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Minimizing the number of periodic points for smooth maps. Non-simply connected case ☆

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ABSTRACT

Let f be a smooth self-map of a closed manifold of dimension $m \geqslant 3$, r be a fixed natural number. In this paper we introduce the topological invariant $NJD_r^m[f]$, which is equal to the minimal number of r-periodic points in the smooth homotopy class of f.

1. Introduction

The classical problem in fixed point theory is to find the minimal number of fixed points in the homotopy class of the given continuous map. If $f: M \to M$ is a map of a compact m-manifold, where $m \geqslant 3$, then it is known that $\min\{\#\mathrm{Fix}(g)\colon g \sim f\}$ is equal to $N(f)$, the Nielsen number of f. Moreover the map realizing the least number of fixed points may be chosen smooth [12] (we call f smooth provided it is C^1). However, such equivalence between continuous and smooth category does not exist if we look for the minimal number of periodic points of the given period.

In 1983 B.J. Jiang introduced in [10] the classical invariant $NF_r(f)$, which gives a lower bound for the number of r-periodic points in the homotopy class of f. Later, in [7], it was proved that in fact $NF_r(f)$ is the best such lower bound, i.e.

$$NF_r(f) = \min\{\#\mathrm{Fix}(g^r)\colon g \sim f\}. \tag{1.1}$$

The natural question is whether there exists a counterpart of such invariant in the smooth category, i.e. what is the minimum in the formula (1.1) for smooth f and g in its smooth homotopy class. In fact, we may consider a bit more general approach. Instead of taking a smooth f and smooth homotopies, we may consider continuous f and search for the minimum over smooth g in the continuous homotopy class. As every smooth homotopy may be approximated by a continuous one, both approaches in the common domain lead to the same result.

In this paper we introduce an invariant $NJD_r[f]$, the Nielsen–Jiang–Dold number, which is the lower bound for the number of r-periodic points in the smooth non-simply connected case. We show (Theorem 4.4) that this invariant is optimal in the following sense:

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(1) $NJD_r[f]$ is less than or equal to the number of fixed points of g^r for any map g smoothly homotopic to the given f,
(2) there exists a map g smoothly homotopic to f with

$$\#\mathrm{Fix}(g^r) = NJD_r[f].$$

Obviously $NJD_r[f] \geqslant NF_r(f)$ and this inequality is usually sharp. The invariants are quite different and the differences in their construction may shed more light on the role of differentiability hypothesis in periodic point theory. It turns out that there are two fundamental obstacles to minimize the number of periodic points. The first one depends on the fundamental group of the manifold (more precisely it is described in the language of Nielsen theory by Reidemeister relations). The second one may be expressed in terms of local fixed point indices of iterations $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$, where x_0 is a periodic point.

If M is simply connected then the Nielsen theory is trivial. On the other hand, the restrictions for indices of iterations of a continuous maps (Dold relations [2]) also turns out to be not an essential obstacle to minimize the number of periodic points. As a result $NF_r(f) \leqslant 1$, i.e. f is always homotopic to a continuous map with no more than one r-periodic point being a fixed point. The situation changes completely if we take a smooth f and consider its smooth homotopy class. Then, we have to take into account strong restrictions for indices of iterations of smooth maps found by Chow, Mallet-Paret and Yorke [1]. Thus, even for a simply connected manifold, the invariant $NJD_r[f]$ is usually much greater than 1 (for the details we refer the reader to [4]).

For the smooth non-simply connected case both types of obstacles must be taken into consideration. In the geometric interpretation of $NJD_r[f]$ the Reidemeister relations are represented by a directed graph and the structure of indices is coded by an associating with each vertex some integer. The minimization under different sets of admissible integers in the vertices gives the value of $NJD_r[f]$.

As, up to now, we know only the description of indices of iterations for smooth maps in $\mathbb{R}^3$ [6], exact calculations are possible for 3-dimensional manifolds and in general they seem to be complicated. However, the invariant is computable for maps with simple Reidemeister relations. In this paper we determine $NJD_r^3[f]$ for self-maps of $\mathbb{R}P^3$ and odd r (which reduces in fact to simply connected case) and for $r = 6$. The last case we use as an illustrative example of the differences between continuous and smooth category.

The paper is organized as follows. After preliminaries, in the second section we prove the congruences (known as Dold relations) for Reidemeister orbits, which gives us the convenient way to write down the sequence of Lefschetz numbers and enables us to define $NJD_r^m[f]$ in combinatorial terms. The definition of $NJD_r^m[f]$ and main theorem are given in Section 4. Section 5 is devoted to a geometric interpretation of the invariant in terms of the so-called Reidemeister graph. In the final section we calculate $NJD_r^m[f]$ for 3-dimensional real projective space.

2. Preliminaries

In this section we give some definitions and statements which will be used throughout the rest of the paper.

2.1. Dold relations

The sequence of indices of iterations must satisfy some congruences, which were found in 1983 by Dold [2]. Let $f: U \to X$, where $U \subset X$ is an open subset of an ENR. We define inductively $U_0 = U$, $U_{n+1} = f^{-1}(U_n)$ i.e. $U_n = \{x \in U\colon x, f(x), \dots, f^n(x) \in U\}$. Assume that the fixed point set $\mathrm{Fix}(f^n) = \{x \in U_n\colon\ f^n(x) = x\}$ is compact for each $n \in \mathbb{N}$. In such a situation the fixed point index $\mathrm{ind}(f^n) = \mathrm{ind}(f^n, U_n)$ is well defined. Dold proved that the sequence of fixed point indices $\{\mathrm{ind}(f^n)\}_{n=1}^{\infty}$ for each $n \in \mathbb{N}$ must satisfy the following congruences (called Dold relations):

$$\sum_{k|n} \mu(n/k)\,\mathrm{ind}(f^k) \equiv 0 \pmod n, \tag{2.1}$$

where μ denotes the classical Möbius function, i.e. $\mu: \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^r$ if k is a product of r different primes, $\mu(k) = 0$ otherwise.

2.2. Periodic expansion

Due to the Dold relations we may decompose every sequence of indices of iterations into a sum of some specific elementary sequences. This decomposition will be called a periodic expansion.

Definition 2.1. For a given k we define the basic sequence:

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k|n, \\ 0 & \text{if } k\nmid n. \end{cases}$$

Notice that each basic sequence reg_k is a periodic sequence of the form: $(0, \dots, 0, k, 0, \dots, 0, k, \dots, \dots)$, where the non-zero entries appear only for indices of the sequence which are multiples of k.

Theorem 2.2. *(Cf. [9].) Any sequence $\psi : \mathbb{N} \to \mathbb{C}$ can be written uniquely in the following form of a periodic expansion*:

$$\psi(n) = \sum_{k|n} a_k \operatorname{reg}_k(n),$$

where $a_n = \frac{1}{n} \sum_{k|n} \mu(k)\psi(n/k)$.

Moreover, ψ takes integer values and satisfies Dold relations iff $a_n \in \mathbb{Z}$ for every $n \in \mathbb{N}$.

Definition 2.3. Assume that a periodic expansion for a sequence $\{\psi(n)\}_{n=1}^{\infty}$ is given. Let $\mathbf{\Delta}(\psi) = \{n \in \mathbb{N}\colon\ a_n \neq 0\}$. The set $\mathbf{\Delta}(\psi)$ will be called *the basic set* for a periodic expansion of ψ. We will also consider *the basic set up to the level r* for ψ defined as $\mathbf{\Delta}_r(\psi) = \{n|r \in \mathbb{N}\colon\ a_n \neq 0\}$.

2.3. $DD^m(p|r)$ sequences

In the problem of minimizing the number of periodic points in a smooth homotopy class the important role play so-called $DD^m(p|r)$ sequences.

Definition 2.4. A sequence of integers $\{c_n\}_{n=1}^{\infty}$ is called $DD^m(p)$ sequence if there are: a C^1 map $\phi : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ is open; and P, an isolated p-orbit of ϕ, such that $c_n = \operatorname{ind}(\phi^n, P)$ (notice that $c_n = 0$ if n is not a multiple of p). The finite sequence $\{c_n\}_{n|r}$ will be called $DD^m(p|r)$ sequence if this equality holds for $n|r$, where r is fixed. In other words, a $DD^m(p)$ sequence is a sequence that can be realized as a sequence of indices of iterations on an isolated p-orbit for some smooth map ϕ.

The following lemma from [4] shows how to obtain the forms of $DD^m(p)$ sequences if we know the forms of $DD^m(1)$ sequences.

Lemma 2.5. *A sequence $\{c_n\}_{n=1}^{\infty}$ is a $DD^m(1)$ sequence if and only if*

$$\tilde{c}_n = \begin{cases} 0 & \text{for } p \nmid n, \\ p c_{n/p} & \text{for } p|n \end{cases}$$

is a $DD^m(p)$ sequence. We will say that a $DD^m(p)$ sequence $\{\tilde{c}_n\}_{n=1}^{\infty}$ comes from a $DD^m(1)$ sequence $\{c_n\}_{n=1}^{\infty}$.

In [6] (cf. Theorem 2.6 below) there is the full description of all possible forms all $DD^3(1)$ sequences and thus, by Lemma 2.5 also $DD^3(p)$ sequences for any given p.

Theorem 2.6. *There are seven kinds of $DD^3(1)$ sequences*:

(A) $c_A(n) = a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$,
(B) $c_B(n) = \operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$,
(C) $c_C(n) = -\operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n)$,
(D) $c_D(n) = a_d \operatorname{reg}_d(n)$,
(E) $c_E(n) = \operatorname{reg}_1(n) - \operatorname{reg}_2(n) + a_d \operatorname{reg}_d(n)$,
(F) $c_F(n) = \operatorname{reg}_1(n) + a_d \operatorname{reg}_d(n) + a_{2d} \operatorname{reg}_{2d}(n)$, *where d is odd,*
(G) $c_G(n) = \operatorname{reg}_1(n) - \operatorname{reg}_2(n) + a_d \operatorname{reg}_d(n) + a_{2d} \operatorname{reg}_{2d}(n)$, *where d is odd.*

In all cases $d \geqslant 3$ and $a_i \in \mathbb{Z}$.

Remark 2.7. Lemma 2.5 gives an easy procedure which enables one to obtain all forms of $DD^m(p)$ sequences once we know the forms of $DD^m(1)$ sequences. In order to get any $DD^m(p)$ sequence it is enough to replace all basic sequences $a_k \operatorname{reg}_k$ by $a_k \operatorname{reg}_{pk}$ in the periodic expansion of some $DD^m(1)$ sequence. For example all $DD^3(p)$ sequences which come from the $DD^3(1)$ of the type (A) have the form

$$c_A(n) = a_1 \operatorname{reg}_p(n) + a_2 \operatorname{reg}_{2p}(n),$$

where a_1, a_2 are arbitrary integers.

3. Dold relations for orbits of Reidemeister classes

Let $f : M \to M$ be a continuous self-map of a compact manifold. For each natural number $k \in \mathbb{N}$ we define the set of orbits of Reidemeister classes $\mathcal{OR}(f^k)$ and for each pair of numbers $l|k$ we denote by $i_{k,l} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$ the natural boosting map. This may be regarded as a functor from the category $(\mathbb{N}, |)$ to the set category. If $N^l \subset \mathrm{Fix}(f^l)$, $N^k \subset \mathrm{Fix}(f^k)$ are Nielsen classes representing Reidemeister classes $A^l \subset \mathcal{OR}(f^l)$ and $A^k \subset \mathcal{OR}(f^k)$ respectively, then $N^l \subset N^k$ implies $i_{k,l}(A^l) = A^k$ (cf. [9]).

By $\mathcal{OR}_n(f)$ and $\mathcal{OR}_\infty(f)$ we will denote the following disjoint sums:

$$\mathcal{OR}_n(f) = \bigcup_{k|n} \mathcal{OR}\big(f^k\big),$$

$$\mathcal{OR}_\infty(f) = \bigcup_{k\in\mathbb{N}} \mathcal{OR}\big(f^k\big).$$

Remark 3.1. We consider $\mathcal{OR}(f^k)$ and $\mathcal{OR}(f^l)$, for $k \neq l$, as disjoint sets. However, we will often identify a Reidemeister class $A \in \mathcal{OR}(f^k)$ with the Nielsen class which corresponds to A. In particular we interpret in this way the fixed point index $\mathrm{ind}(f^k; A)$.

For an orbit of Reidemeister classes $B \in \mathcal{OR}_n(f)$ we will denote by $l(B)$ the unique natural number for which $B \in \mathcal{OR}(f^{l(B)})$.

We say that for two orbits of Reidemeister classes $A \in \mathcal{OR}(f^k)$ and $B \in \mathcal{OR}(f^l)$, B *is preceding* A if $l|k$ and $i_{k,l}(B) = A$. We write then $B \preccurlyeq A$. We use also the notation $B \prec A$, where $B \prec A$ if $B \preccurlyeq A$ but $A \neq B$.

Since each orbit $A \in \mathcal{OR}(f^k)$ is an open and closed subset in $\mathrm{Fix}(f^k)$, the fixed point index $\mathrm{ind}(f^k; A)$ is defined.

Lemma 3.2. *Let $f : X \to X$ be a self-map of a compact polyhedron (or ENR) and let S be a closed–open invariant subset of* $\mathrm{Fix}(f^r)$, *where r is a given fixed natural number. Then the sequence* $\mathrm{ind}(f^n; S)$, *for each $n|r$, satisfies Dold relations i.e.*

$$\sum_{k|n} \mu(n/k) \cdot \mathrm{ind}\big(f^k; S\big) \equiv 0 \pmod n.$$

Proof. Since S is a closed–open subset of $\mathrm{Fix}(f^r)$, there is an open subset $U' \subset X$ satisfying $\mathrm{Fix}(f^r) \cap U' = S$. Since S is invariant, there is a neighborhood $U \supset S$ such that $\bigcup_{i=0}^r f^i(U) \subset U'$. Then the formula (2.1) is valid for the restriction $f_U : U \to X$, since $\mathrm{Fix}((f^r)_U) = S$ is compact. Now the sequence $\{\mathrm{ind}((f_U)^k)\}$, where $k|n$, satisfies the Dold congruences (2.1). It remains to notice that

$$\mathrm{ind}\big((f_U)^k\big) = \mathrm{ind}\big(f^k; S\big)$$

for all $k|r$. □

Lemma 3.3. *For each orbit $A \in \mathcal{OR}(f^n)$*

$$\sum_{B\preccurlyeq A} \mu\big(n/l(B)\big) \cdot \mathrm{ind}\big(f^{l(B)}; B\big) \equiv 0 \pmod n.$$

Proof. We notice that each orbit of Reidemeister classes $A \in \mathcal{OR}(f^n)$ may be regarded as the closed–open subset of $\mathrm{Fix}(f^r)$. By Lemma 3.2 the sequence $\{\mathrm{ind}(f^k; A)\}_{k|n}$ satisfies Dold congruences. On the other hand, the following equalities hold

$$\mathrm{ind}\big(f^k; A\big) = \mathrm{ind}\big(f^k; A \cap \mathrm{Fix}\big(f^k\big)\big) = \mathrm{ind}\Bigg(f^k; \bigcup_{B\preccurlyeq A,\, l(B)=k} B\Bigg)$$
$$= \sum_{B\preccurlyeq A,\, l(B)=k} \mathrm{ind}\big(f^k; B\big).$$

The last equality holds, since the Nielsen classes in $\mathrm{Fix}(f^k)$ are mutually disjoint.

Finally, by the above equality,

$$\sum_{B\preccurlyeq A} \mu\big(n/l(B)\big) \cdot \mathrm{ind}\big(f^{l(B)}; B\big) = \sum_{k|n} \mu(n/k)\Bigg(\sum_{B\preccurlyeq A,\, l(B)=k} \mathrm{ind}\big(f^k; B\big)\Bigg)$$
$$= \sum_{k|n} \mu(n/k)\,\mathrm{ind}\big(f^k; A\big) \equiv 0 \pmod n,$$

where the last congruence follows from Lemma 3.2. □

Definition 3.4. Let μ^* be the Möbius function on the partially ordered set $(\mathcal{OR}_\infty(f), \preccurlyeq)$, i.e. $\mu^* : \mathrm{Int}(\mathcal{OR}_\infty(f)) \to \mathbb{Z}$, where $\mathrm{Int}(\mathcal{OR}_\infty(f))$ denotes the set of all intervals in $\mathcal{OR}_\infty(f)$, and μ^* is defined by two properties:

- $\mu^*(B, B) = 1$,
- $\mu^*(B, A) = -\sum_{\{C:\ B \preccurlyeq C \prec A\}} \mu^*(B, C)$.

Remark 3.5. Let $A \in \mathcal{OR}(f^a)$, $B \in \mathcal{OR}(f^b)$. Then

$$\mathrm{Int}(B, A) = \begin{cases} \{i_{d,b}(B):\ b|d|a\} & \text{for } B \preccurlyeq A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 3.6. *Let $A \in \mathcal{OR}(f^a)$ and $B \in \mathcal{OR}(f^b)$. If $B \preccurlyeq A$, then $\mu^*(B, A) = \mu(a/b)$.*

Proof. We fix B and apply induction by $A = i_{a,b}(B)$. For $A = B$ we have that $\mu^*(A, A) = \mu(a/a) = 1$. Assume that the equality holds for orbits $C \in \mathcal{OR}(f^c)$ which satisfy $B \preccurlyeq C \prec A$. We will show that then the equality holds also for A. There is

$$\begin{aligned} \mu^*(B, A) &= -\sum_{B \preccurlyeq C \prec A} \mu^*(B, C) = -\sum_{b|c|a;\, c \neq a} \mu(c/b) \\ &= -\sum_{d|\frac{a}{b};\, d \neq \frac{a}{b}} \mu(d) = \mu(a/b) - \sum_{d|\frac{a}{b}} \mu(d) = \mu(a/b), \end{aligned}$$

where the third equality was obtained by substituting $d = \frac{c}{b}$ and the last equality follows from the fact that $\sum_{d|k} \mu(d) = 0$ for $k > 1$. $\square$

Now we are ready to formulate and prove Dold relations for orbits of Reidemeister classes. Let us mention that there are similar congruences for Reidemeister numbers in [3].

Theorem 3.7. *For a fixed $A \in \mathcal{OR}(f^n)$ we have*

$$\sum_{B \preccurlyeq A} \mu^*(B, A) \cdot \mathrm{ind}\big(f^{l(B)}; B\big) \equiv 0 \pmod{n}. \tag{3.1}$$

Proof. By Lemma 3.6 $\mu^*(B, A) = \mu^*(n/l(B))$ and the theorem follows from Lemma 3.3. $\square$

Definition 3.8. For $B \in \mathcal{OR}(f^k)$ we define the function $\mathrm{Reg}_B : \mathcal{OR}_\infty(f) \to \mathbb{Z}$ putting

$$\mathrm{Reg}_B(A) = \begin{cases} k & \text{for } B \preccurlyeq A, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.9. Let us notice that for the orbits $A \in \mathcal{OR}(f^a)$, $B \in \mathcal{OR}(f^b)$ satisfying $B \preccurlyeq A$

$$\mathrm{Reg}_B(A) = \mathrm{reg}_b(a).$$

Let us consider a little more general situation. Let $I : \mathcal{OR}_\infty(f) \to \mathbb{Z}$ be a function satisfying for each given $A \in \mathcal{OR}(f^n)$

$$\sum_{B \preccurlyeq A} \mu^*(B, A) \cdot I(B) \equiv 0 \pmod{n}.$$

We will say then that I satisfies the *Dold congruences*.

Theorem 3.10. *Let I satisfy the Dold congruences. Then for each $B \in \mathcal{OR}_\infty(f)$ a unique integer number a_B is defined, such for any given $A \in \mathcal{OR}_\infty(f)$ the following equality holds*:

$$I(A) = \sum_{\{B:\ B \preccurlyeq A\}} a_B \,\mathrm{Reg}_B(A). \tag{3.2}$$

We will call such representation of I generalized periodic expansion.

Proof. We may rewrite the formula (3.2) as $I(A) = \sum_{\{B:\ B \preccurlyeq A\}} a_B l(B)$. Then, by the Möbius inversion formula for partially ordered sets (cf. for example [13]), we obtain that it is equivalent, for $B \preccurlyeq A$, to

$$a_A l(A) = \sum_{B \preccurlyeq A} \mu^*(B, A) I(B). \tag{3.3}$$

As a consequence, for each A the unique a_A is defined. Moreover, $A \in \mathcal{OR}(f^n)$, so $l(A) = n$. Then by the assumption that I satisfies Dold relations we get that a_A is integer. □

Corollary 3.11. *Let $f : M \to M$ be a self-map of a compact manifold. Since by Theorem 3.7 $I(A) = \operatorname{ind}(f^n; A)$, where $A \in \mathcal{OR}(f^n)$, satisfies the Dold congruences, there exist unique numbers $a_B \in \mathbb{Z}$ such that*

$$\operatorname{ind}(f^n; A) = \sum_B a_B \cdot \operatorname{Reg}_B(A) \tag{3.4}$$

for all $A \in \mathcal{OR}(f^n)$.

4. Lower bound for the number of points in Fix(f^r) for smooth f

In this section we introduce the invariant $NJD_r^m[f]$, the lower bound for the number of r-periodic points of a map smoothly homotopic to the given smooth self-map $f : M \to M$ of a smooth closed connected and possibly non-simply connected manifold.

We assume that the considered manifold M is closed, however all results remain true if M has a nonempty boundary but f has no periodic points on the boundary.

First we will analyze the impact of a single isolated orbit. This will motivate the definition of the invariant. Then we prove that this invariant is optimal. In the last subsection we will give an upper bound for the number $NJD_r^m[f]$.

We fix a number $r \in \mathbb{N}$. We will assume that Fix(f^r) is finite, otherwise we may by Kupka–Smale Theorem (cf. [14]) approximate f by a smooth map which has a finite number of r-periodic points (for the given r).

4.1. An isolated orbit

This subsection gives a motivation for the construction of the invariant $NJD_r^m[f]$.

First of all let us analyze the impact of an orbit of periodic points $a = \{x_1, \ldots, x_{l_a}\} \subset \operatorname{Fix}(f^r)$, $l_a | r$. The orbit a determines:

(1) an orbit of Reidemeister classes $A \in \mathcal{OR}(f^{l_a})$ i.e. the orbit of Nielsen classes containing the points of a,
(2) a $DD^m(l_a|r)$ sequence c_a given by

$$c_a(n) = \operatorname{ind}(f^n; a)$$

where $l_a | n | r$.

Let us notice that then for each orbit $A \in \mathcal{OR}(f^n)$ with $n|r$:

$$\#\operatorname{Fix}(f^r; A) = \sum_a l_a, \tag{1'}$$

$$\operatorname{ind}(f^n; A) = \sum_a c_a(n), \tag{2'}$$

where in both cases the summation runs over the set of orbits of points contained in A.

Now we are going to reverse the above approach. Namely, we represent $\operatorname{ind}(f^n; A)$ as the sum of the type (2′), which allows us to obtain the least sum $\sum_a l_a$.

4.2. Invariant $NJD_r^m[f]$

Each orbit of Reidemeister classes $A \in \mathcal{OR}(f^k)$ has its own fixed point index $\operatorname{ind}(f^k; A)$ and the sum of the indices gives $L(f^k)$. On the other hand, we can extend, in a natural way, the definition of $DD^m(p)$ sequences from the set of natural numbers onto the set $\mathcal{OR}_\infty(f)$ – we say that each such sequence is *attached* at some orbit (see Definition 4.1 below). Finally, we decompose the function

$$\{L(f^n)\}_{n|r} = \left\{\sum_A \operatorname{ind}(f^n; A)\right\}_{n|r}$$

into the minimal sum of $DD^m(p)$ sequences attached at some orbits and obtain $NJD_r^m[f]$.

Definition 4.1. For a fixed $DD^m(h)$ sequence c and an orbit $H \in \mathcal{OR}(f^h)$ we define the function $C_H : \mathcal{OR}_\infty(f) \to \mathbb{Z}$ by

$$C_H(A) = \begin{cases} c(n) & \text{for } H \preccurlyeq A;\ A \in \mathcal{OR}(f^n), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We say then that the $DD^m(h)$ sequence c is *attached* at the orbit $H \in \mathcal{OR}(f^h)$.

For example $\mathrm{Reg}_H : \mathcal{OR}_\infty(f) \to \mathbb{Z}$ may be regarded as the basic sequence reg_h attached at the orbit $H \in \mathcal{OR}(f^h)$.

Remark 4.2. C_H may be written in the form of generalized periodic expansion. What is more, this expansion is strictly related to the periodic expansion of the sequence $\{c(n)\}_n$ in the formula (4.1). Namely, if

$$c(n) = \sum_{d \in \mathbf{\Delta}(c)} a_d \, \mathrm{reg}_d(n), \tag{4.2}$$

then

$$C_H(A) = \sum a_d \, \mathrm{Reg}_B(A), \tag{4.3}$$

where the sum is extended over the set $\{B \in \mathcal{OR}(f^d)\colon\ d \in \mathbf{\Delta}(c) \text{ and } H \preccurlyeq B\}$.

By Corollary 3.11 $\mathrm{ind}(f^n; A) = \sum_B a_B \, \mathrm{Reg}_B(A)$ for every orbit of Reidemeister classes $A \in \mathcal{OR}(f^n)$, where $n|r$.
Now we consider a splitting of $\mathrm{ind}(f^n; A)$ (we omit below the variable A):

$$\sum_B a_B \, \mathrm{Reg}_B = C_{H_1} + \cdots + C_{H_s}, \tag{4.4}$$

where C_{H_i} denotes a $DD^m(h_i)$ sequence c_i attached at the class $H_i \in \mathcal{OR}(f^{h_i})$.
Notice that by (4.3) both sides of the equality (4.4) are combinations of some Reg_B.

Definition 4.3. We define $NJD_r^m[f]$, the Nielsen–Jiang–Dold number, by

$$NJD_r^m[f] = \text{minimal sum } h_1 + \cdots + h_s,$$

such that the equality (4.4) holds.

If m, the dimension of the manifold is known from the context, we will write just $NJD_r[f]$.
Let us remark that both sides of the equality (4.4) have the form of generalized periodic expansion. Thus the problem of finding $NJD_r[f]$ has a combinatorial nature, once we know the form of generalized periodic expansion of Lefschetz numbers (i.e. left-hand side of (4.4)) and the forms of $DD^m(h_i)$ sequences (i.e. right-hand side of (4.4)).
Now we come to the main result of the paper, which states that $NJD_r[f]$ is the minimum of the number of r-periodic points for all smooth maps in the homotopy class of f.

Theorem 4.4 *(Main Theorem). The number $NJD_r[f]$ satisfies*:

(1) *$NJD_r[f]$ is the homotopy invariant,*
(2) *$\#\mathrm{Fix}(f^r) \geqslant NJD_r[f]$,*
(3) *if $m \geqslant 3$ then f is homotopic to a smooth map g realizing the number $NJD_r[f]$ i.e. $\#\mathrm{Fix}(g^r) = NJD_r[f]$.*

Proof. (1) In the definition of $NJD_r[f]$ we use only fixed point index and Reidemeister classes and these are homotopy invariants.
(2) By item (2′) of Section 4.1, assuming f is smooth, we get that $\mathrm{ind}(f^n; A)$ is the sum of $DD^m(l_a|r)$ sequences, realized by the orbits a of f. As a consequence by the equality $\#\mathrm{Fix}(f^r) = \sum_{A \in \mathcal{OR}(f^r)} \sum_{a \subset A} l_a$, we get that

$$\#\mathrm{Fix}\big(f^r\big) = \sum_a l_a \geqslant NJD_r[f].$$

(3) In this part of the proof we will make use of two procedures: Smooth Creating Procedure and Canceling Lemma, which for the sake of clarity are listed separately in Section 4.3.
Let us suppose that the splitting of $\mathrm{ind}(f^n; A)$:

$$\sum_B a_B \, \mathrm{Reg}_B = C_{H_1} + \cdots + C_{H_s} \tag{4.5}$$

is the minimal one i.e. $h_1 + \cdots + h_s = NJD_r[f]$.
Recall that the function C_{H_i} is obtained by attaching a sequence c_i at the orbit $H_i \in \mathcal{OR}(f^{h_i})$. First we assume that:

(∗) for each C_{H_i}, we may find in the orbit $H_i \in \mathcal{OR}(f^{h_i})$ an orbit of points $a_i = \{a_i^1, \ldots, a_i^{h_i}\}$ such that f is smooth near each a_i and

$$\mathrm{ind}\big(f^n; a_i\big) = c_i(n).$$

We notice that then for any $A \in \mathcal{OR}(f^n)$ and $i = 1, \dots, s$

$$C_{H_i}(A) = \operatorname{ind}\big(f^n; A \cap a_i\big) \quad \text{for } n|r.$$

We will show that $S = a_1 \cup \dots \cup a_s$ satisfies the assumptions of Canceling Lemma 4.6. As (1) and (2) of Lemma 4.6 are obvious, we verify the assumption (3) in the form given by (4.6).

$$\begin{aligned}\operatorname{ind}\big(f^n; A \cap S\big) &= \operatorname{ind}\big(f^n; A \cap (a_1 \cup \dots \cup a_s)\big)\\ &= \operatorname{ind}\big(f^n; A \cap a_1\big) + \dots + \operatorname{ind}\big(f^n; A \cap a_s\big) = C_{H_1}(A) + \dots + C_{H_s}(A)\\ &= \sum_B a_B \operatorname{Reg}_B(A) = \operatorname{ind}\big(f^n; A\big).\end{aligned}$$

Now Canceling Lemma 4.6 gives a homotopy from f to a map f_1 with $\operatorname{Fix}(f_1^r) = S$. Notice that f_1 is smooth in some neighborhood W of the orbits $a_1, \dots, a_s$ and f_1^r has no fixed points outside W. Thus, if f_1 is not smooth as the global map, we approximate it by a smooth map g constantly equal to f_1 on W without adding any new r-periodic points in the compact set $M \setminus W$. Hence f is homotopic to smooth g and

$$\#\operatorname{Fix}\big(g^r\big) = \#S = \#\{a_1 \cup \dots \cup a_s\} = h_1 + \dots + h_s = NJD_r[f].$$

To end the proof it is enough to notice that we may provide the condition $(*)$. Indeed, by Smooth Creating Procedure (Theorem 4.5), we may find a homotopy from f to a map f' such that for any i, in the orbit $H_i' \in \mathcal{OR}(f'^{h_i})$ corresponding to $H_i \in \mathcal{OR}(f^{h_i}) = \mathcal{OR}(f'^{h_i})$, there is an isolated orbit of points $a_i = \{a_i^1, \dots, a_i^{h_i}\}$ (items (1)–(4) and (6)). What is more, by item (5), f' is smooth near the orbit and $\operatorname{ind}(f'^n; a_i) = c_i(n)$. As a result, $(*)$ is satisfied for f' and we may apply the above to deform f' to a map g with $\#\operatorname{Fix}(g^r) = NJD_r[f'] = NJD_r[f]$. □

4.3. Procedures

Now we will present two results which were used in the proof of our main Theorem 4.4: Smooth Creating Procedure and Canceling Lemma [7].

Due to Smooth Creating Procedure we may create an orbit in the homotopy class of f, by a use of homotopy f_t which is constant is the small neighborhood of periodic points of f (up to the given period r) and such that f_1^r is smooth near the created orbit and may be given there by an arbitrarily prescribed formula.

Theorem 4.5 *(Smooth Creating Procedure). Given numbers $p, r \in \mathbb{N}$, $p|r$ and a map $f: M \to M$, where* $\dim M \geqslant 3$, *such that* $\operatorname{Fix}(f^r)$ *is finite and a point $x_0 \notin \operatorname{Fix}(f^r)$. Then there is a homotopy $\{f_t\}_{0 \leqslant t \leqslant 1}$ satisfying*:

(1) $f_0 = f$.
(2) *$\{f_t\}$ is constant in a neighborhood of* $\operatorname{Fix}(f^r)$.
(3) *$f_1^p(x_0) = x_0$ and $f_1^i(x_0) \neq x_0$ for $i = 1, \dots, p-1$.*
(4) *The orbit $\mathcal{O} = \{x_0, f_1(x_0), \dots, f_1^{p-1}(x_0)\}$ is isolated in* $\operatorname{Fix}(f_1^r)$.
(5) *f_1 realizes given $DD^m(p|r)$ sequence $\{\tilde{c}_n\}_{n|r}$ on $\mathcal{O}$, i.e. f_1 is smooth in a neighborhood of $\mathcal{O}$ and $\tilde{c}_n = \operatorname{ind}(f_1^n; \mathcal{O})$ for $n|r$.*
(6) *The orbit $\mathcal{O}$ may represent an arbitrarily prescribed orbit in $\mathcal{OR}(f_1^p)$.*

The items (1)–(4) are the part of Theorem 3.3 in [7] (called Creating Procedure). The statement of the item (5) is the same as Proposition 3.6 in [4]. Finally, the item (6) follows from the item (5) of Theorem 3.2 in [7] (called Addition Procedure).

The following lemma enables one to remove subsets of periodic points which have indices of iterations equal to zero.

Lemma 4.6 *(Canceling Lemma). ([7, Lemma 5.4]) Let f be a continuous self-map of M. Suppose that $S \subset$* $\operatorname{Fix}(f^r)$ *satisfies*:

(1) *S is finite and f-invariant i.e. $f(S) = S$.*
(2) $\operatorname{Fix}(f^r) \setminus S$ *is compact.*
(3) $\operatorname{ind}(f^n; A \setminus S) = 0$ *for any $n|r$ and any orbit of Reidemeister classes $A \subset \mathcal{OR}(f^n)$.*

Then there is a homotopy f_t, starting from $f_0 = f$, constant near S and such that $\operatorname{Fix}(f_1^r) = S$.

Remark 4.7. Let us notice that the assumption (3) of Lemma 4.6 is equivalent to the following condition: for any $n|r$ and any orbit of Reidemeister classes $A \subset \mathcal{OR}(f^n)$, we have

$$\operatorname{ind}\big(f^n; A \cap S\big) = \operatorname{ind}\big(f^n; A\big). \tag{4.6}$$

4.4. An estimation of $NJD_r^m[f]$

Definition 4.8. We define $\mathrm{d}(A)$, the *depth* of the class $A \in \mathcal{OR}(f^n)$, as the least $k|n$ with $A \in \operatorname{im} i_{n,k}$.

Definition 4.9. An orbit $A \in \mathcal{OR}(f^n)$ is called irreducible if $\mathrm{d}(A) = n$, i.e. if it is not preceded by an orbit of a smaller depth. For a given r, we will denote the set of irreducible orbits as $\mathcal{IOR}_r(f)$.

Theorem 4.10. *Let $f: M \to M$ be a self-map of a closed manifold of dimension $\geqslant 3$, let $r \in \mathbb{N}$ be a fixed number. Then f is homotopic to a smooth map g satisfying*

$$\#\mathrm{Fix}(g^r) \leqslant \sum_B \mathrm{d}(B), \tag{4.7}$$

where the summation runs over the set of all orbits of Reidemeister classes B satisfying: $l(B)$ divides r and $a_B \neq 0$ in the formula (3.4).

Proof. Let $\mathrm{ind}(f^n; A) = \sum_B a_B \operatorname{Reg}_B(A)$, where $a_B \neq 0$. Let us notice that each $DD^3(p)$ sequence is also $DD^m(p)$ sequence for $m \geqslant 3$, thus in dimensions greater that 3 we may use the sequences described in Theorem 2.6 for estimating $NJD_r[f]$. For a class $B \in \mathcal{OR}(f^k)$ we fix $B' \in \mathcal{OR}(f^{\mathrm{d}(B)})$, $B' \preccurlyeq B$. Then $a_B \operatorname{Reg}_B$ can be regarded as the attachment of the sequence $a_B \operatorname{reg}_k$ of the type (D) or (A) at the class B'. We can do this independently for each B. The obtained set of attachments gives us in Definition 4.3 the sum $\sum_B l(B') = \sum_B \mathrm{d}(B)$, which means that $NJD_r[f] \leqslant \sum_B \mathrm{d}(B)$. □

5. Reidemeister graph

In this short section we present a natural geometric interpretation of $NJD_r[f]$, which makes the calculations more convenient and helps to imagine the obtained data.

5.1. Construction of Reidemeister graph

First of all let us notice that the set $\mathcal{OR}_\infty(f) = \bigcup_{k\in\mathbb{N}} \mathcal{OR}(f^k)$ (disjoint sum) is a partially ordered set by the relation "$\preccurlyeq$". The set determines a directed graph in which vertices are orbits of Reidemeister classes and a (unique) directed edge from B to A corresponds to the relation $B \prec A$. If we associate with each vertex $A \in \mathcal{OR}(f^k)$ the number a_A from the generalized periodic expansion then we get, what we will call, *graph of orbits of Reidemeister classes* (briefly Reidemeister graph) $\mathcal{GOR}(f)$.

For a fixed integer r we denote by $\mathcal{GOR}(f;r)$ the full subgraph whose vertices are elements of $\mathcal{OR}(f^k)$ for $k|r$. Let us remark that $\mathcal{GOR}(f;r)$ carries all data needed to determine $NJD_r[f]$.

Remark 5.1. Assume that $\mathcal{GOR}(f;r)$ has some connected components $G_1, \ldots, G_v$. By Definition 4.3 to get $NJD_r[f]$ we have to split the sum $\sum a_B \operatorname{Reg}_B$ into the sum of functions C_{h_i} with the minimal value of $h_1 + \cdots + h_s$. As $\sum_B a_B \operatorname{Reg}_B(A) = \sum_{1\leqslant i\leqslant v} \sum_{B\in G_i} a_B \operatorname{Reg}_B(A)$, it is evident that we may find minimal splitting $h_{i_1} + \cdots + h_{i_{s_i}}$ for each G_i separately and then add the obtained sums: $\sum_i h_{i_1} + \cdots + h_{i_{s_i}}$.

5.2. Continuous category versus smooth category

If f is a continuous map, then the minimal number of points in $\mathrm{Fix}(g^r)$ for all g homotopic to f is equal to $NF_r(f)$ – the invariant introduced by Jiang in [10]. Below we explain the differences between $NF_r(f)$ and $NJD_r[f]$ in terms of Reidemeister graph. First, we remind the reader the definition of $NF_r(f)$. We call a subset $S \subset \mathcal{OR}_n(f)$ *Preceding System* if each essential orbit in $\mathcal{OR}_n(f)$ is preceded by an orbit in S. S is called *Minimal Preceding System* (MPS) if the sum of the depth of elements in S

$$\sum_{H\in S} \mathrm{d}(H)$$

is minimal. The number $NF_r(f)$ is defined as the above least sum i.e. the sum of depth of orbits in an MPS. Of course, always $NJD_r[f] \geqslant NF_r(f)$.

Now, notice that for calculating $NF_r(f)$ we do not care about the values of indices at vertices of the graph, the only information we need is whether the indices are non-zero (the class is essential) or not. Calculating $NJD_r[f]$ we have to realize also indices in each vertex B, which are expressed by the coefficients a_B at Reg_B.

If, during the calculation of $NJD_r[f]$, we attach in each $H \in \mathcal{OR}(f^l)$ of a given MPS some $DD^m(l|r)$ sequence, that may be not enough, because some coefficients a_B at Reg_B may be not realized. As a consequence, usually $NJD_r[f] > NF_r(f)$ and the equality holds only in very special situations.

5.3. $NJD_r[f]$ for simply connected manifolds

The invariant $D_r^m[f]$, which was defined in [4], is equal to the least number of points in $\mathrm{Fix}(g^r)$ for all smooth self-maps g of a simply connected manifold, homotopic to f. If our manifold M is simply connected, then there is the equality $D_r^m[f] = NJD_r^m[f]$ and in such a case all Reidemeister orbits consist of one element. Then $\mathcal{GOR}(f;r)$ is a connected graph, in which the relation "$\preccurlyeq$" is isomorphic to "$|$". As a consequence we get, by Remark 3.9 for fixed $B \in \mathcal{OR}(f^k)$ and every $A \in \mathcal{OR}(f^n)$, that $\mathrm{Reg}_B(A) = \mathrm{reg}_k(n)$.

6. The least number of points in $\mathrm{Fix}(f^r)$ for a smooth self-map of $\mathbb{R}P^3$

We start with recalling some basic information about real projective spaces $\mathbb{R}P^m$ for $m \geqslant 3$, m odd. We may define $\mathbb{R}P^m$ as the quotient space of S^m by the antipodal action of $\mathbb{Z}_2$. Thus we get the universal covering $p : S^m \to \mathbb{R}P^m$ and the fundamental group $\pi_1\mathbb{R}P^m = \mathbb{Z}_2$. Since m is odd, a twist of S^m gives an isotopy from id_{S^m} to $-\mathrm{id}_{S^m}$ which induces a cyclic isotopy of $\mathbb{R}P^m$ based at $\mathrm{id}_{\mathbb{R}P^m}$. This proves that $\mathbb{R}P^m$ is a Jiang space and all, i.e. both, Nielsen classes of the given self-map f of $\mathbb{R}P^m$ have equal indices (cf. [10]).

Since $\mathbb{R}P^m$ is oriented, for m odd, the degree $d = \deg(f)$ of a map $f : \mathbb{R}P^m \to \mathbb{R}P^m$ is defined. Then for the Lefschetz number $L(f)$ we have the equality: $L(f) = 1 - d$. Moreover if d is odd then the homotopy group homomorphism $f_\# : \pi_1\mathbb{R}P^m \to \pi_1\mathbb{R}P^m$ is the isomorphism, hence $\mathcal{R}(f^n) = \mathbb{Z}_2$ for all $n \in \mathbb{N}$ ($\mathcal{R}(f^n)$ denotes the set of Reidemeister classes of f^n). On the other hand, when d is even then $f_\#$ is the zero map and $\mathcal{R}(f^n) = \{*\}$, a set which consist of one point [8].

6.1. d is even

Let d be an even number. Then $\mathcal{R}(f^n) = \{*\}$ for all n, hence the Reidemeister graph looks in the same way as in the simply connected case. Moreover, we notice that if $g : S^m \to S^m$ is a map also of degree d, then $L(f^n) = L(g^n) = 1 - d^n$ hence the Reidemeister graphs of the two maps are isomorphic, and thus $NJD_r^m[f] = D_r^m[g]$. On the other hand, in dimension 3 the complete description of $D_r^3[g]$ for self-maps of S^3 is given in [5]. As a consequence, we have the explicit formulae for $NJD_r^3[f]$ for even d (Theorems 4.2 and 4.6 in [5]).

6.2. d is odd

The case of odd d is much more complicated. The map $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ of odd degree d is homotopic to a map induced by an odd degree self-map of S^m, also of degree d and the induced homomorphism $f_\# : \pi_1\mathbb{R}P^m \to \pi_1\mathbb{R}P^m$ is the identity on $\mathbb{Z}_2$. This implies that the Reidemeister action is trivial:

$$\alpha * \beta = \alpha + \beta - f_\#\alpha = \alpha + \beta - \alpha = \beta,$$

hence $\mathcal{R}(f^n) = \mathbb{Z}_2$, for each n. For the same reason each orbit of Reidemeister classes consists of a single element and thus $\mathcal{OR}(f^n) = \mathbb{Z}_2$.

Now we consider the map $i_{k,l} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$. The above remarks and the formula

$$i_{k,l}[\alpha] = \left[\alpha + f_\#^l\alpha + f_\#^{2l}\alpha + \cdots + f_\#^{k-l}\alpha\right] = [\alpha + \cdots + \alpha] = [k/l \cdot \alpha]$$

imply

$$i_{k,l} = \begin{cases} \mathrm{id}_{\mathbb{Z}_2} & \text{for } k/l \text{ odd}, \\ 0 & \text{for } k/l \text{ even}. \end{cases} \tag{6.1}$$

Let us denote $\mathcal{OR}(f^l) = \{l', l''\}$, $\mathcal{OR}(f^k) = \{k', k''\}$, where l' and k' correspond to neutral element in $\mathbb{Z}_2$. Then:

$$i_{k,l}(l') = k', \tag{6.2}$$

$$i_{k,l}(l'') = \begin{cases} k'' & \text{if } \frac{k}{l} \text{ is odd}, \\ k' & \text{if } \frac{k}{l} \text{ is even}. \end{cases} \tag{6.3}$$

Now we will give an upper bound for the number $NJD_r[f]$ for a map $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ of odd degree d. This will be the sum in the right-hand side of (4.7) of Theorem 4.10 where the summation is extended onto all orbits of Reidemeister classes (including those with $a_B = 0$).

We will represent the given number $r \in \mathbb{N}$ as $r = 2^{e(r)} \cdot r_{\mathrm{odd}}$, where r_{odd} is an odd integer. Let $\zeta(n)$ denote the number of all divisors of the natural number n.

Theorem 6.1. *The map $f : \mathbb{R}P^3 \to \mathbb{R}P^3$, of odd degree d, is homotopic to a smooth map g such that* $\mathrm{Fix}(g^r) \leqslant \zeta(r) + (2^{e(r)+1} - 1)\zeta(r_{\mathrm{odd}})$.

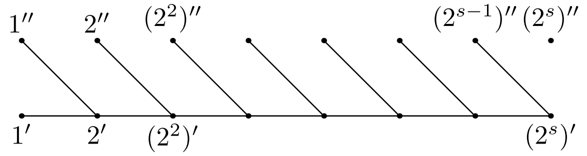

Fig. 1.

Proof. It is enough to show that the number $\zeta(r) + (2^{e(r)+1} - 1) \cdot \zeta(r_{\mathrm{odd}})$ is not less than the sum $\sum_B \mathrm{d}(B)$ in Theorem 4.10, where the summation runs over the set of all orbits of Reidemeister classes B satisfying: $l(B)$ divides r.

As we know by the above considerations $\mathcal{OR}(f^k) = \mathbb{Z}_2 = \{k', k''\}$, furthermore $k' = i_{k,1}(1')$ by (6.2). Thus $\mathrm{d}(k') = 1$ and by (6.3) $\mathrm{d}(k'') = 2^{e(k)}$.

We get

$$\sum_A \mathrm{d}_A = \sum_{k|r} \bigl(\mathrm{d}(k') + \mathrm{d}(k'')\bigr) = \sum_{k|r} \bigl(1 + 2^{e(k)}\bigr) = \zeta(r) + \sum_{k|r} 2^{e(k)}$$
$$= \zeta(r) + \sum_{v=0}^{e(r)} \Bigl(\sum_{p|r_{\mathrm{odd}}} 2^v\Bigr) = \zeta(r) + \zeta(r_{\mathrm{odd}}) \cdot \Bigl(\sum_{v=0}^{e(r)} 2^v\Bigr)$$
$$= \zeta(r) + \zeta(r_{\mathrm{odd}}) \cdot \bigl(2^{e(r)+1} - 1\bigr). \qquad \square \tag{6.4}$$

Remark 6.2. The estimation from Theorem 6.1 may be improved. Let us consider $r = 2^s$, $\mathcal{OR}(f^{2^s}) = \{(2^s)', (2^s)''\}$, $s = 0, 1, \dots, e(r)$. Attaching in the orbits $1'', 2'', 4'', \dots, (2^{e(r)-1})''$ sequences of the type (A) we may realize also $a_B \operatorname{Reg}_B$ for $B \in \{2', 4', \dots, (2^{e(r)})'\}$. As a consequence, the orbits $\{2', 4', \dots, (2^{e(r)})'\}$ may be removed from the sum $\sum_{k|r}(\mathrm{d}(k') + \mathrm{d}(k''))$ in (6.4). Since the depth of each of these orbits equals 1, we get

$$NJD_r[f] \leqslant \zeta(r) + \bigl(2^{e(r)+1} - 1\bigr) \cdot \zeta(r_{\mathrm{odd}}) - e(r). \tag{6.5}$$

Now we give the precise value of $NJD_r[f]$ for some special values of r (d is still odd).

6.2.1. $r = 2^s$

We will use the following convention drawing Reidemeister graphs: if $A \prec B \prec C$ then we omit the edge from A to C, understanding that there is the connection between these two vertices through B.

In this case $i_{k,l} = 0$, hence the Reidemeister graph is given in Fig. 1.

Remark 6.2 gives

$$NJD_r[f] \leqslant \zeta(r) + \bigl(2^{e(r)+1} - 1\bigr)\zeta(r_{\mathrm{odd}}) - e(r)$$
$$= (s+1) + \bigl(2^{s+1} - 1\bigr) - s = 2^{s+1} = 2r. \tag{6.6}$$

On the other hand, by [8] $NF_{2^s}(f) = 2^{s+1} = 2r$. Now (6.6) and the inequality $NF_r(f) \leqslant NJD_r[f]$ imply $NF_r(f) = NJD_r[f] = 2r$. By the above calculations we see that in this case the least number of periodic points (in the continuous homotopy class) may be realized by a smooth map, which is rather an exceptional situation.

6.2.2. $r = 6$

Now we illustrate the theory on some particular example. We take f, a self-map of $\mathbb{R}P^3$ of odd degree, we fix $r = 6$ and we calculate both $NF_6(f)$ and $NJD_6[f]$. In each case we show in details the method of computation, which will reveal the differences between the continuous and smooth cases.

Theorem 6.3. *Let $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ be a map of odd degree d. Then:*

in the continuous case

$$NF_6(f) = \begin{cases} 0 & \text{for } d = 1, \\ 2 & \text{for } d = -1, \\ 4 & \text{otherwise}, \end{cases}$$

in the smooth case

$$NJD_6[f] = \begin{cases} 0 & \text{for } d = 1, \\ 2 & \text{for } d = -1, \\ 7 & \text{for } d = 3, \\ 8 & \text{otherwise}. \end{cases}$$

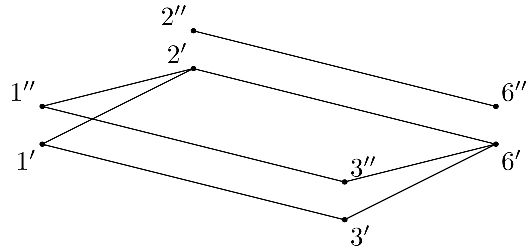

Fig. 2.

Proof. First of all we draw the graph $\mathcal{GOR}(f;6)$, see Fig. 2. We remind the reader that k', k'' denote the orbits of Reidemeister classes in $\mathcal{OR}(f^k)$.

First we will consider the continuous case.

- Let $d=+1$. Then $L(f^k)=0$ for all $k\in\mathbb{N}$, hence all Nielsen classes are inessential and $NF_k(f)=0$ for all $k\in\mathbb{N}$.
- Let $d=-1$. Then $L(f^k)=1-(-1)^k$, hence $L(f^k)=2$ for k odd and $L(f^k)=0$ for k even. In particular for $k|6$ we have essential classes only for $k=1,3$. Then the set $\mathcal{OR}(f)=\{1',1''\}$ is the (unique) MPS which implies $NF_6(f)=\#\mathcal{OR}(f)=2$.
- Let $d\neq\pm1$. Then all involved Nielsen classes are essential. It follows from the graph that the (unique) MPS is $\{1',1'',2''\}$, hence $NF_6(f)=1+1+2=4$.

Now we pass to the smooth case. By Corollary 3.11 we may represent $\operatorname{ind}(f^k;A)$ in the form of generalized periodic expansion. Because for a given k there are only two orbits, $A\in\{k',k''\}$, it has the form:

$$\operatorname{ind}\bigl(f^k;k^*\bigr)=\sum_{l^*}a_{l^*}\operatorname{Reg}_{l^*}\bigl(k^*\bigr), \tag{6.7}$$

where $k^*\in\{k',k''\}$, $l^*\in\{l',l''\}$.

Our aim is to find the decomposition of $\sum_{l^*}a_{l^*}\operatorname{Reg}_{l^*}(k^*)$ into the minimal (in the sense of Definition 4.3) sum of functions C_{H_i} defined in (4.1). Let us remind that each C_{H_i} is a $DD^3(h_i|r)$ sequence attached at H_i. We have quite a lot of information about the form of C_{H_i} in our case. As the dimension of the manifold is equal to 3 here, we may use the description of $DD^3(1)$ sequences given in Theorem 2.6. There are seven types of such sequences (A)–(G).

By Remark 2.7 any $DD^3(p)$ sequence may be obtained from (A)–(G) by replacing all $a_k\operatorname{reg}_k$ by $a_k\operatorname{reg}_{pk}$, so there are also seven types of $DD^3(p)$ sequences. We will say that the given $DD^3(p)$ sequence is of the type (X), where $X\in\{A,B,C,D,E,F,G\}$, if it comes from $DD^3(1)$ sequence of the type (X).

Thus each C_{H_i} may be written as $C^X_{i^*}$, a $DD^3(i|r)$ sequence c_X of the type (X) attached at the class i^*.

- Let $d=1$. Then, as we have seen, all the classes are inessential, hence $NJD_6[f]=0$.
- Let $d=-1$. Then, as above, the only essential classes are $\{1',3',1'',3''\}$. Moreover $L(f^3)=L(f)=2$, hence, each of these four classes has index $+1$. We notice that
$$\operatorname{ind}\bigl(f^k,k^*\bigr)=\operatorname{Reg}_{1'}\bigl(k^*\bigr)+\operatorname{Reg}_{1''}\bigl(k^*\bigr)-\operatorname{Reg}_{2'}\bigl(k^*\bigr).$$
We may realize it by $C^A_{1'}=\operatorname{Reg}_{1'}-\operatorname{Reg}_{2'}$ and $C^A_{1''}=\operatorname{Reg}_{1''}$, two sequences of the type (A) attached at $1'$ and $1''$ respectively.
This implies that $NJD_6[f]=2$.
- Let $d=3$. We get
$$\begin{aligned}\operatorname{ind}\bigl(f^k;k^*\bigr)={}&\\ &-\operatorname{Reg}_{1'}\bigl(k^*\bigr)-\operatorname{Reg}_{1''}\bigl(k^*\bigr)-\operatorname{Reg}_{2'}\bigl(k^*\bigr)-4\operatorname{Reg}_{3'}\bigl(k^*\bigr)-4\operatorname{Reg}_{3''}\bigl(k^*\bigr)-56\operatorname{Reg}_{6'}\bigl(k^*\bigr)\\ &-2\operatorname{Reg}_{2''}\bigl(k^*\bigr)-60\operatorname{Reg}_{6''}\bigl(k^*\bigr),\end{aligned} \tag{6.8}$$
where in the second row of (6.8) there is a decomposition of the big component and in the third, the small one.
We will show that the contribution of the big component to $NJD_6[f]$ is equal to 3 and the small one is equal to 4, thus by Remark 5.1 $NJD_6[f]=3+4=7$.

The big component

$$\begin{aligned}&-\operatorname{Reg}_{1'}\bigl(k^*\bigr)-\operatorname{Reg}_{1''}\bigl(k^*\bigr)-\operatorname{Reg}_{2'}\bigl(k^*\bigr)-4\operatorname{Reg}_{3'}\bigl(k^*\bigr)\\ &\quad-4\operatorname{Reg}_{3''}\bigl(k^*\bigr)-56\operatorname{Reg}_{6'}\bigl(k^*\bigr),\end{aligned} \tag{6.9}$$

may be rewritten in the form:

$$\begin{aligned}&\mathrm{Reg}_{1'} - 4\,\mathrm{Reg}_{3'} - 56\,\mathrm{Reg}_{6'}\\ &\quad - 2\,\mathrm{Reg}_{1'} - \mathrm{Reg}_{2'}\\ &\quad - \mathrm{Reg}_{1''} - 4\,\mathrm{Reg}_{3''}.\end{aligned}$$

As a result, we may realize generalized periodic expansion of (6.9) in the form

$$C^F_{1'}(k^*) + C^A_{1'}(k^*) + C^C_{1''}(k^*),$$

i.e. as the sum of three $DD^3(1|r)$ sequences of the type (F), (A), (C) respectively, attached at $1'$ or $1''$. Namely:

$c_F = \mathrm{reg}_1 - 4\,\mathrm{reg}_3 - 56\,\mathrm{reg}_6$, attached at $1'$,
$c_A = -2\,\mathrm{reg}_1 - \mathrm{reg}_2$, attached at $1'$,
$c_C = -\mathrm{reg}_1 - 4\,\mathrm{reg}_3$, attached at $1''$. Finally, the contribution of the big component to $NJD_6[f]$ is equal to $1+1+1=3$ because the smaller decomposition is impossible (we must use at least two $DD^3(1|6)$ sequences to realize indices at $1'$ and $1''$ but it is immediate that two such sequences are not enough).

Now let us consider the impact on $NJD_6[f]$ which comes from the small component. Since there are only two orbits (vertices) in the small component, two $DD^3(2|6)$ sequences, of the type (A) and (D), attached at $2''$ will do, so $NJD_6[f] \leqslant 2+2=4$. On the other hand, suppose that one such $DD^3(2|6)$ sequence attached at $2''$ realizes both coefficients in the periodic expansion of the small component. Since $\mathrm{ind}(f^2; 2'') = \frac{1-3^3}{2} = -4$, $a_1 = -2$ and thus this sequence must be of the type (A):

$$c_A = a_1\,\mathrm{reg}_2 + a_2\,\mathrm{reg}_4,$$

hence it will not realize $a_{6''} = 60 \neq 0$.

Finally, the small component gives the contribution equal to 4, and thus $NJD_6[f] = 3+4=7$.

Now we consider the remaining cases of large d i.e.

- $d \notin \{-1, +1, 3\}$.

First of all we notice that the contribution of the smaller component is 4 (we use the same arguments as above).

Now we show that the contribution of the large component is at least 4. Notice that:

$$\begin{aligned}a_{1'} &= a_{1''} = \frac{1-d}{2} \notin \{0, -1, +1\},\\ a_{3'} &= a_{3''} = 1/2\left(\frac{1-d^3-(1-d)}{3}\right) = \frac{d^3-d}{6} \neq 0.\end{aligned}$$

Thus, we must use at least one $DD^3(1|6)$ sequence of the type (A) attached at $1'$ to realize index at $1'$, but this is not enough to realize index at $3'$ (as $a_{3'} \neq 0$), so we need one more sequence attached at $1'$ to do so. For the same reason we have to use at least two sequences to realize indices at $1''$ and $3''$.

On the other hand, we see that four sequences will do. In fact,

$$\begin{aligned}&a_{1'}\,\mathrm{Reg}_{1'} + a_{1''}\,\mathrm{Reg}_{1''} + a_{2'}\,\mathrm{Reg}_{2'} + a_{3'}\,\mathrm{Reg}_{3'} + a_{3''}\,\mathrm{Reg}_{3''} + a_{6'}\,\mathrm{Reg}_{6'} =\\ &\quad \mathrm{Reg}_{1'} + a_{3'}\,\mathrm{Reg}_{3'} + a_{6'}\,\mathrm{Reg}_{6'} \qquad (= C^F_{1'})\\ &\quad + a_{3''}\,\mathrm{Reg}_{3''} \qquad (= C^D_{1''})\\ &\quad + (a_{1'}-1)\,\mathrm{Reg}_{1'} + a_{2'}\,\mathrm{Reg}_{2'} \qquad (= C^A_{1'})\\ &\quad + a_{1''}\,\mathrm{Reg}_{1''} \qquad (= C^A_{1''}),\end{aligned}$$

which gives the contribution $1+1+1+1=4$. As a result, taking into account both components $NJD_6[f] = 4+4=8$. □

6.2.3. d is odd, r is odd

If r is odd then all $i_{k,l}$ are isomorphisms and the Reidemeister graph splits into two connected components. An example, $\mathcal{GOR}(f; 15)$, is given in Fig. 3.

Let us represent the graph of orbits of Reidemeister classes $\mathcal{GOR}(f, r) = C' \cup C''$ as the union of connected components.

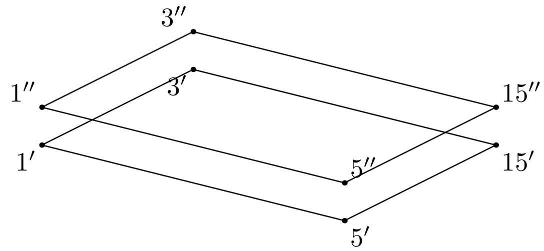

Fig. 3.

The formula (6.7) takes the following forms

$$\operatorname{ind}(f^k; k') = \sum_{l'|k} a_{l'} \operatorname{reg}_{l'}(k'), \tag{6.10}$$

$$\operatorname{ind}(f^k; k'') = \sum_{l''|k} a_{l''} \operatorname{reg}_{l''}(k''). \tag{6.11}$$

The coefficients $a_{l'}$, $a_{l''}$ are equal and non-zero (for all $k|r$).

In fact, since $\mathbb{R}P^3$ is a Jiang space, $\operatorname{ind}(f^l; l') = \operatorname{ind}(f^l; l'')$ implies $a_{l'} = a_{l''}$. On the other hand, $L(f^k) = 1 - d^k = L(\tilde{f}^k)$, where $\tilde{f}: S^3 \to S^3$ denotes a map of degree d. Now by Theorem 1.2 in [11] all coefficients in the expansion

$$L(\tilde{f}^k) = \sum_{l|k} a_l \operatorname{reg}_l(k)$$

are non-zero, hence $a_{l'} = a_{l''} = a_l/2$ are also non-zero.

As a result, we may apply the simply connected methods applicable to S^3 to calculate minimal decompositions for (6.10) and (6.11). Using Definition 2.3 we define the set $G = \mathbf{\Delta}_r(\{L(\tilde{f}^n)\}_n) \setminus \{1,2,4\} = \mathbf{\Delta}_r(\{L(f^n)\}_n) \setminus \{1,2,4\}$.

By Theorem 4.10 in [4] we get

$$NJD_r[\tilde{f}] = \begin{cases} \#G & \text{if } |L(\tilde{f})| \leqslant \#G, \\ \#G+1 & \text{otherwise.} \end{cases}$$

Thus, the same result holds for each component C' and C'' each of which gives the same contribution to $NJD_r[f]$. We obtain

$$NJD_r[f] = \begin{cases} 2\#G & \text{if } |L(f)| \leqslant 2\#G, \\ 2(\#G+1) & \text{otherwise.} \end{cases}$$

Now taking into account that for S^3 and odd r there is $\#G = \zeta(r) - 1$ (cf. [5]), we get

$$NJD_r[f] = \begin{cases} 2\zeta(r) - 2 & \text{if } -2\zeta(r) + 3 \leqslant d \leqslant 2\zeta(r) - 1, \\ 2\zeta(r) & \text{otherwise.} \end{cases} \tag{6.12}$$

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