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Comment on the orthogonality of the Macdonald functions of imaginary order

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ABSTRACT

Recently, Yakubovich [Opuscula Math. 26 (2006) 161–172] and Passian et al. [J. Math. Anal. Appl. 360 (2009) 380–390] have presented alternative proofs of an orthogonality relation obeyed by the Macdonald functions of imaginary order. In this note, we show that the validity of that relation may be also proven in a simpler way by applying a technique occasionally used in mathematical physics to normalize scattering wave functions to the Dirac delta distribution.

1. Introduction

In recent papers, Yakubovich [1] and Passian et al. [2] proved the following orthogonality relation for the Macdonald functions of imaginary order:

$$\int_0^\infty \mathrm{d}x\, \frac{K_{\mathrm{i}\nu}(x)K_{\mathrm{i}\nu'}(x)}{x} = \frac{\pi^2}{2\nu\sinh(\pi\nu)}\delta(\nu-\nu') \quad (\nu,\nu'>0). \tag{1.1}$$

In the first of these works, a proof relied on advanced techniques of the theory of distributions. An approach adopted in the second paper was a two-step one. At first, two heuristic arguments making the relation (1.1) plausible were presented. One of these arguments was based on an integral relation between the Macdonald functions of imaginary order and Mehler's conical functions, for which a counterpart orthogonality relation had been known for a long time. The second argument given in support of the validity of Eq. (1.1) exploited the fact that the Laplace transform of $K_{\mathrm{i}\nu}(x)$ is a known elementary function. Subsequently, a sophisticated proof of the relation (1.1), different from the one in Ref. [1], was presented.

It is the purpose of this note to present still another proof of the relation (1.1). The approach we adopt here is known in mathematical physics, where it is occasionally used for normalization of scattering states to the Dirac delta distribution. In some sense, it is akin to the standard method used to prove weighted orthogonality relations for eigenfunctions of regular Sturm–Liouville problems (cf., e.g., Ref. [3, Section 7.1]). It has several advantages. First, it is *elementary* compared to the methods used in Refs. [1,2]. Second, it is *constructive*: one *derives* the orthogonality relation. Finally, it is *general* and may be used to obtain counterpart orthogonality relations not only for $K_{\mathrm{i}\nu}(x)$, but also for other special functions (e.g., Ref. [4]).

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2. A summary of relevant properties of the Macdonald functions of imaginary order

Before we proceed to the merit, in this short section we shall summarize these properties of the Macdonald functions of imaginary order which will be exploited later in Section 3. The formulas presented below have been excerpted from the collection of Magnus et al. [5].

The function $K_{\mathrm{i}\nu}(x)$ is a particular solution to the modified Bessel differential equation

$$x^2\frac{\mathrm{d}^2F(x)}{\mathrm{d}x^2}+x\frac{\mathrm{d}F(x)}{\mathrm{d}x}+\left(\nu^2-x^2\right)F(x)=0. \tag{2.1}$$

Other particular solutions to Eq. (2.1) are the modified Bessel functions of the first kind

$$I_{\pm\mathrm{i}\nu}(x)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+1\pm\mathrm{i}\nu)}\left(\frac{x}{2}\right)^{2k\pm\mathrm{i}\nu}. \tag{2.2}$$

The relationship between the three functions is

$$K_{\mathrm{i}\nu}(x)=\frac{\pi}{2\mathrm{i}}\frac{I_{-\mathrm{i}\nu}(x)-I_{\mathrm{i}\nu}(x)}{\sinh(\pi\nu)}, \tag{2.3}$$

from which it follows immediately that

$$K_{\mathrm{i}\nu}(x)=K_{-\mathrm{i}\nu}(x). \tag{2.4}$$

For large positive values of x, the function $K_{\mathrm{i}\nu}(x)$ has the asymptotic representation

$$K_{\mathrm{i}\nu}(x)\overset{x\to\infty}{\sim}\sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left[1+O\left(x^{-1}\right)\right], \tag{2.5}$$

while for $x\to+0$ from Eqs. (2.3) and (2.2) and with the aid of the known relationship

$$\left|\Gamma(\mathrm{i}\nu)\right|=\sqrt{\frac{\pi}{\nu\sinh(\pi\nu)}}\quad(\nu\in\mathbb{R}) \tag{2.6}$$

one deduces that

$$K_{\mathrm{i}\nu}(x)\overset{x\to+0}{\sim}\sqrt{\frac{\pi}{\nu\sinh(\pi\nu)}}\cos\left[-\nu\ln\frac{x}{2}+\arg\Gamma(\mathrm{i}\nu)\right]+O\left(x^2\sin\left[-\nu\ln\frac{x}{2}+\arg\Gamma(2+\mathrm{i}\nu)\right]\right)\quad(\nu\in\mathbb{R}). \tag{2.7}$$

3. Derivation of the orthogonality relation for the Macdonald functions of imaginary order

To derive the orthogonality relation for the Macdonald functions of imaginary order, we proceed as follows. If $K_{\mathrm{i}\nu}(x)$, with $\nu\in\mathbb{R}$, is substituted for $F(x)$ into Eq. (2.1), this results in the differential identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}K_{\mathrm{i}\nu}(x)}{\mathrm{d}x}\right)+\left(\frac{\nu^2}{x}-x\right)K_{\mathrm{i}\nu}(x)=0. \tag{3.1}$$

The counterpart identity for the function $K_{\mathrm{i}\nu'}(x)$, with $\nu'\in\mathbb{R}$, is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}K_{\mathrm{i}\nu'}(x)}{\mathrm{d}x}\right)+\left(\frac{\nu'^2}{x}-x\right)K_{\mathrm{i}\nu'}(x)=0. \tag{3.2}$$

Next, we premultiply the first of the above equations by $K_{\mathrm{i}\nu'}(x)$, the second one by $K_{\mathrm{i}\nu}(x)$, subtract and integrate the result over x from $x=\xi>0$ to $x=\infty$. After obvious rearrangements, this gives

$$\left(\nu^2-\nu'^2\right)\int_{\xi}^{\infty}\mathrm{d}x\,\frac{K_{\mathrm{i}\nu}(x)K_{\mathrm{i}\nu'}(x)}{x}=\int_{\xi}^{\infty}\mathrm{d}x\left[K_{\mathrm{i}\nu}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}K_{\mathrm{i}\nu'}(x)}{\mathrm{d}x}\right)-K_{\mathrm{i}\nu'}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}K_{\mathrm{i}\nu}(x)}{\mathrm{d}x}\right)\right]. \tag{3.3}$$

The integral on the right-hand side of Eq. (3.3) is easily evaluated by parts; one obtains

$$\left(\nu^2-\nu'^2\right)\int_{\xi}^{\infty}\mathrm{d}x\,\frac{K_{\mathrm{i}\nu}(x)K_{\mathrm{i}\nu'}(x)}{x}=\left[x\left(K_{\mathrm{i}\nu}(x)\frac{\mathrm{d}K_{\mathrm{i}\nu'}(x)}{\mathrm{d}x}-K_{\mathrm{i}\nu'}(x)\frac{\mathrm{d}K_{\mathrm{i}\nu}(x)}{\mathrm{d}x}\right)\right]_{x=\xi}^{\infty}. \tag{3.4}$$

By virtue of Eq. (2.5), the expression in the bracket on the right-hand side of the above relation vanishes in the upper limit. Hence, we obtain

$$\int_{\xi}^{\infty} \mathrm{d}x \, \frac{K_{\mathrm{i}\nu}(x) K_{\mathrm{i}\nu'}(x)}{x} = -\xi \, \frac{K_{\mathrm{i}\nu}(\xi) \frac{\mathrm{d}K_{\mathrm{i}\nu'}(\xi)}{\mathrm{d}\xi} - K_{\mathrm{i}\nu'}(\xi) \frac{\mathrm{d}K_{\mathrm{i}\nu}(\xi)}{\mathrm{d}\xi}}{\nu^2 - \nu'^2} \tag{3.5}$$

and consequently

$$\int_{0}^{\infty} \mathrm{d}x \, \frac{K_{\mathrm{i}\nu}(x) K_{\mathrm{i}\nu'}(x)}{x} = -\lim_{\xi \to +0} \xi \, \frac{K_{\mathrm{i}\nu}(\xi) \frac{\mathrm{d}K_{\mathrm{i}\nu'}(\xi)}{\mathrm{d}\xi} - K_{\mathrm{i}\nu'}(\xi) \frac{\mathrm{d}K_{\mathrm{i}\nu}(\xi)}{\mathrm{d}\xi}}{\nu^2 - \nu'^2}. \tag{3.6}$$

Using the asymptotic representation (2.7) and elementary trigonometric identities transforms Eq. (3.6) into

$$\begin{aligned} \int_{0}^{\infty} \mathrm{d}x \, \frac{K_{\mathrm{i}\nu}(x) K_{\mathrm{i}\nu'}(x)}{x} &= \frac{\pi}{2\sqrt{\nu\nu' \sinh(\pi\nu) \sinh(\pi\nu')}} \\ &\quad \times \lim_{\xi \to +0} \left\{ \frac{\sin[-(\nu - \nu') \ln \frac{\xi}{2} + \arg \Gamma(\mathrm{i}\nu) - \arg \Gamma(\mathrm{i}\nu')]}{\nu - \nu'} \right. \\ &\quad \left. + \frac{\sin[-(\nu + \nu') \ln \frac{\xi}{2} + \arg \Gamma(\mathrm{i}\nu) + \arg \Gamma(\mathrm{i}\nu')]}{\nu + \nu'} \right\}. \end{aligned} \tag{3.7}$$

To evaluate the limit on the right-hand side of Eq. (3.7), we observe that if $f(\eta)$ is a real analytic function of $\eta \in \mathbb{R}$, such that $f(0) = 0$ (which implies that $\lim_{\eta \to 0} f(\eta)/\eta$ is finite), then in the distributional sense it holds that

$$\lim_{a \to \infty} \frac{\sin[a\eta + f(\eta)]}{\pi\eta} = \frac{1}{2\pi} \lim_{a \to \infty} \int_{-a - f(\eta)/\eta}^{a + f(\eta)/\eta} \mathrm{d}\alpha \, \mathrm{e}^{\mathrm{i}\alpha\eta} = \frac{1}{2\pi} \lim_{a \to \infty} \int_{-a}^{a} \mathrm{d}\alpha \, \mathrm{e}^{\mathrm{i}\alpha\eta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \mathrm{e}^{\mathrm{i}\alpha\eta}. \tag{3.8}$$

In the expression at the extreme right of the above chain of equalities one immediately recognizes the well-known Fourier representation of the Dirac delta distribution $\delta(\eta)$ (cf., e.g., [6, Section 4.5]), so that, provided the function $f(\eta)$ satisfies the above constraints, one has

$$\lim_{a \to \infty} \frac{\sin[a\eta + f(\eta)]}{\pi\eta} = \delta(\eta). \tag{3.9}$$

As for $\xi \to +0$ it holds that $-\ln(\xi/2) \to \infty$, with the help of the above relationship Eq. (3.7) becomes

$$\int_{0}^{\infty} \mathrm{d}x \, \frac{K_{\mathrm{i}\nu}(x) K_{\mathrm{i}\nu'}(x)}{x} = \frac{\pi^2}{2\sqrt{\nu\nu' \sinh(\pi\nu) \sinh(\pi\nu')}} \left[\delta\left(\nu - \nu'\right) + \delta\left(\nu + \nu'\right)\right]. \tag{3.10}$$

Exploiting in Eq. (3.10) the following basic property of the delta distribution [6, Section 4.4]:

$$g\left(\eta'\right)\delta\left(\eta - \eta'\right) = g(\eta)\delta\left(\eta - \eta'\right), \tag{3.11}$$

one eventually arrives at the sought orthogonality relation

$$\int_{0}^{\infty} \mathrm{d}x \, \frac{K_{\mathrm{i}\nu}(x) K_{\mathrm{i}\nu'}(x)}{x} = \frac{\pi^2}{2\nu \sinh(\pi\nu)} \left[\delta\left(\nu - \nu'\right) + \delta\left(\nu + \nu'\right)\right]. \tag{3.12}$$

If we impose the constraint $\nu, \nu' > 0$, then $\nu + \nu' > 0$ and consequently in the distributional sense we have

$$\delta\left(\nu + \nu'\right) = 0 \quad \left(\nu, \nu' > 0\right). \tag{3.13}$$

It is then evident that under the above restriction the orthogonality relation (3.12) turns into the one in Eq. (1.1).

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