



## Note

## A note on total reinforcement in graphs

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## ABSTRACT

In this note we prove a conjecture and improve some results presented in a recent paper of Sridharan et al. [N. Sridharan, M.D. Elias, V.S.A. Subramanian, Total reinforcement number of a graph, *AKCE Int. J. Graphs Comb.* 4 (2) (2007) 197–202].

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## 1. Introduction

A *total dominating set*, abbreviated as TDS, of a graph  $G$  is a set  $S$  of vertices of  $G$  such that every vertex in  $G$  is adjacent to a vertex in  $S$ . Every graph without isolated vertices has a TDS, since  $V(G)$  is such a set. The *total domination number* of  $G$ , denoted by  $\gamma_t(G)$ , is the minimum cardinality of a TDS of  $G$ . A TDS of  $G$  of cardinality  $\gamma_t(G)$  is called a  $\gamma_t(G)$ -set. Total domination was introduced by Cockayne et al. [3] and is now well studied in graph theory. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [6,7]. A recent survey of total domination in graphs can be found in [8].

A *dominating set*  $S$  in a graph  $G$  is a set of vertices of  $G$  such that each vertex not in  $S$  is adjacent to a vertex of  $S$ . The *domination number* of  $G$ ,  $\gamma(G)$ , equals the minimum cardinality of a dominating set. Kok and Mynhardt [9] introduced the *reinforcement number*  $r(G)$  of a graph as the minimum number of edges that have to be added to the graph in order to decrease the domination number. Since the domination number of every graph  $G$  is at least 1, by convention Kok and Mynhardt defined  $r(G) = 0$  if  $\gamma(G) = 1$ .

This concept of the reinforcement number in a graph was further considered for several domination variants, including *fractional domination* and *independent domination*. See, for example, [2,4,5,11] and elsewhere. Sridharan et al. [10] introduced the concept of *total reinforcement* in graphs. The *total reinforcement number*,  $r_t(G)$ , of a graph with no isolated vertex is the minimum number of edges that need to be added to the graph in order to decrease the total domination number. Since the total domination number of every graph  $G$  with no isolated vertex is at least 2, by convention  $r_t(G) = 0$  if  $\gamma_t(G) = 2$ . Total reinforcement in trees was recently studied by Blair et al. in [1].

In this paper we prove a conjecture due to Sridharan, Elias, and Subramanian [10] on the total reinforcement number of a graph, as well as improving some results presented in [10].

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### 1.1. Notation

For notation and graph theory terminology not defined herein, we refer the reader to [6]. Let  $G = (V, E)$  be a graph with vertex set  $V$  of order  $n = |V|$  and edge set  $E$  of size  $m = |E|$ , and let  $v$  be a vertex in  $V$ . The *open neighborhood* of  $v$  is  $N_G(v) = \{u \in V \mid uv \in E\}$  and the *closed neighborhood* of  $v$  is  $N_G[v] = \{v\} \cup N_G(v)$ . If the graph  $G$  is clear from the context, we simply write  $N(v)$  and  $N[v]$  rather than  $N_G(v)$  and  $N_G[v]$ , respectively. For a set  $S \subseteq V$ , its *open neighborhood* is the set  $N(S) = \bigcup_{v \in S} N(v)$  and its *closed neighborhood* is the set  $N[S] = N(S) \cup S$ . The degree of a vertex  $v$  in  $G$  is  $d_G(v) = |N_G(v)|$ . The maximum degree among the vertices of  $G$  is denoted by  $\Delta(G)$ . A cycle on  $n$  vertices is denoted by  $C_n$ , while a path on  $n$  vertices is denoted by  $P_n$ .

If  $X$  and  $Y$  are subsets of vertices in a graph  $G$ , then the set  $X$  totally dominates  $Y$  in  $G$  if  $Y \subseteq N(X)$ . In particular, if  $Y = V$ , then  $X$  is a TDS in  $G$ . An *efficient total dominating set* in a graph  $G = (V, E)$  is a TDS  $S$  in  $G$  for which the open neighborhoods  $N(v)$ , where  $v \in S$ , form a partition of  $V$ . Thus if  $S$  is an efficient total dominating set in  $G$ , then  $|N(v) \cap S| = 1$  for every vertex  $v \in S$ .

Suppose that  $S \subseteq V$  and that  $v \in S$ . The  *$S$ -private neighborhood* of  $v$ , denoted by  $\text{pn}(v, S)$ , consists of all vertices in the open neighborhood of  $v$  but not in the open neighborhood of  $S \setminus \{v\}$ ; that is,  $\text{pn}(v, S) = N(v) \setminus N(S \setminus \{v\})$ . Thus if  $u \in \text{pn}(v, S)$ , then  $N(u) \cap S = \{v\}$ . We call a vertex  $u \in \text{pn}(v, S)$  an  *$S$ -private neighbor* of  $v$ . The  *$S$ -external private neighborhood* of  $v$  is defined by  $\text{epn}(v, S) = \text{pn}(v, S) \cap (V \setminus S)$  and its  *$S$ -internal private neighborhood* by  $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$ . Hence,  $\text{pn}(v, S) = \text{epn}(v, S) \cup \text{ipn}(v, S)$ . We call a vertex in the set  $\text{epn}(v, S)$  an *external  $S$ -private neighbor* of  $v$ , and a vertex in  $\text{ipn}(v, S)$  is called an *internal  $S$ -private neighbor* of  $v$ .

### 1.2. Known results

Sridharan et al. [10] studied the total reinforcement number of a graph. In particular, they obtained the following results.

**Theorem A ([10]).** Let  $G \neq mK_2$  be a graph with  $\gamma_t(G) \geq 3$  and with maximum degree  $\Delta$ , and let  $D$  be an arbitrary  $\gamma_t(G)$ -set. If  $r_t(G) = \Delta$ , then the graph  $G$  satisfies the following conditions.

- (1)  $G[D] = mK_2$  and  $|V(G)| = 2m\Delta$  for some integer  $m \geq 2$ .
- (2) Every vertex in  $D$  is of maximum degree  $\Delta$ .
- (3) Every vertex in  $V(G) \setminus D$  is adjacent to exactly one vertex in  $D$ .

**Theorem B ([10]).** If  $G$  is a graph of order  $n$  with  $\gamma_t(G) \geq 3$  and with maximum degree  $\Delta$ , then  $r_t(G) \leq \min\{\Delta, \lfloor (n-2)/2 \rfloor\}$ .

Sridharan et al. [10] made the following conjecture.

**Sridharan–Elias–Subramanian Conjecture ([10]).** The three necessary conditions for  $r_t(G) = \Delta(G)$  given in Theorem A are also sufficient conditions.

## 2. The main result

We now state and prove our results.

**Theorem 1.** If  $G$  is a graph of order  $n$  with  $\gamma_t(G) \geq 3$ , then  $r_t(G) \leq n/\gamma_t(G)$ . Further if equality holds, then  $G$  has an efficient total dominating set all of whose vertices have the same degree in  $G$ .

**Proof.** Suppose that  $G = (V, E)$  and let  $D$  be a  $\gamma_t(G)$ -set. Then,  $\text{pn}(v, D) \neq \emptyset$  for every vertex  $v \in D$ . By assumption,  $|D| \geq 3$ . Let  $v$  be a vertex in  $D$  with  $|\text{pn}(v, D)|$  a minimum. Since

$$\bigcup_{w \in D} \text{pn}(w, D) \subseteq V,$$

we have that

$$|D| \cdot |\text{pn}(v, D)| \leq \sum_{w \in D} |\text{pn}(w, D)| = \left| \bigcup_{w \in D} \text{pn}(w, D) \right| \leq n, \quad (1)$$

and so  $|\text{pn}(v, D)| \leq n/|D| = n/\gamma_t(G)$ . We now construct a graph  $H$  from  $G$  as follows. If  $\text{ipn}(v, D) = \emptyset$ , then let  $u$  be an arbitrary vertex in  $D \setminus \{v\}$  and let  $H$  be the graph obtained from  $G$  by joining  $u$  to every vertex in  $\text{pn}(v, D)$ . If  $|\text{ipn}(v, D)| \geq 1$ , suppose that  $u \in \text{ipn}(v, D)$ . If  $|\text{ipn}(v, D)| = 1$ , let  $H$  be the graph obtained from  $G$  by joining  $u$  to every vertex in  $\text{epn}(v, D)$  and to one vertex in  $D \setminus \{u, v\}$ . If  $|\text{ipn}(v, D)| \geq 2$ , let  $H$  be the graph obtained from  $G$  by joining  $u$  to every vertex in  $\text{pn}(v, D) \setminus \{u\}$ . In all three cases, the set  $D \setminus \{v\}$  is a total dominating set in  $H$  and we add at most  $|\text{pn}(v, D)|$  edges to  $G$ .

Hence,  $r_t(G) \leq |\text{pn}(v, D)| \leq n/\gamma_t(G)$ . Further if  $r_t(G) = n/\gamma_t(G)$ , then in particular  $|\text{pn}(v, D)| = n/|D| = n/\gamma_t(G)$ . This implies equality throughout the inequality chain (1). Therefore,  $|\text{pn}(v, D)| = |\text{pn}(w, D)|$  for all  $w \in D$  and

$$\bigcup_{w \in D} \text{pn}(w, D) = V.$$

This in turn implies that  $D$  is an efficient total dominating set in  $G$  all of whose vertices have the same degree in  $G$ .  $\square$

As a consequence of [Theorem 1](#), we have the following result.

**Theorem 2.** *Let  $G$  be a graph with  $\gamma_t(G) \geq 3$  and with maximum degree  $\Delta$ . Then,  $r_t(G) \leq \Delta$ , with equality if and only if  $G$  has an efficient total dominating set all of whose vertices have degree  $\Delta$  in  $G$ .*

**Proof.** Let  $G = (V, E)$  have order  $n$  and let  $D$  be a  $\gamma_t(G)$ -set. Then,

$$\bigcup_{v \in D} N(v) = V,$$

and so,

$$n = |V| = \left| \bigcup_{v \in D} N(v) \right| \leq \sum_{v \in D} d(v) \leq |D| \cdot \Delta, \quad (2)$$

or, equivalently,  $n/\gamma_t(G) = n/|D| \leq \Delta$ . Hence by [Theorem 1](#),  $r_t(G) \leq \Delta$ . This establishes the desired upper bound.

Suppose that  $r_t(G) = \Delta$ . Then,  $r_t(G) = n/\gamma_t(G) = \Delta$ . By [Theorem 1](#),  $r_t(G) = n/\gamma_t(G)$  implies that  $G$  has an efficient total dominating set all of whose vertices have the same degree in  $G$ . Further,  $n/\gamma_t(G) = \Delta$  implies that we must have equality throughout the inequality chain (2). Therefore,  $d(v) = \Delta$  for every vertex  $v \in D$ . Hence,  $G$  has an efficient total dominating set all of whose vertices have degree  $\Delta$  in  $G$ .

Conversely, let  $G$  be a graph with  $\gamma_t(G) \geq 3$  and with maximum degree  $\Delta$  and suppose that  $G$  has an efficient total dominating set  $S$  all of whose vertices have degree  $\Delta$  in  $G$ . Then,  $V = N_G(S)$ , and so,  $n = |V| = |N_G(S)| = \Delta|S|$ . Let  $E'$  be a minimum set of edges that need to be added to  $G$  in order to decrease the total domination number. Then,  $r_t(G) = |E'|$ . Suppose that  $H = G + E'$  and let  $S'$  be a  $\gamma_t(H)$ -set. Then,  $V = N_H(S')$ , and so  $|N_H(S')| = |V| = n = \Delta|S|$ . Let  $V' = V \setminus N_G(S')$  denote the set of vertices not totally dominated by the set  $S'$  in  $G$ . Then,  $|V'| \leq |E'|$ . Thus since  $|S'| < |S|$ , we have that  $|N_H(S')| = |N_G(S')| + |V'| \leq \Delta|S'| + |E'| \leq \Delta(|S| - 1) + |E'|$ . Hence,  $\Delta|S| = |N_H(S')| \leq \Delta(|S| - 1) + |E'|$ , and so  $r_t(G) = |E'| \geq \Delta$ . As shown earlier,  $r_t(G) \leq \Delta$  for all graphs  $G$  with no isolated vertex. Consequently,  $r_t(G) = \Delta$ .  $\square$

A proof of the Sridharan–Elias–Subramanian conjecture is an immediate consequence of [Theorem 2](#) since a graph  $G$  has an efficient total dominating set all of whose vertices have maximum degree  $\Delta$  in  $G$  if and only if the three conditions given in [Theorem A](#) hold.

As observed earlier, if  $G$  is a graph of order  $n$  with  $\gamma_t(G) \geq 3$  and with maximum degree  $\Delta$ , then  $n/\gamma_t(G) \leq \Delta$ . Further,  $n/\gamma_t(G) \leq n/3$ . Hence, [Theorem 1](#) is a stronger result than that of [Theorem B](#).

The total reinforcement number of a path and a cycle can now be computed by using [Theorem 2](#). By convention,  $r_t(P_n) = r_t(C_n) = 0$  for  $n = 3$  or  $n = 4$ .

**Corollary 3.** For  $n \geq 5$ ,  $r_t(P_n) = r_t(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{4} \\ 1 & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$

**Proof.** Suppose that  $n \geq 5$  and that  $G \in \{P_n, C_n\}$ . By [Theorem 2](#),  $r_t(G) \leq \Delta(G) = 2$ . Further suppose that  $r_t(G) = 2$ . Then, by [Theorem 2](#),  $G$  has an efficient total dominating set all of whose vertices have degree 2 in  $G$ . This implies that  $n \equiv 0 \pmod{4}$ .  $\square$

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