



## Combinatorics

## A lower bound on the total outer-independent domination number of a tree

*Une borne inférieure pour le cardinal des sous-ensembles totalement dominants et extérieurement-indépendants d'un arbre*

Marcin Krzywkowski

Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland

## ARTICLE INFO

## Article history:

Received 28 February 2010

Accepted after revision 19 November 2010

Available online 18 December 2010

Presented by the Editorial Board

## ABSTRACT

A total outer-independent dominating set of a graph  $G$  is a set  $D$  of vertices of  $G$  such that every vertex of  $G$  has a neighbour in  $D$ , and the set  $V(G) \setminus D$  is independent. The total outer-independent domination number of a graph  $G$ , denoted by  $\gamma_t^{oi}(G)$ , is the minimum cardinality of the total outer-independent dominating set of  $G$ . We prove that for every nontrivial tree  $T$  of order  $n$  with  $l$  leaves we have  $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$ , and we characterize the trees attaining this lower bound.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## R É S U M É

Un sous-ensemble totalement dominant et extérieurement indépendant d'un graphe est un sous-ensemble  $D$  des sommets de  $G$  tel que chaque sommet de  $G$  ait un voisin dans  $D$  et l'ensemble  $V(G) \setminus D$  soit indépendant. Le plus petit cardinal d'un tel sous-ensemble est noté  $\gamma_t^{oi}(G)$ . Nous démontrons que pour tout arbre  $T$  non trivial, d'ordre  $n$  avec  $l$  feuilles, nous avons  $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$ . De plus, nous caractérisons les arbres réalisant cette borne inférieure.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

Let  $G = (V, E)$  be a graph. By the neighbourhood of a vertex  $v$  of  $G$  we mean the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is the cardinality of its neighbourhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The path on  $n$  vertices we denote by  $P_n$ . We say that a subset of  $V(G)$  is independent if there is no edge between every two its vertices. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves.

A subset  $D \subseteq V(G)$  is a dominating set of  $G$  if every vertex of  $V(G) \setminus D$  has a neighbour in  $D$ , while it is a total dominating set of  $G$  if every vertex of  $G$  has a neighbour in  $D$ . The domination (total domination, respectively) number of  $G$ , denoted by  $\gamma(G)$  ( $\gamma_t(G)$ , respectively), is the minimum cardinality of a dominating (total dominating, respectively) set

E-mail address: fevernova@wp.pl.

of  $G$ . Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2]. For a comprehensive survey of domination in graphs, see [3,4].

A subset  $D \subseteq V(G)$  is a total outer-independent dominating set, abbreviated TOIDS, of  $G$  if every vertex of  $G$  has a neighbour in  $D$ , and the set  $V(G) \setminus D$  is independent. The total outer-independent domination number of  $G$ , denoted by  $\gamma_t^{oi}(G)$ , is the minimum cardinality of a total outer-independent dominating set of  $G$ . A total outer-independent dominating set of  $G$  of minimum cardinality is called a  $\gamma_t^{oi}(G)$ -set. The study of total outer-independent domination in graphs was initiated in [5].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every nontrivial tree  $T$  of order  $n$  with  $l$  leaves we have  $\gamma_t(T) \geq (n - l + 2)/2$ . They also characterized the extremal trees.

We prove the following lower bound on the total outer-independent domination number of a tree. For every nontrivial tree  $T$  of order  $n$  with  $l$  leaves we have  $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$ . We also characterize the trees attaining this lower bound.

## 2. Results

We begin with the following two straightforward observations.

**Observation 1.** Every support vertex of a graph  $G$  is in every  $\gamma_t^{oi}(G)$ -set.

**Observation 2.** For every connected graph  $G$  of diameter at least three there exists a  $\gamma_t^{oi}(G)$ -set that contains no leaf.

We show that if  $T$  is a nontrivial tree of order  $n$  with  $l$  leaves, then  $\gamma_t^{oi}(T)$  is bounded below by  $(2n - 2l + 2)/3$ . For the purpose of characterizing the trees attaining this bound we introduce a family  $\mathcal{T}$  of trees  $T = T_k$  that can be obtained as follows. Let  $T_1$  be a path  $P_4$  with support vertices labelled  $x$  and  $y$ , and let  $A(T_1) = \{x, y\}$ . Let  $H$  be a path  $P_3$  with a leaf labelled  $u$ , and the support vertex labelled  $v$ . If  $k$  is a positive integer, then  $T_{k+1}$  can be obtained recursively from  $T_k$  by one of the following operations.

- Operation  $\mathcal{O}_1$ : Attach a vertex by joining it to any vertex of  $A(T_k)$ . Let  $A(T_{k+1}) = A(T_k)$ .
- Operation  $\mathcal{O}_2$ : Attach a copy of  $H$  by joining  $u$  to any leaf of  $T_k$ . Let  $A(T_{k+1}) = A(T_k) \cup \{u, v\}$ .

Now we prove that for every tree  $T$  of the family  $\mathcal{T}$ , the set  $A(T)$  defined above is a TOIDS of minimum cardinality equal to  $(2n - 2l + 2)/3$ .

**Lemma 3.** If  $T \in \mathcal{T}$ , then the set  $A(T)$  defined above is a  $\gamma_t^{oi}(T)$ -set of size  $(2n - 2l + 2)/3$ .

**Proof.** We use the terminology of the construction of the trees  $T = T_k$ , the set  $A(T)$ , and the graph  $H$  defined above. To show that  $A(T)$  is a  $\gamma_t^{oi}(T)$ -set of cardinality  $(2n - 2l + 2)/3$  we use the induction on the number  $k$  of operations performed to construct  $T$ . If  $T = T_1 = P_4$ , then  $(2n - 2l + 2)/3 = (8 - 4 + 2)/3 = 2 = |A(T)| = \gamma_t^{oi}(T)$ . Let  $k \geq 2$  be an integer. Assume that the result is true for every tree  $T' = T_k$  of the family  $\mathcal{T}$  constructed by  $k - 1$  operations. Let  $n'$  be the order of the tree  $T'$  and  $l'$  the number of its leaves. Let  $T = T_{k+1}$  be a tree of the family  $\mathcal{T}$  constructed by  $k$  operations.

If  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_1$ , then  $n = n' + 1$ . Observe that  $A(T')$  contains no leaf. Thus  $l = l' + 1$ . It is easy to see that  $A(T) = A(T')$  is a TOIDS of the tree  $T$ . Thus  $\gamma_t^{oi}(T) \leq |A(T)| = |A(T')| = \gamma_t^{oi}(T')$ . Of course,  $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T')$ . This implies that  $\gamma_t^{oi}(T) = |A(T)| = |A(T')| = (2n' - 2l' + 2)/3 = (2n - 2 - 2l + 2 + 2)/3 = (2n - 2l + 2)/3$ .

Now assume that  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_2$ . We have  $n = n' + 3$  and  $l = l'$ . It is easy to see that  $A(T) = A(T') \cup \{u, v\}$  is a TOIDS of the tree  $T$ . Thus  $\gamma_t^{oi}(T) \leq |A(T)| = |A(T')| + 2 = \gamma_t^{oi}(T') + 2$ . Let us denote by  $w$  the neighbour of  $u$  other than  $v$  and by  $x$  a neighbour of  $w$  other than  $u$ . First assume that there exists a  $\gamma_t^{oi}(T)$ -set that does not contain  $w$ . Thus  $u, v \in D$ . It is easy to see that  $D \setminus \{u, v\}$  is a TOIDS of the tree  $T'$ . Now assume that every  $\gamma_t^{oi}(T)$ -set contains  $w$ . Since  $\text{diam}(T) \geq 3$ , let  $D$  be a  $\gamma_t^{oi}(T)$ -set that contains no leaf. Thus  $u, v \in D$ . If  $x \in D$ , then it is easy to see that  $D \setminus \{u, v\}$  is a TOIDS of the tree  $T'$ . Now suppose that  $x \notin D$ . Since  $T' \in \mathcal{T}$ , we have  $T' \neq P_2$ . This implies that  $d_{T'}(x) = d_T(x) \geq 2$ . Since  $x \notin D$  and the set  $V(T) \setminus D$  is independent, every neighbour of  $x$  belongs to the set  $D$ . Let us observe that  $D \cup \{x\} \setminus \{w\}$  is a TOIDS of the tree  $T$  that does not contain  $w$ , a contradiction. Since in every case  $D \setminus \{u, v\}$  is a TOIDS of the tree  $T'$ , we get  $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$ . Now we conclude that  $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 2$ . We get  $\gamma_t^{oi}(T) = |A(T)| = \gamma_t^{oi}(T') + 2 = |A(T') \cup \{u, v\}| = (2n' - 2l' + 2)/3 + 2 = (2n - 6 - 2l + 2 + 6)/3 = (2n - 2l + 2)/3$ .  $\square$

Now we establish the main result, a lower bound on the total outer-independent domination number of a tree together with the characterization of the extremal trees.

**Theorem 4.** If  $T$  is a nontrivial tree of order  $n$  with  $l$  leaves, then  $\gamma_t^{oi}(T) \geq (2n - 2l + 2)/3$  with equality if and only if  $T \in \mathcal{T}$ .

**Proof.** If  $\text{diam}(T) = 1$ , then  $T = P_2$ . We have  $(2n - 2l + 2)/3 = (4 - 4 + 2)/3 = 2/3 < 2 = \gamma_t^{oi}(T)$ . If  $\text{diam}(T) = 2$ , then  $T$  is a star  $K_{1,m}$ . We have  $n = m + 1$  and  $l = m$ . Now we get  $(2n - 2l + 2)/3 = (2m + 2 - 2m + 2)/3 = 4/3 < 2 = \gamma_t^{oi}(T)$ . Now let us

assume that  $\text{diam}(T) = 3$ . Thus  $T$  is a double star. If  $T = P_4$ , then  $T \in \mathcal{T}$ , and by Lemma 3 we have  $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$ . Now assume that  $T$  is a double star different than  $P_4$ . By Observation 1, for any double star  $T^*$  of the family  $\mathcal{T}$  both support vertices belong to every  $\gamma_t^{oi}(T^*)$ -set. Lemma 3 implies that they belong to the set  $A(T^*)$  defined earlier. Therefore the tree  $T$  can be obtained from  $P_4$  by proper numbers of operations  $\mathcal{O}_1$  performed on the support vertices. Thus  $T \in \mathcal{T}$ . By Lemma 3 we have  $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$ .

Now we assume that  $\text{diam}(T) \geq 4$ . Thus the order of the tree  $T$  is an integer  $n \geq 5$ . We obtain the result by induction on the number  $n$ . Assume that the theorem is true for every tree  $T'$  of order  $n' < n$  with  $l'$  leaves.

First assume that some support vertex of  $T$ , say  $x$ , is adjacent to at least two leaves. One of them let us denote by  $y$ . Let  $T' = T - y$ . We have  $n' = n - 1$  and  $l' = l - 1$ . Since every  $\gamma_t^{oi}(T')$ -set, as well as every  $\gamma_t^{oi}(T)$ -set, contains every support vertex, it is easy to observe that  $\gamma_t^{oi}(T) = \gamma_t^{oi}(T')$ . Now we get  $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') \geq (2n' - 2l' + 2)/3 = (2n - 2 - 2l + 2 + 2)/3 = (2n - 2l + 2)/3$ . If  $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$ , then obviously  $\gamma_t^{oi}(T') = (2n' - 2l' + 2)/3$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . By Observation 1, the vertex  $x$  is in every TOIDS of the tree  $T'$ . Lemma 3 implies that  $x \in A(T')$ . Therefore the tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ . Henceforth, we can assume that every support vertex of  $T$  is adjacent to exactly one leaf.

We now root  $T$  at a vertex  $r$  of maximum eccentricity  $\text{diam}(T)$ . Let  $v$  be a support vertex at maximum distance from  $r$ ,  $u$  be the parent of  $v$ , and  $w$  be the parent of  $u$  in the rooted tree. By  $T_x$  let us denote the subtree induced by a vertex  $x$  and its descendants in the rooted tree  $T$ . We distinguish between the following two cases:  $d_T(u) \geq 3$  and  $d_T(u) = 2$ .

**Case 1.**  $d_T(u) \geq 3$ . First assume that  $u$  has a child  $b \neq v$  that is a support vertex. Let  $T' = T - T_v$ . We have  $n' = n - 2$  and  $l' = l - 1$ . Let  $D$  be a  $\gamma_t^{oi}(T)$ -set that contains no leaf. Thus  $u, v, b \in D$ . Of course,  $D \setminus \{v\}$  is a TOIDS of the tree  $T'$ . Therefore  $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1$ . Now we get  $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 1 \geq (2n' - 2l' + 2)/3 + 1 = (2n - 4 - 2l + 2 + 2 + 3)/3 = (2n - 2l + 3)/3 > (2n - 2l + 2)/3$ .

Now assume that  $v$  is the only one support vertex among the descendants of  $u$ . Thus  $u$  is a parent of a leaf, say  $x$ . Let  $T' = T - x$ . We have  $n' = n - 1$  and  $l' = l - 1$ . Let  $D$  be any  $\gamma_t^{oi}(T)$ -set. We have  $u, v \in D$ . It is easy to see that  $D$  is a TOIDS of the tree  $T'$ . Therefore  $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T)$ . Now we get  $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') \geq (2n' - 2l' + 2)/3 = (2n - 2 - 2l + 2 + 2)/3 = (2n - 2l + 2)/3$ . If  $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$ , then obviously  $\gamma_t^{oi}(T') = (2n' - 2l' + 2)/3$ . By the inductive hypothesis we have  $T' \in \mathcal{T}$ . It follows from the definition of the family  $\mathcal{T}$  that for every tree  $T^* \in \mathcal{T}$  the set  $A(T^*)$  does not contain any leaf. Lemma 3 implies that  $A(T')$  is a TOIDS of the tree  $T'$ . Since  $v$  has to have a neighbour in  $A(T)$ , we have  $u \in A(T')$ . Therefore the tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_1$ . Thus  $T \in \mathcal{T}$ .

**Case 2.**  $d_T(u) = 2$ . We consider the following two possibilities:  $d_T(w) = 2$  and  $d_T(w) \geq 3$ . First assume that  $d_T(w) = 2$ . The parent of  $w$  let us denote by  $x$ . If  $d_T(x) = 1$ , then  $T = P_5$ . We have  $(2n - 2l + 2)/3 = (10 - 4 + 2)/3 = 8/3 < 3 = \gamma_t^{oi}(T)$ . Now assume that  $T \neq P_5$ . Thus  $d_T(x) \geq 2$ . First let us prove that there exists a  $\gamma_t^{oi}(T)$ -set that does not contain  $w$ . Assume that there exists a  $\gamma_t^{oi}(T)$ -set  $D$  that contains  $w$ . If  $x \notin D$ , then every neighbour of  $x$  belongs to  $D$  as the set  $V(T) \setminus D$  is independent. It is easy to see that  $D \cup \{x\} \setminus \{w\}$  is a TOIDS of the tree  $T$  of cardinality  $|D| = \gamma_t^{oi}(T)$ . Thus  $D \cup \{x\} \setminus \{w\}$  is a  $\gamma_t^{oi}(T)$ -set that does not contain  $w$ . If  $x \in D$ , then no neighbour of  $x$  besides  $w$  belongs to the set  $D$ , otherwise  $D \setminus \{w\}$  is a TOIDS of the tree  $T$  of cardinality  $|D| = \gamma_t^{oi}(T) - 1$ , a contradiction. Let  $y$  be any neighbour of  $x$  besides  $w$ . Observe that  $D \cup \{y\} \setminus \{w\}$  is a TOIDS of the tree  $T$  of cardinality  $|D| = \gamma_t^{oi}(T)$ . Thus  $D \cup \{y\} \setminus \{w\}$  is a  $\gamma_t^{oi}(T)$ -set that does not contain  $w$ . Now we conclude that there exists a  $\gamma_t^{oi}(T)$ -set that does not contain  $w$ . Let  $D$  be such a set. Of course, we have  $u, v \in D$ . Let  $T' = T - T_u$ . We have  $n' = n - 3$  and  $l' = l$ . Let us observe that  $x \in D$  as  $w \notin D$  and the set  $V(T) \setminus D$  is independent. Thus  $D \setminus \{u, v\}$  is a TOIDS of the tree  $T'$ . Therefore  $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$ . Now we get  $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 2 \geq (2n' - 2l' + 2) + 2 = (2n - 6 - 2l + 2 + 6)/3 = (2n - 2l + 2)/3$ . If  $\gamma_t^{oi}(T) = (2n - 2l + 2)/3$ , then we easily get  $\gamma_t^{oi}(T') = (2n' - 2l' + 2)/3$ . By the inductive hypothesis we get  $T' \in \mathcal{T}$ . The tree  $T$  can be obtained from  $T'$  by operation  $\mathcal{O}_2$ . Thus  $T \in \mathcal{T}$ .

Now assume that  $d_T(w) \geq 3$ . First assume some descendant of  $w$  is a leaf. Let  $D$  be a  $\gamma_t^{oi}(T)$ -set that contains no leaf. Thus  $v, u, w \in D$ . The descendant of  $v$  let us denote by  $z$ . Let  $T' = T - z$ . We have  $n' = n - 1$  and  $l' = l$ . It is easy to see that  $D \setminus \{v\}$  is a TOIDS of the tree  $T'$ . Therefore  $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 1$ . Now we get  $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 1 \geq (2n' - 2l' + 2)/3 + 1 = (2n - 2 - 2l + 2 + 3)/3 = (2n - 2l + 3)/3 > (2n - 2l + 2)/3$ .

Now assume that among the descendants of  $w$  there is no leaf. Let  $x$  be a descendant of  $w$  different from  $u$ . Let  $T' = T - T_u$ . We have  $n' = n - 3$  and  $l' = l - 1$ . Let  $D$  be a  $\gamma_t^{oi}(T)$ -set that contains no leaf. We have  $u, v, x \in D$ . Observe that  $D \setminus \{u, v\}$  is a TOIDS of the tree  $T'$ . Therefore  $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$ . Now we get  $\gamma_t^{oi}(T) \geq \gamma_t^{oi}(T') + 2 \geq (2n' - 2l' + 2)/3 + 2 = (2n - 6 - 2l + 2 + 2 + 6)/3 = (2n - 2l + 4)/3 > (2n - 2l + 2)/3$ .  $\square$

## References

- [1] M. Chellali, T. Haynes, A note on the total domination number of a tree, *J. Combin. Math. Combin. Comput.* 58 (2006) 189–193.
- [2] E. Cockayne, R. Dawes, S. Hedetniemi, Total domination in graphs, *Networks* 10 (1980) 211–219.
- [3] T. Haynes, S. Hedetniemi, P. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [4] T. Haynes, S. Hedetniemi, P. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.
- [5] M. Krzywkowski, Total outer-independent domination in graphs, submitted for publication.

