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New Proofs of Some Fibonacci Identities

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Abstract

Lucas proved in 1876 several identities for Fibonacci numbers. We give elementary and short proofs of them.

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By Fibonacci sequence we mean the sequence $\{F_n\}_{n=1}^{\infty}$ such that $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$, for $n \geq 3$. The elements of this sequence are called Fibonacci numbers. Lucas proved in 1876 that for every positive integer n we have $F_{2n+1} = F_n^2 + F_{n+1}^2$, $F_{2n} = F_{n+1}^2 - F_{n-1}^2$, $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$, $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$, see [1], pages 69, 71, and 79. We give combinatorial proofs of these identities which are elementary and short.

Let us consider dominoes of dimensions 2×1 and an area of dimensions $2 \times n$, where n is a positive integer. Squares of our area are signed as follows: upper from left to right by integers from 1 to n, and lower from left to right by symbols from $1'$ to n'. By the i-th column we mean the pair of squares i and i'. By the position of a domino we mean the set of squares on which this domino lies. The covering of the area is the set of positions of dominoes which cover this area. Two coverings are distinguish if and only if proper sets of positions are different. Let the sequence $\{a_n\}_{n=1}^{\infty}$ be such that a_n is the number of distinguish coverings of the area of dimensions $2 \times n$. For example, $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, see Figure 1. We also define $a_0 = 1$.

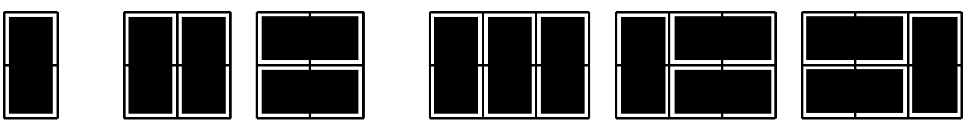

Figure 1

In the following lemma we give a recursive formula for a_n.

Lemma 1 *For every positive integer $n \geq 3$ we have $a_n = a_{n-2} + a_{n-1}$.*

Proof. If we want to cover the area of dimensions $2 \times n$, then the domino on the square 1 lies horizontally or vertically, see Figure 2.

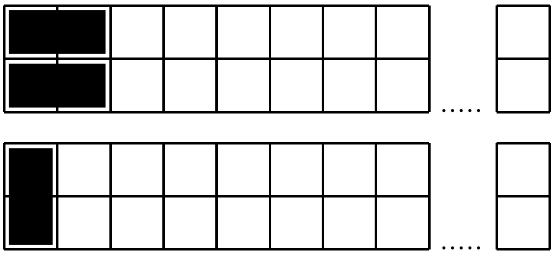

Figure 2

If it lies horizontally, then we have to cover the rest area of dimensions $2\times (n-2)$. There are a_{n-2} possibilities of doing that. If it lies vertically, then we have to cover the rest area of dimensions $2 \times (n-1)$. There are a_{n-1} possibilities of doing that. Adding these numbers, we get $a_n = a_{n-2} + a_{n-1}$. ■

Now we prove a relation between elements of sequences $\{a_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$.

Lemma 2 *If n is a positive integer, then $a_n = F_{n+1}$.*

Proof. We have $a_1 = 1 = F_2$, $a_2 = 2 = F_3$, and the same recurrent formula effects for both sequences $\{a_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$, so $a_n = F_{n+1}$. ■

Now let us prove the following formula for a Fibonacci number with an odd index.

Theorem 3 (Lucas, 1876) *For every positive integer n we have*

$$F_{2n+1} = F_n^2 + F_{n+1}^2.$$

Proof. By Lemma 2, the identity above is equivalent to $a_{2n} = a_{n-1}^2 + a_n^2$. Let us consider two halfs of the area of dimensions $2 \times 2n$ (two areas of dimensions $2 \times n$ each). If we cover the area of dimensions $2 \times 2n$, then its halfs have common dominoes or do not have common dominoes, see Figure 3.

In the first case it remains to cover independently two areas of dimensions $2\times\ (n-\ 1)$ each, there are a_{n-1}^2 possibilities of doing that. In the second case we have to cover independently two areas of dimensions $2 \times n$ each, there are a_n^2 possibilities of doing that. Adding these numbers, we get $a_{2n} = a_{n-1}^2 + a_n^2$. ■

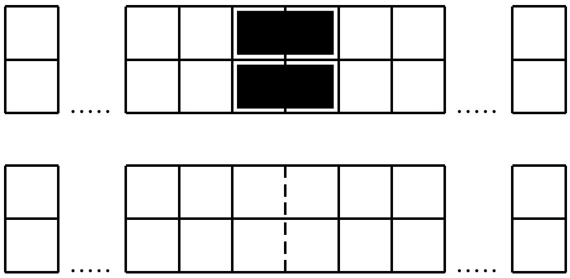

Figure 3

Now we prove a formula for a Fibonacci number with an even index.

Theorem 4 (Lucas, 1876) *If n is a positive integer, then*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

Proof. By Lemma 2, the identity above is equivalent to $a_{2n-1} = a_n^2 - a_{n-2}^2$. Since $a_n^2 - a_{n-2}^2 = (a_{n-2} + a_{n-1})^2 - a_{n-2}^2 = a_{n-1}^2 + 2a_{n-2}a_{n-1}$, it suffices to prove that $a_{2n-1} = a_{n-1}^2 + 2a_{n-2}a_{n-1}$. If we cover the area of dimensions $2 \times (2n-1)$, then there are the following three possibilities respecting the column n: it is covered by a domino lying vertically, or it is covered (together with column $n-1$) by pair of dominoes lying horizontally, or it is covered (together with column $n+1$) by pair of dominoes lying horizontally, see Figure 4.

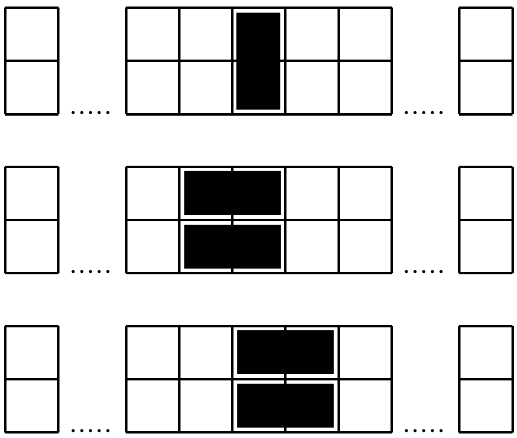

Figure 4

In the first case it remains to cover independently two areas of dimensions $2 \times (n-1)$ each, there are a_{n-1}^2 possibilities of doing that. In the second case it remains to cover independently two areas of dimensions $2 \times (n-2)$ and $2 \times n$, there are $a_{n-2}a_n$ possibilities of doing that. In the third case, by symmetry to the previous case, there are also $a_{n-2}a_n$ possibilities. Adding these numbers, we get $a_{2n-1} = a_{n-1}^2 + 2a_{n-2}a_{n-1}$. ■

Now let us observe that Theorem 4 can be also easily proved using Theorem 3. We have $F_{2n} = F_{2n+1} - F_{2n-1} = F_n^2 + F_{n+1}^2 - F_{n-1}^2 - F_n^2 = F_{n+1}^2 - F_{n-1}^2$. Theorem 3 similarly follows from Theorem 4, as $F_{2n+1} = F_{2n+2} - F_{2n} = F_{n+2}^2 - F_n^2 - F_{n+1}^2 + F_{n-1}^2 = (F_n + F_{n+1})^2 - F_n^2 - F_{n+1}^2 + (F_{n+1} - F_n)^2 = F_n^2 + 2F_nF_{n+1} + F_{n+1}^2 - F_n^2 - F_{n+1}^2 + F_{n+1}^2 - 2F_nF_{n+1} + F_n^2 = F_n^2 + F_{n+1}^2$.

Now let us prove a formula for n first Fibonacci numbers.

Theorem 5 (Lucas, 1876) *For every positive integer n we have*

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

Proof. By Lemma 2, the identity above is equivalent to $\sum_{i=0}^{n-1} a_i = a_{n+1} - 1$. Let us consider all possible coverings of the area of dimensions $2 \times (n+1)$ excluding the covering in which every domino lies vertically. There are $a_{n+1} - 1$ such coverings. Now let us count these coverings in different way. Since we exclude the covering in which every domino lies vertically, every considered covering has at least one pair of dominoes lying horizontally (one above another). Let us consider the pair of indices of columns which are covered by first (considering from left side) pair of dominoes lying horizontally. "The smallest" such possible pair is $(1, 2)$, "the next" possible is $(2, 3)$, and "the greatest" possible is $(n, n + 1)$, see Figure 5.

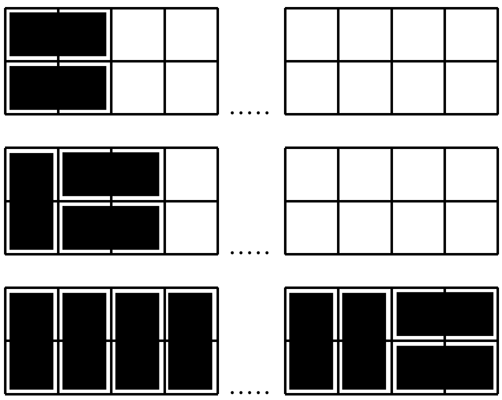

Figure 5

There are a_i possibilities of covering the area of dimensions $2 \times (n + 1)$ in such way that the first pair of dominoes lying horizontally covers the columns $n - i$ and $n - i + 1$. Thus the number of distinguish coverings of the area of dimensions $2 \times (n+1)$ excluding the covering in which every domino lies vertically is equal to $\sum_{i=0}^{n-1} a_i$. ■

Now we prove the following formula for n first Fibonacci numbers with even indices.

Theorem 6 (Lucas, 1876) *If n is a positive integer, then*

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1.$$

Proof. The identity above is equivalent to $\sum_{i=1}^{n} a_{2i-1} = a_{2n} - 1$. Let us consider all possible coverings of the area of dimensions $2 \times 2n$ excluding the covering in which every domino lies horizontally. There are $a_{2n} - 1$ such coverings. Let us count these coverings in different way. Since we exclude the covering in which every domino lies horizontally, every considered covering has a domino lying vertically. Let us consider the column which is covered by the first (considering from the left side) domino lying vertically. The index of this column is odd, because the part of the area on the left side of that column has an even length, as it is covered only by dominoes lying horizontally. The smallest such possible index is 1, the next possible is 3, and the greatest possible is $2n-1$, see Figure 6.

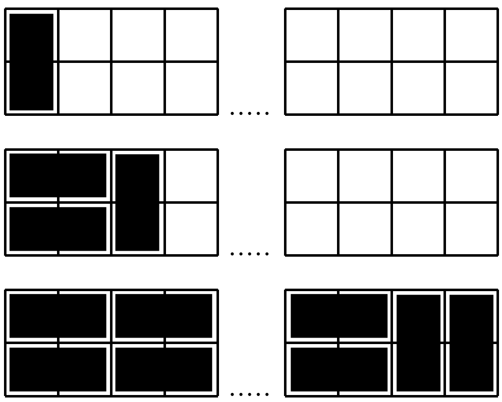

Figure 6

There are a_{2i-1} possibilities of covering the area of dimensions $2 \times 2n$ in such way that the first (considering from the left side) domino lying vertically covers the column $2n-2i-1$. Thus the number of coverings of the area of dimensions $2 \times 2n$ excluding the covering in which every domino lies horizontally is equal to $\sum_{i=1}^{n} a_{2i-1}$. ■

Now we prove a formula for n first Fibonacci numbers with odd indices.

Theorem 7 (Lucas, 1876) *For every positive integer n we have*

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}.$$

Proof. The identity above is equivalent to $\sum_{i=0}^{n-1} a_{2i} = a_{2n-1}$. Let us consider all possible coverings of the area of dimensions $2 \times (2n-1)$. There are a_{2n-1} such coverings. Let us count these coverings in different way. Every counted covering has a domino lying vertically, because our area has an odd length. Let us consider the column which is covered by first (considering from the left side) domino lying vertically. The index of this column is odd, because the part of the area on the left side of that column has an even length, as it is covered only by dominoes lying horizontally. The smallest such possible index is 1, the next possible is 3, and the greatest possible is $2n-1$, see Figure 7.

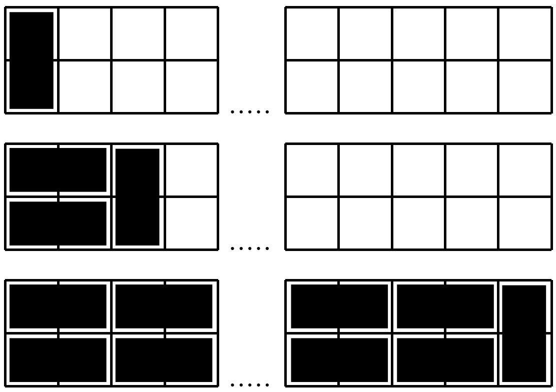

Figure 7

There are a_{2i} possibilities of covering the area of dimensions $2 \times (2n-1)$ in such way that the first (considering from the left side) domino lying vertically covers the column $2n-2i-1$. Thus there are $\sum_{i=0}^{n-1} a_{2i}$ distinguish coverings of the area of dimensions $2 \times (2n-1)$. ■

It is easy to see that Theorem 5 follows from Theorems 6 and 7, as $\sum_{i=1}^{2n} F_i = \sum_{i=1}^{n} F_{2i-1} + \sum_{i=1}^{n} F_{2i} = F_{2n} + F_{2n+1} - 1 = F_{2n+2} - 1$. Similarly, Theorems 6 and 7 follow from each other.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, Canada, 2001.

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