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MINIMIZATION OF THE NUMBER OF PERIODIC POINTS FOR SMOOTH SELF-MAPS OF CLOSED SIMPLY-CONNECTED 4-MANIFOLDS

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Abstract. Let M be a smooth closed simply-connected m-dimensional manifold, f be a smooth self-map of M and r be a given natural number. The invariant $D_r^m[f]$ defined by the authors in [Forum Math. 21 (2009)] is equal to the minimum of $\#\mathrm{Fix}(g^r)$ over all maps g smoothly homotopic to f. In this paper we calculate the invariant $D_r^4[f]$ for the class of smooth self-maps of 4-manifolds with fast grow of Lefschetz numbers and for r being a product of different primes.

1. **Introduction.** One of the fundamental problems in periodic point theory is to find minimal number of periodic points in the homotopy class of a given map. Let f be a self-map of a compact manifold M. B. Jiang introduced in 1983 the invariant $NF_r(f)$ which estimates from above $\#\mathrm{Fix}(g^r)$ for all g homotopic to f [14]. J. Jezierski proved in 2006 that the invariant is the best estimation if the dimension of M is at least 3 [12]. This means that $NF_r(f)$ is equal to the minimal number of elements in $\mathrm{Fix}(g^r)$ over all g homotopic to f. In the last years the invariant was computed in many special cases, see for example: [10], [13], [16], [18].

In the recent papers [4], [6] the authors developed the theory for the smooth (i.e. C^1) category, searching for the minimum in smooth homotopy class. As a result, two counterparts of $NF_r(f)$ were found: $D_r^m[f]$ for simply-connected manifolds [4] and its generalization $NJD_r^m[f]$ for non simply-connected ones [6]. The crucial demanding for effective computation of the invariants is the knowledge of all sequences of local fixed point indices of iterations at a periodic p-orbit for smooth maps in the given dimension m, called $DD^m(p)$ sequences. This information was provided in dimension 3 in the paper [9], which made it possible to compute the value of $D_r^3[f]$ for $S^2 \times I$ [4], S^3 [5], a two-holed 3-dimensional closed ball [3] and $NJD_r^3[f]$ for $\mathbb{R}P^3$ [7]. Recently, in [8] we provided the list of all possible sequences of local indices of iterations in arbitrary dimension, which allows one to calculate the invariants for self-maps of higher dimensionional manifolds. In this paper we partially realize this

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programme for simply-connected manifolds and $m = 4$. We calculate $D^4_r[f]$ under the assumption that the so-called *periodic expansion* of $\{L(f^n)\}_{n=1}^{\infty}$, the sequence of the Lefschetz numbers of iterations, has only non-zero coefficients. This property holds for example for maps with fast grow of the sequence of Lefschetz numbers, such as self-maps of S^4 with degree d satisfying $|d| > 1$.

The paper is organized in the following way. First, in Section 2 we give the definition of $D^m_r[f]$ which is expressed in terms of $DD^m(p)$ sequences. Next, in Section 3 we provide the list of all $DD^m(1)$ sequences and prove that in order to calculate $D^4_r[f]$ it is enough to use only $DD^4(1)$ sequences. Finally, in Section 4 we calculate $D^4_r[f]$ for r being a product of different primes (Theorem 4.8).

2. **The invariant $D^m_r[f]$.** The notion of *Differential Dold* sequences (DD sequences in short) introduced in [4] is used in the definition of the invariant $D^m_r[f]$. A $DD^m(p)$ sequence is a sequence of integers that can be locally realized as a sequence of indices on an isolated p-orbit for some smooth map.

Definition 2.1. A sequence of integers $\{c_n\}_{n=1}^{\infty}$ is called a $DD^m(p)$ sequence if there is a C^1 map $\phi : U \to \mathbb{R}^m$ $(U \subset \mathbb{R}^m)$ and its isolated p-orbit P such that $c_n = \operatorname{ind}(\phi^n, P)$. If this equality holds for $n|r$, where r is fixed, then the finite sequence $\{c_n\}_{n|r}$ will be called a $DD^m(p|r)$ sequence.

Let r be fixed. The minimal decomposition of the sequence of Lefchetz numbers of iterations into $DD^m(p|r)$ sequences gives the value of $D^m_r[f]$.

Definition 2.2. Let $\{L(f^n)\}_{n|r}$ be a finite sequence of Lefschetz numbers. We decompose $\{L(f^n)\}_{n|r}$ into the sum:

$$L(f^n) = c_1(n) + \ldots + c_s(n), \tag{1}$$

where c_i is a $DD^m(l_i|r)$ sequence for $i = 1, \ldots, s$. Each such decomposition determines the number $l = l_1 + \ldots + l_s$. We define the number $D^m_r[f]$ as the smallest l which can be obtained in this way.

The invariant $D^m_r[f]$ is equal to the minimal number of r-periodic points in smooth homotopy class of f.

Theorem 2.3. ([4]) *Let M be a smooth closed connected and simply-connected manifold of dimension $m \geq 3$ and $r \in \mathbb{N}$ a fixed number. Then,*

$$D^m_r[f] = \min\{\#\mathrm{Fix}(g^r) : g \text{ is smoothly homotopic to } f\}.$$

Periodic expansion is a convenient method of storing the data connected with the sequence of indices of iterations. Each such sequence can be expanded as a combination of some basic periodic sequences $\{\mathrm{reg}_k\}_n$ taken with integral coefficients.

Definition 2.4. For a given k we define the basic sequence:

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if} \quad k|n, \\ 0 & \text{if} \quad k \nmid n. \end{cases}$$

A sequence of indices of iterations (as well as a sequence of Lefchetz numbers of iterations) may be written down in the form of *periodic expansion* (cf. [15]), namely:

$$\operatorname{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \mathrm{reg}_k(n), \tag{2}$$

where $a_n = \frac{1}{n}\sum_{k|n}\mu(k)\mathrm{ind}(f^{(n/k)}, x_0)$, a_n are integers, μ is the classical Möbius function, i.e. $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^s$ if k is a product of s different primes, $\mu(k) = 0$ otherwise.

The fact that the coefficients a_n are integers follows from the result of Dold [2].

The invariant $D_r^m[f]$ is defined in terms of $DD^m(p)$ sequences. On the other hand, it is enough to know only the forms of $DD^m(1)$ sequences, because every $DD^m(p)$ sequence can be obtained from some $DD^m(1)$ one.

Definition 2.5. We will say that the $DD^m(p)$ sequence $\{\tilde{c}_n\}_n$ comes from the given $DD^m(1)$ sequence $\{c_n\}_n$ with the periodic expansion $c_n = \sum_{d=1}^{\infty} a_d \mathrm{reg}_d(n)$ if the periodic expansion of $\{\tilde{c}_n\}_n$ has the form:

$$\tilde{c}_n = \sum_{d=1}^{\infty} a_d \mathrm{reg}_{pd}(n).$$

Theorem 2.6 ([4])**.** *Every $DD^m(p)$ sequence comes from some $DD^m(1)$ sequence.*

3. **Local indices of iterations in dimension** 4**.** In this section we give the complete list of all sequences of local indices of iterations of a smooth map in dimension 4 i.e. the list of all $DD^4(1)$ sequences. Let us mention here that the forms of indices of iterations for continuous maps are known since 1991 [1], and recently indices of iterations have been found also for other important classes of maps, such as holomorphic maps [22] and planar homeomorphisms [17], [21].

Definition 3.1. Let H be a finite subset of natural numbers, we introduce the following notation.

By $\mathrm{LCM}(H)$ we mean the least common multiple of all elements in H with the convention that $\mathrm{LCM}(\emptyset) = 1$. We define the set $\overline{H}$ by: $\overline{H} = \{\mathrm{LCM}(Q) : Q \subset H\}$.

For natural s we denote by $L(s)$ any set of natural numbers of the form $\overline{L}$ with $\#L = s$ and $1, 2 \notin L$.

By $L_2(s)$ we denote any set of natural numbers of the form $\overline{L}$ with $\#L = s + 1$ and $1 \notin L$, $2 \in L$.

Theorem 3.2 (**Main Theorem I in** [8])**.** *Let f be a C^1 self-map of $\mathbb{R}^m$, where m is even. Then the sequence of local indices of iterations $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ has one of the following forms:*

$$(A^e)\ \ \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-4}{2})} a_k \mathrm{reg}_k(n).$$

$$(B^e)\ \ \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m-2}{2})} a_k \mathrm{reg}_k(n).$$

$$(C^e), (D^e), (E^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-2}{2})} a_k \mathrm{reg}_k(n),$$

where

$$a_1 = \begin{cases} 1 & \textit{in the case } (C^e), \\ -1 & \textit{in the case } (D^e), \\ 0 & \textit{in the case } (E^e). \end{cases}$$

$$(F^e) \ \ \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m}{2})} a_k \mathrm{reg}_k(n),$$

where $a_1 = 1$.

By $[d, l]$ we denote the least common multiple of d and l.

Theorem 3.3. *The list of all* $DD^4(1)$ *sequences is the following:*
(A) $\quad c_A(n) = a_1\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n)$;
(B) $\quad c_B(n) = a_1\mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n)$;
$(C-E)_{odd}$

$$c_X(n) = \varepsilon_X\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n) + a_d\mathrm{reg}_d(n) + a_{2d}\mathrm{reg}_{2d}(n),$$

where $\varepsilon_X \in \{-1, 0, 1\}$, $X \in \{C, D, E\}$, d *is odd.*
$(C-E)_{even}$

$$c_X(n) = \varepsilon_X\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n) + a_d\mathrm{reg}_d(n),$$

where $\varepsilon_X \in \{-1, 0, 1\}$, $X \in \{C, D, E\}$, d *is even.*
(F) $\ c_F(n) = \mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n) + a_l\mathrm{reg}_l(n) + a_{[d,l]}\mathrm{reg}_{[d,l]}(n)$,
In all cases $d, l \geq 3$ *and* $a_i \in \mathbb{Z}$.

Proof. We apply Theorem 3.2 for $m = 4$, obtaining the corresponding parts of the thesis. For example, to obtain the case (F), we use (F^e) and get:

$$L(\frac{m}{2}) = L(2) = \overline{\{d, l\}} = \mathrm{LCM}\{Q \subset \{d, l\}\} = \{1, d, l, [d, l]\}.$$

□

Corollary 1. *Let us notice that any* $DD^4(1)$ *sequence has one of the following forms:*

1. $a_1\mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n)$;
   for $a_1, a_d \in \mathbb{Z}$.
2. $\varepsilon\mathrm{reg}_1(n) + a_2\mathrm{reg}_2(n) + a_d\mathrm{reg}_d(n) + \gamma_d a_{2d}\mathrm{reg}_{2d}(n)$;
   for $a_2, a_d \in \mathbb{Z}$, $\varepsilon = 0, \pm 1$, $\gamma_d = 0$ *if* d *is even and* $\gamma_d = 1$ *if* d *is odd.*
3. $\mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n) + a_l\mathrm{reg}_l(n) + a_{[d,l]}\mathrm{reg}_{[d,l]}(n)$;
   for $a_d, a_l \in \mathbb{Z}$, $d, l \geq 3$.

The next two lemmas show that during the calculation of $D^4_r[f]$ we may consider only $DD^4(1)$ sequences, which makes the computation much easier.

Lemma 3.4 (Remark 4.6 in [4])**.** *For* $m \geq 3$ *in Definition 2.2 of* $D^m_r[f]$ *we can equivalently use only* $DD^m(p|r)$ *sequences such that* $p < 2^{[\frac{m+1}{2}]}$.

Lemma 3.5. *To calculate* $D^4_r[f]$ *it is enough to consider only* $DD^4(1)$ *sequences.*

Proof. By Lemma 3.4 it is enough to consider only such $DD^4(p|r)$ sequences for which $p \leq 3$.

We show that
(1) every $DD^4(2|r)$ sequence is a sum of at most two $DD^4(1|r)$ sequences.
(2) every $DD^4(3|r)$ sequence is a sum of at most three $DD^4(1|r)$ sequences.

Proof of **(1)**. Using Theorem 2.6 we find the forms of all $DD^4(2|r)$ sequences, each of which comes from some $DD^4(1|r)$ sequences of one of the types (A)-(F). Next, we represent each $DD^4(2|r)$ sequence as a sum of at most two $DD^4(1|r)$ sequences.

(A) $a_2\text{reg}_2(n) + a_4\text{reg}_4(n)$ is in fact the $DD^4(1|r)$ sequence of the type $(D)_{even}$.
(B) $a_2\text{reg}_2(n) + a_{2d}\text{reg}_{2d}(n)$ the same argument as above is true.
(C-E) For d odd we have that $[4, 2d] = 4d$, then
$\varepsilon_X\text{reg}_2(n) + a_4\text{reg}_4(n) + a_{2d}\text{reg}_{2d}(n) + a_{4d}\text{reg}_{4d}(n) =$
$\text{reg}_1(n) + a_4\text{reg}_4(n) + a_{2d}\text{reg}_{2d}(n) + a_{4d}\text{reg}_{4d}(n) \quad (F)$
$-\text{reg}_1(n) + \varepsilon_X\text{reg}_2(n) \quad (A)$
where on the right-hand side of the above formula we indicated that the first sum is realized by a sequence of the type (F) and the second by (A).

In the same way we deal with the case of d even (every sequence is a sum of a sequence of the type (F) and (A)).

(F) Notice that $[2d, 2l] = 2[d, l]$, thus we get
$\text{reg}_2(n) + a_{2d}\text{reg}_{2d}(n) + a_{2l}\text{reg}_{2l}(n) + a_{2[d,l]}\text{reg}_{2[d,l]}(n) =$
$-\text{reg}_1(n) + \text{reg}_2(n)+ \quad (A)$
$+\text{reg}_1(n) + a_{2d}\text{reg}_{2d}(n) + a_{2l}\text{reg}_{2l}(n) + a_{[2d,2l]}\text{reg}_{[2d,2l]}(n). \quad (F)$

Proof of **(2)**. Let us now consider a $DD^4(3|r)$ sequence.

Notice that by Theorem 2.6 and Corollary 1 it has always the form with no more than four basic sequences reg_i, i.e.

$$a_p\text{reg}_p(n) + a_q\text{reg}_q(n) + a_r\text{reg}_r(n) + a_s\text{reg}_s(n),$$

where $p, q, r, s \geq 3$. Then we may represent this sequence as a sum of three $DD^4(1|r)$ sequences in the following way:
$-\text{reg}_1(n) + a_p\text{reg}_s+ \quad (B)$
$+a_q\text{reg}_q(n)+ \quad (D)$
$+\text{reg}_1(n) + a_r\text{reg}_r(n) + a_s\text{reg}_s(n) \quad (F)$
This completes the proof. □

4. **Calculation of the invariant.** We work under the following standing assumptions

Standing Assumptions

1. *$f : M^4 \to M^4$ is a smooth self-map of a smooth closed connected and simply-connected 4-manifold,*
2. *$r = p_1 \dots p_s$ is a product of different prime numbers,*
3. *in the periodic expansion of Lefschetz numbers*

$$L(f^k) = \sum_{i=1}^{\infty} a_i\text{reg}_i(k)$$

$a_i \neq 0$ for all $i \neq 1$ dividing r.

Remark 1. The assumption (3) is satisfied for a self-map $f : S^4 \to S^4$ with $|\deg(f)| > 1$ [20]. In general, it often takes place if the growth of $\{L(f^k)\}_k$ is quick.

We will find the formula for $D_r^4[f]$, under the above assumptions.

It turns out that first it is convenient to find the minimal decomposition of the sum

$$\sum_{i|r} a_i\text{reg}_i$$

into $DD^4(1|r)$ sequences *modulo reg_1* i.e. we require that the equality holds only for all divisors $i|r$ different than 1. In other words, we will temporarily ignore the coefficient at reg_1.

Lemma 4.1. *The two following numbers are equal:*

1. *the minimal number of summands in the decomposition of the sum*
$$\sum_{i|r} a_i \mathrm{reg}_i$$
modulo reg_1 *into* $DD^4(1|r)$ *sequences,*
2. *the minimal number* $h(s)$ *determining the family of pairs of subsets of* $I_s = \{1,\dots,s\}$*:*
$$\{A_1, B_1\}, \{A_2, B_2\}, \dots, \{A_{h(s)}, B_{h(s)}\}$$
such that
$$\bigcup_{i=1}^{h(s)} \{A_i, B_i, A_i \cup B_i\} = 2^{I_s} \setminus \{\emptyset\}$$
i.e. for each nonempty subset $X \subset I_s$ *there is an* i *such that either* $X = A_i$ *or* $X = B_i$ *or* $X = A_i \cup B_i$.

Proof. Let us notice that to get the minimal decomposition of
$$\sum_{i|r} a_i \mathrm{reg}_i \quad \text{modulo} \quad \mathrm{reg}_1,$$
we should use as much as possible the most "greedy" $DD^4(1|r)$ sequences, with the greatest number of basic expressions reg_i i.e. of the type (2) or (3) of Corollary 1. In both of these cases we have the sequences of the form:
$$\varepsilon \mathrm{reg}_1 + a_d \mathrm{reg}_d + a_l \mathrm{reg}_l + \gamma a_{[d,l]} \mathrm{reg}_{[d,l]}, \tag{3}$$
where d, l are divisors of r different than 1, $\gamma \in \{0,1\}$.

Since $r = p_1 \cdots p_s$ is a product of different primes, there is a bijection $G : 2^{I_s} \to \mathrm{Div}(r)$ between $\mathrm{Div}(r)$, the set of all divisors of r, and the family of all subsets of $I_s = \{1,\dots,s\}$:
$$\{1,\dots,s\} \supset A \to \Pi_{i\in A}\, p_i \in \mathrm{Div}(r),$$
with the convention that $\Pi_{i\in\emptyset} p_i = 1$. Moreover
$$G(A \cup B) = [G(A), G(B)].$$

As a result, every triple of divisors d, l, $[d,l]$ determining the sequence (3) is associated with a triple of subsets of I_s: A_j, B_j, $A_j \cup B_j$.

Now, a decomposition of the sum $\sum_{1\neq i|r} a_i \mathrm{reg}_i(k)$ into $h(s)$ $DD^4(1|r)$ sequences of the form (3) is equivalent to the existence of $h(s)$ families of subsets of I_s
$$\{A_1, B_1\}, \{A_2, B_2\}, \dots, \{A_{h(s)}, B_{h(s)}\}$$
such that
$$\bigcup_{i=1}^{h(s)} \{A_i, B_i, A_i \cup B_i\} = 2^{I_s} \setminus \{\emptyset\}$$
i.e. for each nonempty subset $X \subset I_s$ there is an i such $X = A_i$, $X = B_i$ or $X = A_i \cup B_i$. □

Now our problem reduces to the following combinatorial question:

Problem 4.1. *Let* s *be a natural number. Find the minimal number* $h(s)$ *such that there exist* $h(s)$ *families of subsets* $\mathcal{A}_1, \dots, \mathcal{A}_{h(s)} \subset 2^{I_s}$ *satisfying*

1. $\#\mathcal{A}_i \leq 2$ *i.e. each family consists of at most two subsets,*

2. *for each nonempty subset* $X \subset \{1, \dots, s\}$ *there exists* $i \in \{1, \dots, s\}$ *such that* X *is one of the sets* A_i, B_i *or* $A_i \cup B_i$, *where* $\mathcal{A}_i = \{A_i, B_i\}$.

Theorem 4.2. *The minimal number searched in Problem 4.1 is given by the formula*

$$h(s) = \frac{2^s + (-1)^{s+1}}{3}. \tag{4}$$

The proof of Theorem 4.2 is a consequence of the following three lemmas.

Lemma 4.3. *The formula (4) for* $h(s)$ *can be given inductively as follows:*

$$h(2) = 1, \quad h(s+1) = 2 \cdot h(s) + (-1)^s.$$

Proof.

$$2 \cdot h(s) + (-1)^s = 2 \cdot \frac{2^s + (-1)^{s+1}}{3} + (-1)^s$$
$$= \frac{2^{s+1} + 2 \cdot (-1)^{s+1} + 3 \cdot (-1)^s}{3} = \frac{2^{s+1} + (-1)^s}{3} = h(s+1).$$

□

Lemma 4.4. $h(s)$ *given by the formula (4) is less or equal to the minimal number satisfying the conditions in Problem 4.1.*

Proof. We notice that each family containing two subsets $\{A, B\} \subset 2^{I_s}$ determines at most three nonempty subsets $A, B, A \cup B \subset I_s$. Thus, to realize all nonempty subsets in I_s we need at least $(2^s - 1)/3$ pairs. The last means that the minimal number in Problem 4.1 is greater or equal to $(2^s - 1)/3$. On the other hand, the least natural number $\geq (2^s - 1)/3$ is equal to $(2^s - 1)/3$ when s is even and $(2^s + 1)/3$ when s is odd. It remains to notice that in both cases we get $h(s)$. □

Lemma 4.5. *(I) For each* $s \geq 2$ *there exist* $h(s) = \frac{2^s + (-1)^{s+1}}{3}$ *families satisfying the conditions in Problem 4.1.*

(II) Moreover, if s *is even then each family must contain two different subsets, while if* s *is odd then* $h(s) - 1$ *families must contain two different subsets and the last family can contain only one subset consisting of a single, arbitrarily chosen, element.*

Proof. We will show inductively that (for $s \geq 2$): there exists a family $\mathcal{A}_s = \{\{A_i, B_i\} : i = 1, \dots, h(s)\}$ whose elements are nonempty subsets $A_i, B_i \subset I_s$ realizing all nonempty subsets in I_s and moreover

1. $A_i \neq B_i$ if $i = 1, \dots, h(s)$ and s is even,
2. $A_i \neq B_i$ if $i = 1, \dots, h(s) - 1$ and s is odd.
3. $A_{h(s)} = B_{h(s)} = \{s\}$ for s odd.

For $s = 2$ all nonempty subsets of $I_2 = \{1, 2\}$ can be obtained from the family $\{\{1\}, \{2\}\}$ which implies $h(2) = 1$.

Now we assume that for even s a family $\mathcal{A}_s = \{\{A_i, B_i\} : i = 1, \dots, h(s)\}$ where $A_i \neq B_i$ realizes all nonempty subsets in $I_s = \{1, \dots, s\}$. Then the family

$$\mathcal{A}_{s+1} = \{\{A_i, B_i\}, \{A_i \cup \{s+1\}, B_i \cup \{s+1\}\}, \{\{s+1\}\}\} : i = 1, \dots, h(s)\}$$

realizes all nonempty subsets in $I_{s+1} = \{1, \dots, s, s+1\}$. Moreover,

$$\#\mathcal{A}_{s+1} = 2 \cdot \#\mathcal{A}_s + 1 = 2 \cdot h(s) + 1 = 2 \cdot h(s) + (-1)^s = h(s+1)$$

since s is even.

Now, the family

$$\mathcal{A}_{s+2} = \{\{A'_i, B'_i\}, \{A'_i \cup \{s+2\}, B'_i \cup \{s+2\}\}, \{\{s+1\}, \{s+2\}\}\}$$
$$\text{where } \{A'_i, B'_i\} \in \mathcal{A}_{s+1} \setminus \{\{s+1\}\}$$

realizes all subsets in I_{s+2} and moreover

$$\#\mathcal{A}_{s+2} = 2 \cdot (\#\mathcal{A}_{s+1} - 1) + 1 = 2 \cdot h(s+1) - 1 = 2 \cdot h(s+1) + (-1)^{s+1} = h(s+2)$$

since $s+1$ is odd.

This ends the proof of part (I). Part (II) of Lemma 4.5 follows from Lemma 4.4 and the observation that for $s+1$ odd in the above inductive construction, the family $\{\{s+1\}\}$, i.e. the last element in $\mathcal{A}_{s+1}$, consists of one subset containing a single element. It is evident that after a permutation $\{\{s+1\}\}$ can be replaced with $\{\{i\}\}$ for an arbitrarily prescribed $i \in I_{s+1}$. □

Proof of Theorem 4.2

Lemma 4.4 gives

$$h(s) \leq \text{ minimal number in Problem 4.1}$$

while Lemma 4.5 proves the opposite inequality. □

By Theorem 4.2 we obtain

Corollary 2. *The minimal decomposition of the sum*

$$\sum_{i|r} a_i \mathrm{reg}_i$$

modulo reg_1 *into* $DD^4(1|r)$ *sequences contains exactly*

$$h(s) = \frac{2^s + (-1)^{s+1}}{3}$$

sequences.

Moreover, by Lemma 4.5 (II) we get:

(A) if s is even then the minimal decomposition must contain $h(s)$ sequences of the type

$$\varepsilon \cdot \mathrm{reg}_1 + a_d \mathrm{reg}_d + a_l \mathrm{reg}_l + \gamma a_{[d,l]} \mathrm{reg}_{[d,l]}, \tag{5}$$

i.e. of the form (2) or (3) of Corollary 1 ($\gamma \in \{0,1\}$);

(B) if s is odd then the minimal decomposition must contain $h(s)-1$ sequences of the type (5) while the remaining sequence may be $a_1\mathrm{reg}_1(n) + a_d\mathrm{reg}_d(n)$ (i.e. of the type (1) of Corollary 1), where $d \neq 1$ is an arbitrarily prescribed divisor of r.

Remark 2. Let us notice that in all sequences (5), appearing in the minimal decomposition modulo 1 described in Corollary 2, the divisors d, l must be different as they correspond to different subsets in Lemma 4.5, so both $\mathrm{reg}_d(n)$ and $\mathrm{reg}_l(n)$ appear with nonzero coefficients.

Now we are in a position to find the formula for $D_r^4[f]$, i.e. we take into account also the coefficient at reg_1.

Let us remark that $D_r^4[f] \geq h(s)$. In fact, in the minimal realization modulo reg_1 we need $h(s)$ of $DD^4(1|r)$ sequences. The following lemmas make it precise when these sequences are sufficient to obtain the decomposition with $a_1\mathrm{reg}_1$ and when one additional sequence, to realize $a_1\mathrm{reg}_1$, is necessary.

Lemma 4.6. *Assume our Standing Assumptions are satisfied and s is even, then*

$$D^4_r[f] = \begin{cases} h(s) & \text{if } (r \text{ is odd and } L(f) = h(s)) \\ & \text{or } (r \text{ is even and } h(s) - 2 \le L(f) \le h(s)), \\ h(s) + 1 & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2 (A) to realize

$$\sum_{1 \ne i | r} a_i \mathrm{reg}_i$$

we need at least $h(s)$ $DD^4(1|r)$ sequences of the type (2) or (3) of Corollary 1.

If we assume that r is odd then they all must be of the type (3). Then the contribution of each of them to the coefficient at reg_1 is 1. If moreover $L(f) = h(s)$ then $D^4_r[f] = h(s)$, since $a_1 = L(f)$. Otherwise, we need one sequence of the type (1) more to realize the difference $(a_1 - h(s)) \cdot \mathrm{reg}_1(n)$.

Now we consider the case of even r. Then exactly one sequence in the minimal decomposition must be of the type (2) and the remaining $h(s) - 1$ sequences are of the type (3). Their contribution to the coefficient at reg_1 is $(h(s) - 1) + \varepsilon$ where $\varepsilon = 0, +1, -1$. Now, if $h(s) - 2 \le L(f) \le h(s)$, then a_1 can be realized by these sequences. Otherwise we need one more sequence of the type (1). $\square$

Lemma 4.7. *Assume our Standing Assumptions are satisfied and s is odd, then*

$$D^4_r[f] = h(s).$$

Proof. It is enough to show that $\sum_{i|r} a_i \mathrm{reg}_i(n)$ is the sum of exactly $h(s)$ $DD^4(1|r)$ sequences.

Since s is odd, by Corollary 2 (B), $h(s) - 1$ sequences of the types (2) or (3) of Corollary 1 realize

$$\sum_i a_i \mathrm{reg}_i,$$

where the summation runs through the set $\mathrm{Div}(r) \setminus \{1, d\}$, for some $d|r$. Again by Corollary 2 (B), it remains to add one expression of the type (1) realizing the sum $a_1\mathrm{reg}_1 + a_d\mathrm{reg}_d$. $\square$

We sum up our considerations in the following

Theorem 4.8. *Let $f : M^4 \to M^4$ be a smooth self-map of a smooth closed connected and simply-connected 4-manifold, $r = p_1 \dots p_s$ be a product of different prime numbers. We assume that the coefficients a_i in the periodic expansion of $L(f^k) = \sum_{i=1}^{\infty} a_i \mathrm{reg}_i(k)$, are nonzero for all $i|r$, $i \ne 1$. Then*

$$D^4_r[f] = \begin{cases} h(s) & \text{if } (s \text{ is odd }) \text{ or } (r \text{ is odd and } L(f) = h(s)) \\ & \text{or } (r \text{ is even and } h(s) - 2 \le L(f) \le h(s)), \\ h(s) + 1 & \text{otherwise.} \end{cases}$$

where $h(s) = (2^s + (-1)^{s+1})/3$.

Remark 3. If in Theorem 4.8 we drop the part (3) of the Standing Assumption according which $a_i \ne 0$ for all $i \ne 1$ dividing r then the equality becomes the inequality and we get the estimation for $D^4_r[f]$ from above.

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