K_D [2] are symmetric, because the sets of target objects (maximally entangled states, private states) are symmetric, and the tools for distillation are the same for Alice and for Bob. The so called distance EM's [3] are symmetric, because the set of separable states is symmetric, and distance is invariant under unitaries, (which is even more general than symmetry under exchange of subsystems). As for the convex roof measures [4, 5], they are defined by the measures of entanglement on pure states. However *any* EM on pure state must be symmetric, as it is function of eigenvalues of subsystem, which by Schmidt decomposition are the same for both subsystems.

There are quantities related to entanglement, such as one-way distillable entanglement, that are manifestly nonsymmetric [6]. However they are not true EM's in the sense that they can be increased by local operations and classical communication.

Therefore we would like to rise the question: *Can entanglement be asymmetric?* In other words, can a state ρ_{AB} have more entanglement of some type than the state $\sigma_{AB} = V\rho_{AB}V$ where V is unitary operation that swaps subsystems?

A closely related question is: *Can we swap a given state by LOCC?* Indeed, if there exists a measure that is nonsymmetric, it would increase under swap on some state, hence one couldn't swap it by LOCC. On the other hand, one can see that existence on states that are not swapable by LOCC leads to a nonsymmertic EM. At first, the question may seem trivial: it is known that swap is highly nonlocal gate, which if applied to halves of singlets (produced locally), can create 2 bits of entanglement. However, the other halves of singlets stay untouched: we act with swap on two halves and with *identity* on two other halves. Swap *itself* cannot create entanglement out of separable states (cf. [7, 8]), because $V\psi\otimes\phi = \phi\otimes\psi$.

In this paper we will show that indeed, entanglement can feel where is left and where right-hand-side of the system. More specifically, we will first show that in general, one cannot swap states by LOCC. We will then exhibit an asymmetric measure of entanglement. Interestingly, the impossibility of swapping we will proven by use of a usual symmetric EM.

We then consider asymptotic setup, and conjecture, that one cannot swap even in the regime of many copies, and allowing for (asymptotically vanishing) error. We define two asymptotic symmetry/asymmetry measures and show that they coincided. Moreover we exhibit connection between asymmetry and bound entanglement: if there is nonzero asymmetry of entanglement of a given state, then the state necessarily contains bound entanglement. We give also quantitative relations between asymmetry and bound entanglement contents.

Existence of non-swapable states.

We will now prove that for a large class of states one can swap them by LOCC only when one can swap them by local unitaries. The class consists of all states that have full Schmidt rank [9]. Equivalently, such states can be characterized by a measure of entanglement introduced by Gour [10]. The measure for pure state is given by

$$G(\psi_{AB}) = d(\det\rho_A)^{\frac{1}{d}} \tag{1}$$

where ρ_A is reduced density matrix of ψ. For mixed states G is given by

$$G(\rho) = \inf\sum_i p_i G(\psi_i) \tag{2}$$

where infimum is taken over decompositions $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$. (This is standard convex roof procedure [4, 5].) Note that $G(\psi)$ is nonzero if and only if ψ has maximal Schmidt rank. It follows that our class of mixed states is characterized by $G(\rho) > 0$. Thus, we will prove that if $G > 0$, then swapping by LOCC means swapping by product unitary.

In particular, it follows that if state with $G > 0$ has different entropies of subsystems, it cannot be swapped by LOCC, since clearly local unitaries cannot change local entropy. Moreover, for two-qubit states, the condition $G > 0$ is equivalent to entanglement so that we obtain that any two qubit state is LOCC swapable iff it is swapable by $U_A \otimes U_B$.

Our main result is contained in the following theorem

Theorem 1 *Consider state ρ acting on $C^d \otimes C^d$, for which $G > 0$ (equivalently, with Schmidt rank equal to d). Then, if such state can be swapped by LOCC, then it can be also swapped by some product unitary operation $U_A \otimes U_B$.*

To prove this theorem we need two lemmas.

Lemma 1 *For any state ρ on $C^d \otimes C^d$, and trace preserving separable operation $\Lambda(\cdot) = \sum_i A_i \otimes B_i(\cdot)A_i^\dagger \otimes B_i^\dagger$ there holds*

$$\sum_i p_i G(\sigma_i) \leq \sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} G(\rho) \tag{3}$$

where $\sigma_i = \frac{1}{p_i} A_i \otimes B_i(\rho) A_i^\dagger \otimes B_i^\dagger$, $p_i = \mathrm{Tr}(A_i \otimes B_i(\rho) A_i^\dagger \otimes B_i^\dagger)$.

Remark. Similar result (with equality) was obtained for concurrence in [11]. In the proof we will use, in particular, techniques from the proof of monotonicity of convex roof EM's under LOCC [4, 12].

Proof. Consider optimal decomposition $\rho = \sum_j q_j |\psi_j\rangle\langle\psi_j|$, so that $G(\rho) = \sum_j q_j G(\psi_j)$. One finds that

$$\sigma_i = \sum_j \frac{q_j p_i^{(j)}}{p_i} \left(\frac{1}{p_i^{(j)}} X_i |\psi_j\rangle\langle\psi_j| X_i^\dagger \right) \tag{4}$$

$$\equiv \sum_j r_j^{(i)} |\phi_j^{(i)}\rangle\langle\phi_j^{(i)}| \tag{5}$$

where we have denoted $X_i = A_i \otimes B_i$, $p_i^{(j)} = \mathrm{Tr}(X_i |\psi_j\rangle\langle\psi_j| X_i^\dagger)$. The coefficients $r_j^{(i)}$ are probabilities for fixed i and $\phi_j^{(i)}$ are normalized states. We then have

$$\sum_i p_i G(\sigma_i) = \sum_i p_i G(\sum_j r_j^{(i)} |\phi_j^{(i)}\rangle\langle\phi_j^{(i)}|) \leq$$
$$\leq \sum_{ij} p_i r_i^{(j)} G(\phi_j^{(i)}) = \sum_{ij} q_j G(X_i \psi_j) \tag{6}$$

where we have used convexity of G and the fact that $G(\alpha\rho) = \alpha G(\rho)$ for $\alpha \geq 0$. Now, as shown in [10] $G(A \otimes B\psi) = |\det A|^{\frac{1}{d}} |\det B|^{\frac{1}{d}}$. It follows that

$$\sum_i p_i G(\sigma_i) \leq \sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} = \sum_j q_j G(\psi_j)$$
$$\sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} G(\rho) \tag{7}$$

This ends the proof of the lemma. ∎

The second lemma we need is as follows

Lemma 2 *For operation Λ from lemma 1 we have $\sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} \leq 1$ with equality if and only if Λ is mixture of product unitary operations.*

Proof. Note that $|\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} = [det(X_i^\dagger X_i)]^{\frac{1}{d^2}}$ where $X_i = A_i \otimes B_i$. We then have

$$[\det(X_i^\dagger X_i)]^{\frac{1}{d^2}} \leq \frac{1}{d} \mathrm{Tr}(X_i^\dagger X_i) \tag{8}$$

as this is actually the inequality between geometric and arithmetic mean of eigenvalues of $X_i^\dagger X_i$ (cf. [13]). It then follows that equality can hold if and only if all eigenvalues are equal i.e. when $X_i^\dagger X_i$ is proportional to identity. Summing up we get

$$\sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} \leq \frac{1}{d^2} \mathrm{Tr} \sum_i (X_i^\dagger X_i) = 1 \tag{9}$$

where used the fact that Λ is trace preserving, so that $\sum_i X_i^\dagger X_i = I$. Equality can hold only when it holds for all terms, which implies that $(A_i \otimes B_i)^\dagger (A_i \otimes B_i)$ is proportional to identity. Hence A_i and B_i are proportional to unitaries. Thus, Λ is mixture of product unitary operations. ∎

Proof of the theorem 1. We assume that $G(\rho) > 0$ and that we can swap ρ by LOCC, i.e. $\Lambda(\rho) = V\rho V$. We will now use notation from the lemmas. Thus we assume that $\sum_i p_i \sigma_i = V\rho V$. Using invariance of G under swap, convexity of G and lemma 1 we obtain

$$G(\rho) = G(V\rho V) = G(\sum_i p_i \sigma_i) \leq \sum_i p_i G(\sigma_i) \leq$$
$$\leq \sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} G(\rho) \tag{10}$$

Since $G(\rho) > 0$ we get $\sum_i |\det A_i|^{\frac{1}{d}} |\det B_i|^{\frac{1}{d}} \geq 1$. Thus in view of lemma 2 we obtain that Λ must be mixture of product unitaries:

$$\Lambda(\rho) = \sum_i p_i U_A^i \otimes U_B^i \rho U_A^{i\dagger} \otimes U_B^{i\dagger} \equiv \sum_i p_i \sigma_i \tag{11}$$

Then the states σ_i have the same von Neumann entropy S as ρ, so that

$$S(\sum_i p_i \sigma_i) = S(V\rho V) = S(\rho) = \sum_i p_i S(\sigma_i) \tag{12}$$

Now, from strict concavity of entropy we obtain that all σ_i's must be the same, so that $V\rho V = U_A^1 \otimes U_B^1 \rho U_A^{1\dagger} \otimes U_B^{1\dagger}$. Thus swap can be performed by local unitary operation. ∎

Examples. From the theorem it follows that all entangled two qubit states are swapable, if they are swapable by $U_A \otimes U_B$ Thus any state with subsystems of different spectra is not LOCC swapable, since local unitaries keep local spectra. Exemplary state is mixture of $|01\rangle$ and $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Let us see, whether the assumption that $G > 0$ is essential. For higher dimensions there are many states that have $G = 0$. One would be tempted to think that for any entangled state that is LOCC swapable, we can swap it by local unitaries. However, it is not true. Consider state on $C^2 \otimes C^4$ system: being a mixture of $\psi_+ = \frac{1}{\sqrt{2}}(||00\rangle + |11\rangle)$ and $\psi = \frac{1}{\sqrt{2}}(||02\rangle + |13\rangle)$. The subsystems have different spectra, so that we cannot swap it by local unitaries. However, the mixture can be reversibly transformed into e.g. ψ_+ by local unitary. Thus it can be swapped.

Asymmetric EM. We take any "distance" $\mathcal{D}$ which is continuous, satisfies $\mathcal{D}(\Lambda(\rho)|\Lambda(\sigma)) \leq \mathcal{D}(\rho,\sigma)$ and $\mathcal{D}(\rho,\sigma) = 0$ if and only if $\rho = \sigma$. We consider associated measure $E^{\mathcal{D}}(\rho) = \inf_{\sigma_{sep}} \mathcal{D}(\rho, \sigma_{sep})$ [3] where infimum is taken over all separable states. Consider then a fixed state σ that cannot be swapped by LOCC. Now, our measure is defined as

$$E_\sigma(\rho) = E^{\mathcal{D}}(\sigma) - \inf_{\Lambda} \mathcal{D}(\sigma, \Lambda(\rho)) \tag{13}$$

where infimum is taken over all LOCC operations Λ. Note that for separable states $E_\sigma = 0$, and that by definition it does not increase under LOCC. We have $E_\sigma(\sigma) = E^{\mathcal{D}}(\sigma)$ while $E_\sigma(V\sigma V) < E^{\mathcal{D}}(\sigma)$. To see it note that if we cannot swap a state exactly, then we also cannot swap it with arbitrary good accuracy according to distance satisfying the above conditions. This follows from compactness of set of separable operations. Thus the second term is nonzero.

Measure of asymmetry of entanglement. We can define a parameter that would report asymmetry of entanglement of a given state.

$$\mathcal{A}_E(\rho) = \inf_{\Lambda} \mathcal{D}(\Lambda(\rho), V\rho V) \tag{14}$$

where infimum is taken over all LOCC operations Λ. Clearly, it is nonzero if and only if a state cannot be swapped by LOCC.

Asymptotics. So far we have talked about exact transformations. It is interesting to ask if the effect survives limit of many copies, where we allow inaccuracies that vanish asymptotically. We have not been able to answer this question, however we think it is most likely, that even asymptotically, in general one cannot swap states by LOCC.

Under such assumption, we can consider a parameter, which will report *asymptotic symmetry* of entanglement.

To define this parameter we need the notion of optimal transition rate of given state ρ to other state σ denoted as $R(\rho \to \sigma)$ which is the maximal ratio $\frac{m}{n}$ of the transformation $\rho^{\otimes n} \to \sigma' \approx \sigma^{\otimes n}$ via some LOCC map [12].

Definition 1 *Let $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be an entangled state. Then swap-symmetry is defined for entangled states as follows:*

$$S_{swap}(\rho) = R(\rho \to V\rho V^\dagger). \tag{15}$$

which is the optimal rate of transition from ρ to $V\rho V$ by means of LOCC.

This quantity is clearly infinite for separable states. However for entangled states it is always finite

Lemma 3 *For entangled state ρ we have*

$$S_{swap}(\rho) \leq 1 \tag{16}$$

Proof. We apply relation between rates and asymptotically continuous entanglement monotones [12]. Consider two state σ and ρ, and an asymptotically continuous entanglement monotone E. Let us assume that $E^\infty(\sigma) > 0$. Then we have

$$R(\rho \to \sigma) \leq \frac{E^\infty(\rho)}{E^\infty(\sigma)} \tag{17}$$

Here we will take $\sigma = V\rho V$ and E to be entanglement of formation E_F. Regularization of E_F is entanglement cost: $E_F^\infty = E_c$ and it was shown in [14] that it is nonzero for any entangled state. Since $E_c(\rho) = E_c(V\rho V)$ we obtain that $R(\rho \to V\rho V) \leq 1$ which ends the proof. ∎

We can also design another quantity, which would also report how much asymmetric is entanglement of a given state. To this end let us consider round-trip-travel rate i.e. the optimal rate of transferring state ρ into itself via some other state σ (cf. [15]). It is formally defined as

$$R(\rho \rightleftharpoons \sigma) = R(\rho \to \sigma) R(\sigma \to \rho) \tag{18}$$

where we use convention $0 \cdot \infty = \infty \cdot 0 = 0$. We now define our second quantity:

Definition 2 *The following quantity*

$$S_{sym}(\rho) = \sup_{\sigma} R(\rho \rightleftharpoons \sigma) \tag{19}$$

where supremum is taken over all symmetric states σ we will call symmetry.

Again, using [14] we can get that for any entangled state $S_{sym} \leq 1$.

However surprisingly, it turns out that the two quantities are equal:

Proposition 1 *The quantities S_{sym} and S_{swap} are equal to each other*

$$S_{sym} = S_{swap} \tag{20}$$

Proof. To see that $S_{sym} \leq S_{swap}$ consider the protocol achieving S_{sym}

$$\rho^{\otimes n} \to \sigma^{\otimes m} \to \rho^{\otimes k} \tag{21}$$

where $\sigma = V\sigma V$. Since the protocol is optimal, we have $k/n \approx S_{sym}$. In the second stage (transforming σ into ρ let us exchange roles of Alice and Bob. Then, instead of $\rho^{\otimes k}$ we will obtain $(V\rho V)^{\otimes k}$. Thus the total protocol will simply swap the state with rate k/n. Thus we can swap at least with rate S_{sym} which proves $S_{swap} \geq S_{sym}$.

To prove converse, it is enough to find a symmetric state σ such that $R(\rho \rightleftharpoons \sigma)$ will be equal to S_{swap}. Clearly, instead of symmetric (i.e. swap invariant state) we can choose a state which can be made symmetric by local unitaries. We will take

$$\sigma = \rho \otimes V\rho V \tag{22}$$

It is easy to see that local swaps produce a symmetric state from σ. We will now express $R(\rho \rightleftharpoons \rho \otimes V\rho V)$ in terms of $S_{swap}(\rho)$. To this end consider the following transformation

$$\rho^{\otimes n} \otimes \rho^{\otimes m} \to (V\rho V)^{\otimes m} \otimes \rho^{\otimes m} = \sigma^{\otimes m} \tag{23}$$

where the rate $m/n \approx S_{swap}$ is possible by definition of S_{swap}. Then we consider transformation that returns to the state ρ:

$$\sigma^{\otimes m} = (V\rho V)^{\otimes m} \otimes \rho^{\otimes m} \to \rho^{\otimes k} \otimes \rho^{\otimes m} \tag{24}$$

where again by definition of S_{swap} the rate $k/m \approx S_{swap}$ is possible. Thus the overall round-trip-travel rate vis state σ satisfies

$$R(\rho \rightleftharpoons \sigma) \leq \frac{k+m}{n+m} \approx \frac{S_{swap}+1}{\frac{1}{S_{swap}}+1} = S_{swap} \tag{25}$$

Since S_{sym} is supremum of such rates, we obtain that $S_{sym} \geq S_{swap}$. This ends the proof. ∎

We thus obtain our asymptotic quantities measuring symmetry/asymmetry.

Definition 3 *The quantity $S_{sym} = S_{swap}$ we will call symmetry of entanglement, and will denote by $\mathcal{S}_E^{as}$. The quantity $\mathcal{A}_E^{as} = 1 - \mathcal{S}_E^{as}$ we will call asymmetry of entanglement.*

Thus, entanglement in a given state is not symmetric when $\mathcal{A}_E^{as} > 0$. We will now argue that states with nonsymmetric entanglement must possess bound entanglement, i.e. for such state distillable entanglement is strictly smaller than entanglement cost $E_D < E_c$. Thus asymptotic asymmetry brings irreversibility. The reason is obvious, reversibility in distillation-creation process means that we can go reversibly from ρ to ρ through maximally entangled state which is symmetric state. Thus $\mathcal{S}_E^{as} = 1$ in such case. We have

Theorem 2 *For entangled states, we have*

$$\frac{E_D}{E_c} \leq \mathcal{S}_E^{as} \leq 1 \tag{26}$$

Equivalently we have

$$\frac{E_b}{E_c} \geq \mathcal{A}_E^{as} \tag{27}$$

where $E_b = E_c - E_D$.

Proof. The optimal rate $R(\rho \rightleftharpoons \psi_+)$ where $\psi_+ = \frac{1}{2}(|00\rangle + |11\rangle)$ is given by

$$R(\rho \rightleftharpoons \psi_+) = \frac{E_D}{E_c} \tag{28}$$

Since maximally entangled state is symmetric, this is rate of a particular protocol of round-trip-travel from ρ to ρ via symmetric state. Thus it is no greater than $\mathcal{S}_E^{as}$ which is supremum of rates over such protocols. ∎

From this theorem it follows that $\mathcal{S}_E^{as}$ is nonzero for distillable states.

Concluding remarks.

In this paper we propose a measure of asymmetry of entanglement for a single copy of quantum state. This proposition is not unique. Other candidate can be the infimum of distance from the set of single copy LOCC swapable states. It appears that the lower bound on this measure in terms of G-concurrence can be found.

We also conjecture that entanglement can be asymmetric in assymptotic regime of many copies i.e. that there exist states with $\mathcal{S}_E^{as} < 1$. One could then ask if V can increase E_D of some distillable states i.e. if $E_D(\rho \otimes V\rho V) > E_D(\rho^{\otimes 2})$.

If however $\mathcal{S}_E^{as} = 1$ for all states one would have that certain nontrivial task can be achieved via LOCC. Moreover the nice correspondence between transposition and swap would hold. As we have mentioned, like $I \otimes T$ is not physical, the operation $I \otimes V$ can not be implemented by means of LOCC i.e. it is not physical with respect to this class of operations. Although transposition is not completely positive it can be performed on a *known* state, as it is positive. If then $\mathcal{S}_E^{as} = 1$ for all states i.e. all states would be swapable, then V like T could be performed on a *known* state (in this case via LOCC operations).

Note that still there are many states which have $\mathcal{S}_E^{as} = 1$ because they are swap invariant. It is then tempting to develop a scheme of symmetry of entanglement with respect to certain group G of unitary transformations (see in this context [16] and [17]). That is $G-$symmetry of a state would be maximal rate of distillation of states which are invariant under actions of G.

As a generalization of our approach one can consider the asymmetry of general quantum correlations by restricting class of allowed operations to so called *closed* LOCC operations [18]. In such case also certain separable states may exhibit asymmetry. Moreover in analogy to asymmetry of entanglement one can also quantify asymmetry of private (cryptographic) correlations.

Finally, we note that quite recently other interesting investigations of notion of exchange of subsystems and swap symmetry have been independently developed [19, 20, 21].

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