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On homotopy Conley index for multivalued flows in Hilbert spaces

Zdzisław Dzedzej [1]
Faculty of Technical Physics and Applied Mathematics
Gdańsk University of Technology
ul. Narutowicza 11/12, 80-952 Gdańsk, Poland
e-mail:zdzedzej@mif.pg.gda.pl

Grzegorz Gabor
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18, 87–100 Toruń, Poland
e-mail: ggabor@mat.uni.torun.pl

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Abstract

An approximation approach is applied to obtain a homotopy version of the Conley type index in Hilbert spaces considered in [6]. The definition given in the paper is more elementary and, as a by-product, gives a natural connection between indices from [13] and [16] in a finite-dimensional case. Some geometric properties from [7] are discussed in an infinite dimensional situation.

1 Preliminaries on set-valued maps

Let X, Y be metric spaces. By a *set-valued map* φ from X into Y (written $\varphi : X \multimap Y$) we mean a map that assigns to each $x \in X$ a *closed nonempty* subset $\varphi(x)$ of Y. If, for any closed (resp. open) set $U \subset Y$, the *preimage* $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is closed (resp. open), then we say that φ is *upper* (resp. *lower*) semicontinuous (written usc(resp. lsc)); a map φ is *continuous* if it is upper and lower semicontinuous simultaneously. The *graph* $\mathrm{Gr}(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ of an upper semicontinuous map φ is closed. A map φ is upper semicontinuous and has compact values (i.e., for each $x \in X$, the set $\varphi(x)$ is compact) if and only if, for any sequence $(x_n, y_n) \in \mathrm{Gr}(\varphi)$ such that $x_n \to x \in X$, there is a subsequence

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(y_{n_k}) such that $y_{n_k} \to y \in \varphi(x)$ (in other words the projection $p_\varphi : \mathrm{Gr}(\varphi) \to X$ is proper ([2])); in this case the *image* $\varphi(K) := \{y \in Y \mid y \in \varphi(x) \text{ for some } x \in K\}$ of any compact $K \subset X$ is compact. We say that a map φ is *compact* if it is upper semicontinuous and $\mathrm{cl}\,\varphi(X)$ is compact; φ is *completely continuous* if the restriction $\varphi|_B$ of φ to any bounded subset $B \subset X$ is compact.

A proper surjection $p : X \to Y$ is a *Vietoris map* if, for each $y \in Y$, the fibre $p^{-1}(y)$ is acyclic in the sense of the Alexander-Spanier cohomology. A map $p : (X, X') \to (Y, Y')$ of pairs $(X, X'), (Y, Y')$ (i.e. $p : X \to Y$ and $p(X') \subset Y'$) is a *Vietoris* map, if p is a Vietoris map and $p^{-1}(Y') = X'$ (observe that the restriction $p' : X' \to Y'$ of p is a Vietoris map, too). A map $\varphi : X \multimap Y$ is *admissible* (in the sense of Górniewicz) if there exist a space Γ, a Vietoris map $p : \Gamma \to X$ and a continuous map $q : \Gamma \to Y$ such that, for every $x \in X$, $\varphi(x) = q(p^{-1}(x))$. It is clear that admissible maps are upper semicontinuous with nonempty compact values.

The class of admissible maps is rich: for example any acyclic map $\varphi : X \multimap Y$ is admissible (φ is *acyclic* if it is upper semicontinuous and, for any $x \in X$, $\varphi(x)$ is acyclic); it is determined by the pair (p_φ, q_φ) where $p_\varphi : \mathrm{Gr}(\varphi) \to X$ and $q_\varphi : \mathrm{Gr}(\varphi) \to Y$ are the restrictions of the projections $X \times Y \to X$ and $X \times Y \to Y$, respectively. Moreover a superposition of acyclic maps is admissible. For more details concerning admissible maps - see [9]. Let us prove the following elementary:

Proposition 1.1 *Let X be a metric space, E - a linear normed space, and $F : X \multimap E$ a map with nonempty values. Then for each $\varepsilon > 0$ there exists a continuous map $f : X \to E$ such that $f(x) \in \mathrm{conv}(F(B_\varepsilon(x)))$.*

Proof. For each $x \in X$ we can choose a point $v_x \in F(x)$. Let $\{\lambda_s\}_{s \in S}$ be a continuous partition of unity subordinated to the covering $\{V_s\}$, which is a locally finite refinement of the covering $\{B_\varepsilon(x)\}_{x \in X}$. For $s \in S$ we fix a point x_s such that $\mathrm{supp}\,\lambda_s \subset V_s \subset B_\varepsilon(x_s)$. Define a continuous map

$$f(x) := \sum_{s \in S} \lambda_s(x) v_s,$$

where $v_s = v_{x_s}$. If $s \in S_x = \{s |\ \lambda_s(x) \neq 0\}$, then $x \in B_\varepsilon(x_s)$. Thus $x_s \in B_\epsilon(x)$, and hence $v_s \in F(B_\varepsilon(x))$. Therefore $f(x) \in \mathrm{conv} F(B_\epsilon(x))$. □

Remark 1.2 *Observe that E could be a topological vector space in the previous proposition. The map f is locally Lipschitz if we take a locally Lipschitz partition of unity.*

We say that a continuous map $f : X \to Y$ is *a graph ε-approximation* of $\varphi : X \multimap Y$ if $f(x) \in B_\varepsilon(\varphi(B_\varepsilon(x)))$ for every $x \in X$. The following is a version of a classical result of A. Cellina [2] combined with the previous observation.

[2] Recall that a continuous map $f : X \to Y$ is *proper* if, for each compact $K \subset Y$, the preimage $f^{-1}(K)$ is compact; it is worth reminding that f is proper if and only if it is *perfect*, i.e. continuous, closed and such that, for any $y \in Y$, $f^{-1}(y)$ is compact. Observe that a continuous surjection $f : X \to Y$ is perfect if and only if the multivalued map $Y \ni y \multimap f^{-1}(y) \subset X$ is upper semicontinuous and has compact values.

Theorem 1.3 *Let $\varphi : X \multimap E$ be usc with convex values, where X is a metric space and E is a Banach space. Then, for every $\varepsilon > 0$, there exists a locally Lipschitz graph ε-approximation f of φ such that $f(x) \in \mathrm{conv}\varphi(B_\varepsilon(x))$ for every $x \in X$.*

Proof. Let $\varepsilon > 0$. From upper semicontinuity of φ it follows that for every $x \in X$ there exists $0 < \delta(x) < \frac{\varepsilon}{2}$ such that $\varphi(B_{\delta(x)}(x)) \subset B_\varepsilon(\varphi(x))$. Consider a locally finite covering $\{V_s\}_{s\in S}$ of X which is a star-refinement of the covering $\{B_{\delta(x)}(x)\}_{x\in X}$, i.e., stars $\mathrm{st}(V_t) = \bigcup\{V_s : V_s \cap V_t \neq \emptyset\}$ refine the covering $\{B_{\delta(x)}(x)\}_{x\in X}$. Let $\{\lambda_s\}_{s\in S}$ be a locally Lipschitz partition of unity subordinated to the covering $\{V_s\}$. For each $s \in S$ we choose a point $x_s \in V_s$ and some $y_s \in \varphi(x_s)$.

Define

$$f(x) := \sum_{s\in S} \lambda_s(x) y_s.$$

Let $S_x = \{s \in S|\ \lambda_s(x) \neq 0\}$ and let $s \in S_x$. Then $x \in V_s$. It implies that $d(x_s, x) < \delta(x_s) < \varepsilon$ and hence $x_s \in B_\varepsilon(x)$. Therefore

$$f(x) = \sum_{s\in S_x} \lambda_s(x) y_s \in \mathrm{conv}\varphi(B_\varepsilon(x)).$$

Moreover, since $x \in \bigcap_{s\in S_x} V_s$, there exists x' such that $\bigcup_{s\in S_x} V_s \subset B_{\delta(x')}(x')$. Thus both $x, x_s \in V_s$, and thus $d(x, x_s) < 2\delta(x') < \varepsilon$. By our choice of $\delta(x')$ we have $y_s \in B_\varepsilon(\varphi(x'))$. But the latter set is convex, thus $f(x) = \sum \lambda_s(x) y_s \in B_\varepsilon(\varphi(x')) \subset B_\varepsilon(\varphi(B_\varepsilon(x))$ and the proof is complete. □

Corollary 1.4 *If φ is completely continuous, then the approximation f in Theorem 1.3 is also completely continuous.*

Proof. For every bounded set $U \subset X$ we have $f(U) \subset \overline{conv}\varphi(B_\varepsilon(U))$, and the latter set is relatively compact. □

2 Multivalued flows

Let X be a metric space.

Definition 2.1 By a *multivalued flow* on X we mean an upper semicontinuous mapping $\varphi : X \times \mathbb{R} \multimap X$ with nonempty and compact values such that, for every $s, t \in \mathbb{R}$ and $x, y \in X$,

(i) $\varphi(x, 0) = \{x\}$;

(ii) if $s \cdot t \geq 0$, then $\varphi(x, t+s) = \varphi(\varphi(x,t) \times \{s\})$;
(iii) $y \in \varphi(x,t)$ if and only if $x \in \varphi(y, -t)$.

Let $\Delta \subseteq \mathbb{R}$. A map $\sigma : \Delta \to X$ is a Δ-*trajectory of* φ if, for every $t, s \in \Delta$, $\sigma(t) \in \varphi(\sigma(s), t-s)$. It is an easy exercise to prove that every trajectory is continuous. Indeed, let us consider a sequence t_n converging to t_0. Let $U \ni \sigma(t_0)$ be open. Since φ is upper semicontinuous, $\varphi^{-1}(U)$ is open and $(\sigma(t_0), 0) \in \varphi^{-1}(U)$, because $\sigma(t_0) \in \varphi(\sigma(t_0), 0)$. There exist $\delta > 0$ and an open set $V \subset X$ such that $(\sigma(t_0), 0) \in V \times (-\delta, \delta) \subset \varphi^{-1}(U)$. Therefore, for a large n, $|t_n - t_0| < \delta$ and then $\sigma(t_n) \in \varphi(\sigma(t_0), t_n - t_0) \subset U$.

Let $x \in N \subseteq X$. The set of all Δ-trajectories *in* N *originating in* x (i.e., such that $0 \in \Delta$, $\sigma(0) = x$ and $\sigma(t) \in N$ for $t \in \Delta$) is denoted by $\mathrm{Tr}_N(\varphi; \Delta, x)$.

Define the *invariant, right-invariant, left-invariant* (with respect to φ) part of N by:

$$\mathrm{Inv}(N, \varphi) := \{x \in N \mid Tr_N(\varphi; \mathbb{R}, x) \neq \emptyset\},$$

$$\mathrm{Inv}^+(N, \varphi) := \{x \in N \mid Tr_N(\varphi; \mathbb{R}_+, x) \neq \emptyset\},$$

$$\mathrm{Inv}^-(N, \varphi) := \{x \in A \mid Tr_N(\varphi; \mathbb{R}_-, x) \neq \emptyset\},$$

respectively.

Definition 2.2 A subset $K \subset X$ is *invariant* (resp. *positively (negatively) invariant*) with respect to φ if $\mathrm{Inv}(K, \varphi) = K$ (resp. $\mathrm{Inv}^+(K, \varphi) = K$ ($\mathrm{Inv}^-(K, \varphi) = K$)).

Note that, given $N \subset X$, the set $K := \mathrm{Inv}(N, \varphi)$ is invariant with respect to φ and it is the maximal invariant subset of N.

Proposition 2.3 ([6], Proposition 3.9) *Let* Λ *be a metric space,* $N \subset X$ *be closed and let* $\eta : X \times \mathbb{R} \times \Lambda \multimap X$ *be a family of multivalued flows (i.e.,* η *is upper semicontinuous and, for each* $\lambda \in \Lambda$, $\eta(\cdot, \lambda) : X \times \mathbb{R} \multimap X$ *is a multivalued flow). Then the graph of the set-valued map*

$$\Lambda \ni \lambda \mapsto \mathrm{Inv}(N, \eta(\cdot, \lambda))$$

is closed, i.e. for any sequence $(x_n, \lambda_n) \in N \times \Lambda$ *such that* $x_n \in \mathrm{Inv}(N, \eta(\cdot, \lambda_n))$, *if* $(x, \lambda) = \lim_{n\to\infty}(x_n, \lambda_n)$, *then* $x \in \mathrm{Inv}(N, \eta(\cdot, \lambda))$.

Definition 2.4 A closed and bounded set $N \subset X$ is an *isolating neighborhood* for φ if $\mathrm{Inv}(N, \varphi) \subset \mathrm{int}\, N$. We say that a set K invariant with respect to φ is *isolated* if there is an isolating neighborhood N for φ such that $K = \mathrm{Inv}(N, \varphi)$.

In particular, if $X = \mathbb{R}^n$, then each isolating neighborhood N for φ is compact and, by Proposition 2.3, $K = \mathrm{Inv}(N, \varphi)$ is closed in N, hence compact.

3 Conley index in Hilbert spaces

We shall assume the following:

Let $\mathbb{H} = (\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and $L : \mathbb{H} \to \mathbb{H}$ a linear bounded operator with spectrum $\sigma(L)$. We assume the following

- $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$ with all subspaces $\mathbb{H}_k$ being mutually orthogonal and of finite dimension;
- $L(\mathbb{H}_0) \subset \mathbb{H}_0$ where $\mathbb{H}_0$ is the invariant subspace of L corresponding to the part of spectrum $\sigma_0(L) = i\mathbb{R} \cap \sigma(L)$ lying on the imaginary axis,
- $L(\mathbb{H}_k) = \mathbb{H}_k$ for all $k > 0$,
- $\sigma_0(L)$ is isolated in $\sigma(L)$, i.e. $\sigma_0(L) \cap \mathrm{cl}\,(\sigma(L) \setminus \sigma_0(L)) = \emptyset$.

Definition 3.1 A multivalued flow $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is called an *L-flow*, if it has the form

$$\varphi(x,t) = e^{tL}x + U(t,x),$$

where $U : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is an admissible map which is completely continuous.

Let Λ be a metric space. By a *family of L-flows* we understand a set-valued map $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ of the form $\eta(x,t,\lambda) = e^{tL}x + U(x,t,\lambda)$, where $U : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ is an admissible completely continuous mapping, such that, for each $\lambda \in \Lambda$, $\eta(\cdot,\lambda) : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is a multivalued flow.

It is clear that, if $\eta : \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ is a family of L-flows, then, for each $\lambda \in \Lambda$, $\eta(\cdot,\lambda) : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is an L-flow. Moreover, each L-flow is an admissible flow.

Proposition 3.2 ([6], Prop.3.15) *If $X \subset \mathbb{H}$ is closed and bounded, then the set-valued map $\Lambda \ni \lambda \mapsto \mathrm{Inv}(X, \eta(\cdot,\lambda)) \subset X$ is usc and it has compact (possibly empty) values.*

Definition 3.3 An usc mapping $f : \mathbb{H} \multimap \mathbb{H}$ is an *L-vector field* if it is of the form $f(x) = Lx + K(x)$, where $K : \mathbb{H} \multimap \mathbb{H}$ is completely continuous with compact convex values, and if f induces an L-flow π on H.

Given an L-vector field $f := L + F : \mathbb{H} \multimap \mathbb{H}$, F having a sublinear growth (i.e., there is a constant $C > 0$ such that, for each $u \in \mathbb{H}$ and $y \in F(u)$, $\|y\| \leq C(1 + \|u\|)$), the standard fixed point argument (see, e.g., [11], Theorem 5.2.2) implies that, for each $x \in \mathbb{H}$, there is a *mild solution* to the Cauchy problem

$$\begin{cases} u' \in f(u) \quad \text{a.e. on} \quad \mathbb{R}; \\ u(0) = x, \end{cases} \tag{1}$$

i.e., a continuous function $u : \mathbb{R} \to \mathbb{H}$ and a locally (Bochner) integrable function $w : \mathbb{R} \to \mathbb{H}$ such that $w(t) \in F(u(t))$ and $u(t) = e^{tL}x + \int_0^t e^{(t-s)L} w(s)\, ds$ for all $t \in \mathbb{R}$.

Let $S(x) \subset C(\mathbb{R}, \mathbb{H})$ ([3]) be the set of all solutions to (1), $x \in \mathbb{H}$.

Consider a map $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ given by the formula

$$\varphi(x,t) := \{u(t) \mid u \in S(x)\}, \quad x \in \mathbb{H},\ t \in \mathbb{R}. \tag{2}$$

It is shown in [6],(Ex. 3.3) that φ is an admissible multivalued flow on $\mathbb{H}$ (we say that φ is *generated by* f).

We consider here only flows generated by L-vector fields. In particular, if F is single-valued and locally Lipschitz, then f generates a usual (single-valued) flow.

Recall that a *suspension* of a pointed space (X, x_0) is the quotient space $(SX, *) := (S^1 \times X)/(S^1 \times \{x_0\} \cup \{s_0\} \times X)$, where S^1 denotes a circle.

Let $\nu : \mathbb{N} \to \mathbb{N}$ be a given map.

Definition 3.4 A pair of sequences $X = ((X_n, x_n)_{n=n(X)}^{\infty}, (\gamma_n))$ is *a spectrum* provided the maps $\gamma_n : S^{\nu(n)} X_n \to X_{n+1}$ are homotopy equivalences for some $n_1 \geq n(X)$ and each $n \geq n_1$.

We can define a natural notion of a *map of spectra* $f : X \to X'$ as a sequence of maps $f_n : X_n \to X'_n; n \geq n_0 = \max\{n(X), n(X')\}$ such that the diagrams

$$\begin{array}{ccc} S^{\nu(n)} X_n & \xrightarrow{\ S^{\nu(n)} f_n\ } & S^{\nu(n)} X'_n \\ \downarrow \gamma_n & \curvearrowright & \downarrow \gamma'_n \\ X_{n+1} & \xrightarrow{\ f_{n+1}\ } & X'_{n+1} \end{array}$$

are homotopy commutative for all $n \geq n_0$.

Two spectra are *homotopy equivalent* if there is $n_1 \geq n_0$ such that f_n are homotopy equivalences for $n \geq n_1$. The equivalence class of this relation is called *the homotopy type of a spectrum* X and is denoted by $[X]$. One observes that the homotopy type of a spectrum X is determined by the homotopy type of the pointed space (X_n, x_n) with n sufficiently large.

We denote by $\underline{0}$ the spectrum such that for each $n \geq 0$ the space X_n consists only of a base point with the only maps $\epsilon_n : X_n \to X_{n+1}$. This is called a *trivial spectrum*.

[3] $C(\mathbb{R}, \mathbb{H})$ stands for the Fréchet space (i.e., locally convex metrizable and complete) of all continuous maps $\mathbb{R} \to \mathbb{H}$ with the topology of the almost uniform convergence.

One can also define usual topological operations like a "wedge sum" and smash product of spectra and on their homotopy types (see [10], Sec.2 for details).

Let now $f = L + K : \mathbb{H} \to \mathbb{H}$ be a single-valued L-vector field and let $\varphi : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ be an L-flow generated by f.

Denote by $\mathbb{H}^n := \bigoplus_{i=0}^{n} \mathbb{H}_i$ and by $P_n : \mathbb{H} \to \mathbb{H}$ an orthogonal projection onto $\mathbb{H}^n$.

Let $\mathbb{H}_n^{\pm} := \mathbb{H}_n \cap \mathbb{H}^{\pm}, n \geq 1$, where $\mathbb{H}^+$ and $\mathbb{H}^-$ denote L-invariant subspaces of H corresponding to parts of the spectrum of L with positive and negative real parts, respectively. Define $\nu : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $\nu(n) = \dim \mathbb{H}_{n+1}^{+}$.

Define $f_n : \mathbb{H}^n \to \mathbb{H}^n$ by $f_n(x) := Lx + P_n(K(x))$ and let $\varphi_n : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n$ be a flow generated by f_n.

Lemma 3.5 ([8], Lemma 4.1) *Let $N \subset \mathbb{H}$ be an isolating neighborhood for φ. Then there exists n_0 such that, for all $n \geq n_0$, the set $N^n = N \cap \mathbb{H}^n$ is an isolating neighborhood for φ_n.*

Thus the isolated invariant set $S_n = \mathrm{Inv}(N^n, \varphi_n)$ admits *an index pair* (P_1, P_2),(see [17]), i.e. a compact pair (P_1, P_2) such that

(i) the set $\overline{P_1 \setminus P_2}$ is an isolating neighborhood for S_n in N^n;

(ii) (positive invariance of P_2 in P_1) if $x \in P_2$ with $\varphi_n(x,t) \in P_1$ for every $t \in [0, t_0]$, then $\varphi_n(x,t) \in P_2$ for every $t \in [0, t_0]$;

(iii) if $x \in P_1$ and there is $t \geq 0$ with $\varphi_n(x,t) \notin P_1$, then there exists $0 \leq t_0 < t$ such that $\varphi_n(x, t_0) \in P_2$.

The classical homotopy Conley index of S_n is the homotopy type of the pointed space $[P_1/P_2, *]$. By the use of the continuation property of the classical Conley index it was proved in [8], that the family of such index pairs (P_1^n, P_2^n) for $n \geq n_0$ forms a spectrum in the above sense. A homotopy type of this spectrum is called *an $\mathcal{LS}$-index* of the isolating neighborhood N.

Let us denote this index by $h_{\mathcal{LS}}(N, \varphi)$. The following two basic properties have been proved in [8].

Proposition 3.6 *(Nontriviality) Let φ be an single-valued L-flow and $N \subset \mathbb{H}$ an isolating neighborhood. If $h_{\mathcal{LS}}(N, \varphi) \neq \underline{0}$, then $\mathrm{Inv}(N, \varphi) \neq \emptyset$.*

Proposition 3.7 *(Continuation) Let Λ be a compact, connected and locally contractible metric space. Assume that φ_λ is a family of single-valued L-flows and let $N \subset \mathbb{H}$ be an isolating* *neighborhood for the flow* φ_λ *for some* $\lambda \in \Lambda$. *Then there is a compact neighborhood* $C \subset \Lambda$ *of* λ *such that*

$$h_{\mathcal{LS}}(N, \varphi_\mu) = h_{\mathcal{LS}}(N, \varphi_\nu) \qquad \textit{for all } \mu, \nu \in C.$$

Let us now consider a multivalued L-vector field $L + F : \mathbb{H} \multimap \mathbb{H}$. Denote by $a(F, \varepsilon)$ the set of all ε-approximations of F in the sense of Theorem 1.3.

Proposition 3.8 *Let* $N = \overline{U} \subset \mathbb{H}$ *be an isolating neighborhood for a multivalued flow generated by* $L + F$. *There exists an* $\varepsilon > 0$ *such that for arbitrary* $f_0, f_1 \in a(F, \varepsilon)$ N *is an isolating neighborhood for a family of L-flows* η_λ *generated by the family of L-vector fields* $\Psi_\lambda = L + (1 - \lambda) f_0 + \lambda f_1$.

Proof. Let $r > 0$ be such that $N \subset B_r(0)$, and find the Urysohn function $u : \mathbb{H} \to [0, 1]$ such that $u(x) = 0$ for $x \in \overline{B_r(0)}$ and $u(x) = 1$ for any $x \in \mathbb{H} \setminus B_{2r}(0)$.

Consider a homotopy $h : \mathbb{H} \times [0, 1] \multimap \mathbb{H}$,

$$h(x, s) := Lx + (1 - u(x)) \left(\overline{\text{conv}} F(B_s(x)) + \overline{B_s(0)} \right) \cap \overline{\text{conv}} F(\overline{B_{2r}(0)}).$$

Since $\overline{\text{conv}} F(\overline{B_{2r}(0)})$ is compact, the map h generates a family η of multivalued L-flows on $\mathbb{H}$. Notice that $h(\cdot, 0) = L + F$ on $\overline{B_r(0)}$. From Proposition 3.2 it follows that the map $s \mapsto\!\!\circ \ \text{Inv}(N, \eta(\cdot, s))$ is usc with compact values.

Now, suppose the contrary to our claim. Then, for a sequence $\varepsilon_n = \frac{1}{n}$, there are approximations $f_0^n, f_1^n \in a(F, \frac{1}{n})$ and numbers $\lambda_n \in [0, 1]$ with $\text{Inv}(N, \gamma_{f_{\lambda_n}}) \not\subset U$, where $\gamma_{f_{\lambda_n}}$ is the flow generated by $L + (1 - \lambda_n) f_0 + \lambda_n f_1$. This implies that there are points $y_n \in \text{Inv}(N, \gamma_{f_{\lambda_n}}) \cap (N \setminus U)$. Note that $f_{\lambda_n}(\cdot) \subset \overline{\text{conv}} F(B_{\frac{1}{n}}(\cdot)) + \overline{B_{\frac{1}{n}}(0)}$ for every $n \geq 1$. Since the map $s \mapsto\!\!\circ \ \text{Inv}(N, \eta(\cdot, s))$ is usc with compact values, there exists a subsequence (f_k), where $f_k := f_{\lambda_{n_k}}$, such that $\text{Inv}(N, \gamma_{f_k}) \subset \text{Inv}(N, \varphi) + B_{\frac{1}{k}}(0)$ for every $k \geq 1$. Indeed, it is sufficient to notice that $\text{Inv}(N, \gamma_{f_k}) \subset \text{Inv}(N, \eta(\cdot, \frac{1}{n_k}))$.

Now, we can choose a sequence $(z_k) \subset \text{Inv}(N, \varphi)$ such that $|z_k - y_{n_k}| < \frac{1}{k}$. Since the set $\text{Inv}(N, \varphi)$ is compact, we can assume that $z_k \to z_0 \in \text{Inv}(N, \varphi)$. So, $y_{n_k} \to z_0$. But then $z_0 \in \text{Inv}(N, \varphi) \cap (N \setminus U)$; a contradiction. □

The above proposition proves that the following crucial notion of this note does not depend on the approximation f.

Definition 3.9 If N is an isolating neighborhood for an L-flow φ generated by $L + F$, then we define *a homotopy index*

$$h(N, \varphi) := h_{\mathcal{LS}}(N, \varphi_f),$$

where φ_f is the flow generated by $L + f$; $f \in a(F, \varepsilon)$ with $\varepsilon > 0$ sufficiently small.

Now we establish some properties of the index. The first one is an obvious consequence of Proposition 3.6.

Proposition 3.10 *If N is an isolating neighborhood for a multivalued L-flow φ and the homotopy index is nontrivial $h(N,\varphi) \neq \underline{0}$, then* $\mathrm{Inv}(N,\varphi) \neq \emptyset$.

Proposition 3.11 *If N_0, N_1 are two isolating neighborhoods for an L-flow φ such that* $\mathrm{Inv}(N_0,\varphi) \subset \mathrm{int}\, N_1, \mathrm{Inv}(N_1,\varphi) \subset \mathrm{int}\, N_0$, *then $h(N_0,\varphi) = h(N_1,\varphi)$.*

Proposition 3.12 *Let $\varphi : \mathbb{H}\times[0,1]\times\mathbb{R} \multimap \mathbb{H}$ be a family of multivalued L-flows and let $N \subset \mathbb{H}$ be an isolating neighborhood for all $\varphi(\cdot,\lambda), \lambda \in [0,1]$. Then $h(N,\varphi(\cdot,0)) = h(N,\varphi(\cdot,1))$.*

Proof. Consider the family of vector fields $\widetilde{F} : \mathbb{H}\times[0,1] \multimap \mathbb{H}$ such that for every $\lambda \in [0,1]$ the multivalued flow $\varphi(\cdot,\lambda,\cdot)$ is generated by the L-vector field $L + \widetilde{F}(\cdot,\lambda) : \mathbb{H} \multimap \mathbb{H}$. Applying Theorem 1.3 to the map $\widetilde{F}$ we obtain, for every $\varepsilon > 0$, a locally Lipschitz compact single-valued map $\widetilde{f} : \mathbb{H}\times[0,1] \to \mathbb{H}$ such that

$$(*) \qquad \widetilde{f}(x,\lambda) \in \overline{\mathrm{conv}}\widetilde{F}(\overline{B_\varepsilon}(x)\times\overline{B_\varepsilon}(\lambda)) + B_\varepsilon(0)) \text{ for all } \; x \in \mathbb{H}, \lambda \in [0,1].$$

Let us fix $\lambda \in [0,1]$. We shall show that N is an isolating neighborhood for the flows generated by $\widetilde{f}(\cdot,\lambda')$, where $\lambda' \in (\lambda-\varepsilon,\lambda+\varepsilon)$, if ε is small enough. Assume that $\overline{N} \subset B_r(0)$.

Let us define a homotopy $h : \mathbb{H}\times[0,1] \multimap \mathbb{H}$ by the formula

$$h(x,s) = Lx + u(x)[(\overline{\mathrm{conv}}\widetilde{F}(\overline{B_s}(x)\times\overline{B_s}(\lambda)) + \overline{B_s}(0)] \cap \overline{\mathrm{conv}}\widetilde{F}(\overline{B_{2r}}(0)\times[0,1]),$$

where $u : \mathbb{H} \to \mathbb{R}$ is an Urysohn function such that $u(x) = 1$ for $|x| \le r$ and $u(x) = 0$ for $|x| \ge 2r$.

Since h is a family of multivalued L-vector fields, it generates a family of multivalued L-flows $\eta(\cdot,s)$. Moreover, $h(\cdot,0) = L + \widetilde{F}(\cdot,\lambda)$. By Prop.3.2 the mapping $s \mapsto\!\!\!\!\multimap \mathrm{Inv}(N,\eta(\cdot,s))$ is usc and $\mathrm{Inv}(N,\eta(\cdot,0)) \subset \mathrm{int}\, N$. Therefore there exists $s > 0$ such that for all $s' \le s$ we have $\mathrm{Inv}(N,\eta(\cdot.s')) \subset \mathrm{int}\, N$. If we choose $0 < \varepsilon_\lambda < \frac{s}{2}$, then for all $\lambda' \in [\lambda-\varepsilon_\lambda, \lambda+\varepsilon_\lambda]$ we obtain by Theorem 1.3 that for $\varepsilon < \varepsilon_\lambda$

$$\widetilde{f}(x,\lambda') \in \overline{\mathrm{conv}}\widetilde{F}(\overline{B}_\varepsilon(x)\times\overline{B}_\varepsilon(\lambda')) + \overline{B}_\varepsilon(0) \subset \overline{\mathrm{conv}}\widetilde{F}(\overline{B}_s(x)\times\overline{B}_s(\lambda)) + \overline{B}_s(0).$$

We can assume that also $B_s(N) \subset B_r(N) \subset B_{2r}(0)$. It follows that the map $L + \widetilde{f}(\cdot,\lambda')$ is a selection of $h(\cdot,s)$. Therefore for the L-flow $\psi(\cdot,\lambda')$ generated by the vector field $L + \widetilde{f}(\cdot,\lambda')$ we have the inclusion $\mathrm{Inv}(N,\psi(\cdot,\lambda')) \subset \mathrm{int}\, N$, i.e., N is an isolating neighborhood.

Intervals $I_\lambda = (\lambda-\varepsilon_\lambda, \lambda+\varepsilon_\lambda) \cap [0,1]$ form an open covering of $[0,1]$. Choosing a finite subcovering $I_{\lambda_1}, ..., I_{\lambda_k}$ we find $\overline{\varepsilon} < \min\{\varepsilon_{\lambda_i}\}$ such that, for $\widetilde{f}$ satisfying (*) with $\varepsilon = \overline{\varepsilon}$, the set

N is an isolating neighborhood for flows generated by $L + \widetilde{f}(\cdot, \lambda)$ for all $\lambda \in [0,1]$. Thus by Prop. 3.7 the homotopy index $h(N, \psi(\cdot, \lambda))$ does not depend on λ.

On the other hand, the approximation $\widetilde{f}$ can be taken with an additional condition satisfied:

$$\widetilde{f}(\cdot, i) \in a(\widetilde{F}(\cdot, i), \varepsilon), \text{ for } i = 0, 1.$$

In order to assure this condition is satisfied, one repeats the proof of Theorem 1.3 with the following modification: For (x, λ) with $\lambda \notin \{0,1\}$ we take $B_{\delta(x,\lambda)}(x, \lambda)$ such that $B_{\delta(x,\lambda)}(x, \lambda) \cap (\mathbb{H} \times \{0,1\}) = \emptyset$, $B_{\delta(x,0)}(x, 0) \cap (\mathbb{H} \times \{1\}) = \emptyset$, $B_{\delta(x,1)}(x, 1) \cap (\mathbb{H} \times \{0\}) = \emptyset$ and for a locally finite covering $\{V_s\}$ of $\mathbb{H} \times [0,1]$ we choose $(x_s, \lambda_s) \in V_s$ such that $\lambda_s = i, i \in \{0,1\}$, if $V_s \cap (\mathbb{H} \times \{i\}) \neq \emptyset$.

This finishes the proof. □

Proposition 3.13 *Let $\varphi : \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ be an L-flow and let N_1, N_2, N be isolating neighborhoods for φ such that $N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 \subset N$ and* $\mathrm{Inv}(N, \varphi) \subset N_1 \cup N_2$. *Then $h(N, \varphi) = h(N_1, \varphi) \vee h(N_2, \varphi)$.*

Proof. The property follows from the obvious observation that for each n $\mathbb{H}^n \cap N_1 \cap N_2 = \emptyset$ and thus the appropriate index pairs (P, Q) defining the classical Conley index for the isolating neighborhood $N \cap \mathbb{H}^n$ can be chosen in the form of disjoint sums $(P_1 \cup P_2, Q_1 \cup Q_2)$, where $(P_1, Q_1), (P_2, Q_2)$ are index pairs for N_1, N_2, respectively. The rest is the definition of the wedge sum of spectra (see [10] for details). □

In [6] a cohomological version of the Conley index for multivalued L-flows in a Hilbert space was established starting from the finite-dimensional case given in [16]. Instead of the homotopy type of index pairs the authors consider the Alexander-Spanier cohomology groups of these pairs. Since all the maps in the spectra are homotopy equivalences for n large enough, the inverse limit of the groups is well-defined

$$CH^q(N, \varphi) = \varprojlim \{H^{q+\rho(n)}(Y_n, Z_n), \widetilde{\gamma_n}\}.$$

Similarly as in the single-valued case (see [10]) we obtain

Proposition 3.14 *Let N be an isolating neighbourhood for a multivalued L-flow φ. Then the cohomology index from [6] is equal to the cohomology of our spectrum: $CH^q(N, \varphi) = H^q(h(N, \varphi)$ for all $q \in \mathbb{Z}$.*

As a by-product we obtain that the cohomology index of Mrozek ([16]) for a multvalued flow φ in $\mathbb{R}^n$ generated by a differential inclusion is just a cohomology of the homotopy index considered by Kunze in [13].

An interesting question appears: can the homotopy index $h(N,\varphi)$ be described using a behavior of an L-vector field $L+F$ on the boundary of a prescribed set of constraints? In [7] the author gave a positive answer for differential inclusions in a finite dimensional space. We show that an infinite-dimensional version of this result is possible.

We will need the following extension result on graph approximations. Recall that $P_n : \mathbb{H} \to \mathbb{H}^n$ denotes the ortogonal projection.

Lemma 3.15 *Let $B = \overline{B(0,r)} \subset \mathbb{H}$ be a closed ball in $\mathbb{H}$ and $B^n := P_n(B) \subset B$. Let $F : B \multimap \mathbb{H}$ be a compact upper semicontinuous map with convex values and $F_n := P_n \circ F$. Then, for every $\varepsilon > 0$ there exists $n_0 \geq 1$ such that for any $n \geq n_0$ there exists a $\delta_n > 0$ such that any continuous (locally Lipschitz) δ_n-approximation $f : B^n \to \mathbb{H}^n$ of F_n over B^n may be extended to a continuous (locally Lipschitz) ε-approximation $g : B \to E$ of F, i.e., $g|_{B^n} = f$.*

Proof. Let $\varepsilon > 0$ be arbitrary. We will proceed in several steps.

Step 1. There exists a locally Lipschitz function $\eta : B \to (0,\infty)$ such that, for every $x \in \mathbb{H}$, there is $x' \in B(x,\varepsilon)$ such that $B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(x'))$.

Indeed, for each $x \in B$ we choose $0 < r_x < \varepsilon$ such that $F(B_{2r_x}(x)) \subset B_{\varepsilon/2}(F(x))$, since F is usc, and take a locally finite and locally Lipschitz partition of unity $\{\lambda\}_{s\in S}$ subordinated to the covering $\{B(x,r_x)\}_{x\in B}$. For each $s \in S$ denote $r_s := r_{x_s}$, where $\operatorname{supp} \lambda_s \subset B(x_s, r_{x_s})$ for some $x_s \in B$.

Define $\eta : B \to (0,\infty)$,

$$\eta(x) := \sum_{s\in S} \lambda_s(x) r_s, \quad x \in B.$$

Obviously, η is locally Lipschitz. Let $x \in B$, and let $S_x := \{s \in S; \lambda_s(x) > 0\}$. Since the partition of unity is locally finite, we can find $s \in S_x$ such that $\eta(x) \leq r_s$. Hence, $||x - x_s|| < r_s < \varepsilon$ and, for any $y \in B_{\eta(x)}(x)$, $||y - x_s|| \leq ||y - x|| + ||x - x_s|| < 2r_s$. Therefore $B_{\eta(x)}(x) \subset B_{2r_s}(x_s)$ and

$$F(B_{\eta(x)}(x)) \subset F(B_{2r_s}(x_s)) \subset B_{\varepsilon/2}(F(x_s)).$$

Hence, putting $x' := x_s$, we obtain

$$B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_\varepsilon(F(x')).$$

Step 2. For any $(x,y) \in B \times B$ we define

$$U(x,y) := [\eta^{-1}((\eta(x)/2,\infty)) \cap B_{\eta(x)/2}(x)] \times B_{\varepsilon/2}(y)$$

and an open neighborhood of the graph of F

$$\mathcal{U} := \bigcup_{(x,y)\in \mathrm{Gr}(F)} U(x,y).$$

Notice that, if $W \subset B$ is any subset, and a continuous map $f : W \to \mathbb{H}$ satisfies $\mathrm{Gr}(f) \subset \mathcal{U}$, then, for each $x \in W$, there exists $(x', y') \in \mathrm{Gr}(F)$ such that $(x, f(x)) \in U(x, y)$. Hence, $f(x) \in B_{\varepsilon/2}(y')$ and $||x - x'|| < \eta(x')/2 < \eta(x)$. This implies that $f(x) \in B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \subset B_{\varepsilon}(F(B_{\varepsilon}(x)))$.

Step 3. There is $n_0 \geq 1$ such that $||P_n(y) - y|| < \varepsilon/4$ for every $n \geq n_0$ and $y \in F(B)$. Fix $n \geq n_0$, and define

$$\widetilde{U}(x, y) := [\eta^{-1}((\eta(x)/2, \infty)) \cap B_{\eta(x)/2}(x)] \times B_{\varepsilon/4}(y)$$

and an open neighborhood of the graph of F_n in $B \times \mathbb{H}$

$$\mathcal{U}_n := \bigcup_{(x,y) \in \mathrm{Gr}(F_n)} \widetilde{U}(x, y).$$

Notice that, if $(u, v) \in \widetilde{U}(x, y)$, then $v \in B_{\varepsilon/4}(y)$ and $y = P_n(y')$ for some $y' \in F(x)$. Hence, $||v - y'|| \leq ||v - y|| + ||y - y'|| < \varepsilon/2$. It implies that $(u, v) \in U(x, y')$ and, consequently, $\mathcal{U}_n \subset \mathcal{U}$.

Using the partition of unity technique, as in Step 1, it is easy to find a continuous function $\rho' : \mathbb{H}^n \to (0, \varepsilon)$ such that, any $\rho'(\cdot)$-approximation $f : B^n \to \mathbb{H}^n$ of F_n, i.e., $f(x) \in B_{\rho'(x)}(F_n(B_{\rho'(x)}(x)))$ for any $x \in B^n$, satisfies $\mathrm{Gr}(f) \subset \mathcal{U}_n$. Analogously, let $\theta : \mathbb{H} \to (0, \varepsilon)$ be a continuous function such that any $\theta(\cdot)$-approximation $f : B \to \mathbb{H}$ of F, satisfies $\mathrm{Gr}(f) \subset \mathcal{U}$ (comp. [12], Prop. 1.2).

Since B^n is compact, there exists $0 < \delta = \delta_n < \min\{\rho'(x); x \in B^n\}$.

Step 4. Now, let $f : B^n \to \mathbb{H}^n$ be any locally Lipschitz δ-approximation of F_n over B^n. Then $\mathrm{Gr}(f) \subset \mathcal{U}_n$. Since B^n is a Lipschitz retract of B, there exists a locally Lipschitz extension $k : B \to \mathbb{H}$ of f. Since $\mathcal{U}_n \subset \mathcal{U}$ and $\mathcal{U}$ is open in $B \times \mathbb{H}$, there is an open neighborhood Ω of B^n in B such that $(x, k(x)) \in \mathcal{U}$ for every $x \in \Omega$. Hence,

$$k(x) \in B_{\varepsilon/2}(F(B_{\eta(x)}(x))) \text{ for every } x \in \Omega.$$

Take an open set $\Omega_0 \subset B$ with $B^n \subset \Omega_0 \subset \overline{\Omega_0} \subset \Omega$ and a locally Lipschitz Urysohn function $\beta : \mathbb{H} \to [0, 1]$ with $\beta(\overline{\Omega_0}) = \{1\}$ and $\beta(\mathbb{H} \setminus \Omega) = \{0\}$. Take any locally Lipschitz $\theta(\cdot)$-approximation $h : B \to \mathbb{H}$ of F, where $\theta(\cdot)$ is from Step 3. Then $\mathrm{Gr}(h) \subset \mathcal{U}$. Define $g : B \to \mathbb{H}$, $g(x) := \beta(x)k(x) + (1 - \beta(x))h(x)$ for every $x \in B$. Obviously, $g|_{B^n} = f$.

Take any x with $\beta(x) > 0$. Then $x \in \Omega$, and

$$\{k(x), h(x)\} \subset B_{\varepsilon/2}(F(B_{\eta(x)}(x))).$$

By Step 1, $\{k(x), h(x)\} \subset B_{\varepsilon}(F(x'))$ for some x' with $||x - x'|| < \varepsilon$. By the convexity of values of F,

$$g(x) \in B_{\varepsilon}(F(x')) \subset B_{\varepsilon}(F(B_{\varepsilon}(x))).$$

If $\beta(x)=0$, then, since $\theta(x)<\varepsilon$ for every $x\in B$, $g(x)=h(x)\in B_\varepsilon(F(B_\varepsilon(x)))$, too. Hence, g is the required approximation. □

In the sequel we will use the following

Theorem 3.16 (comp. [7], Theorem 4.1) *Let $K=\overline{IntK}$ be a subset of a finite dimensional space E and $F:E\multimap E$ be an usc map with compact convex values and a sublinear growth and such that $K^-(F)$ is a closed strong deformation retract of some open neighborhood $V\subset K$ of $K^-(F)$ in K. Assume that* int $T_K(x)\neq\emptyset$ *for every $x\in K\setminus K^-(F)$, and $T_K(\cdot)$ is lsc outside $K^-(F)$.*

Then

$$h(\mathrm{Inv}(K,\varphi),\varphi)=[K/K^-(F),[K^-(F)]],$$

where φ is a multivalued flow generated by F, and $h(\mathrm{Inv}(K,\varphi),\varphi)$ is defined, if K is an isolating neighborhood, as the Conley index for any flow generated by a sufficiently close Lipschitz approximation of F.

Here $T_K(x)$ denotes the Bouligand tangent cone:

$$T_K(x):=\{v\in\mathbb{R}^n|\ \liminf_{h\to0^+}\frac{\mathrm{dist}(x+hv,K)}{h}=0\}.$$

Let $L+F:\mathbb{H}\multimap\mathbb{H}$ be a multivalued L-vector field, and let φ be an L-flow generated by $L+F$. On the boundary of a set $K\subset\mathbb{H}$ of constraints we consider the following *exit set*:

$$K^-(L+F):=\{x_0\in\partial K\mid\forall x\in S(x_0)\forall t>0\ :\ x([0,t])\not\subset K\}.$$

It means that all trajectories starting at points in $K^-(L+F)$ immediately leave the set K. Assume that K is an isolating neighborhood for φ.

Suppose that the pair $(K,K^-(L+F))$ generates a spectrum (K_n/K_n^-), where K_n^- is the exit set for $K_n=K\cap\mathbb{H}^n$ with respect to $L+P_nF$. Moreover, for some $N\geq1$ and each $n\geq N$, let the following regularity conditions be satisfied:

(H1) Each K_n is *epi-Lipschitz* outside K_n^-, i.e., int $T_{K_n}(x)\neq\emptyset$ for every $x\in K_n\setminus K_n^-$.

(H2) K_n is *sleek* outside K_n^-, i.e., $T_{K_n}(\cdot)$ is lsc on $K_n\setminus K_n^-$.

(H3) K_n^- is a strong deformation retract of some open neighborhood V_n of K_n^- in K_n.

Denote by $[K,L+F]$ the homotopy type of the spectrum (K_n/K_n^-).

Theorem 3.17 *Under the above assumtions,*

$$h(K, \varphi) = [K, L + F].$$

Proof. Let $\varepsilon > 0$ be such that $h(K, \varphi) := h_{\mathcal{LS}}(K, \varphi_f)$, for any $f \in a(F, \varepsilon)$. Let $B = \overline{B(0, r)}$ be a ball in $\mathbb{H}$ such that $K \subset \operatorname{int} B$, and let $n_0 \geq N$ be such that K_n is an isolating neighborhood for each $n \geq n_0$, and n_0 is as in Lemma 3.15. We want to find $f \in a(F, \varepsilon)$ such that $h_{\mathcal{LS}}(K, \varphi_f)$ is a homotopy type of a spectrum $(Y_n/Z_n)_{n \geq n_0}$, and $([Y_n/Z_n, [Z_n]]) = ([K_n/K_n^-, [K_n]])$.

From Theorem 3.16 it follows that there exists a δ_{n_0}-approximation $g_{n_0} : B^{n_0} \to \mathbb{H}^{n_0}$ of $P_{n_0} F|_{B^{n_0}}$ such that its Conley index $[Y_{n_0}/Z_{n_0}, [Z_{n_0}]]$ equals $[K_{n_0}/K_{n_0}^-, [K_{n_0}]]$. We extend g to an ε-approximation $f : B \to \mathbb{H}$ of F. Since the spectrum $(Y_n/Z_n)_{n \geq n_0}$ for $L + f$ is uniquely determined up to a homotopy type by $(Y_{n_0}/Z_{n_0}, [Z_{n_0}])$, and $(Y_{n_0}/Z_{n_0}, [Z_{n_0}])$ is homotopy equivalent to $(K_{n_0}/K_{n_0}^-, [K_{n_0}])$, we obtain $h(K, \varphi) = [K, L + F]$. □

4 Conley index for finite dimensional gradient differential inclusions

Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function satisfying

(3) $$\sup_{y \in \partial f(u)} |y| \leq c(1 + |u|) \text{ for some } c \geq 0 \text{ and every } u \in \mathbb{R}^d.$$

Then the function $\Phi : \mathbb{R}^d \to \mathbb{R}$,

(4) $$\Phi(u) = \frac{1}{2}\langle Lu, u\rangle + f(u), \quad \text{for } u \in \mathbb{R}^d,$$

is locally Lipschitz, and the Clarke generalized gradient

$$F(u) := \partial\Phi(u) = Lu + \partial f(u)$$

is well defined (see [1], [4] for definitions and properties of the gradient). Moreover, $F : \mathbb{R}^d \multimap \mathbb{R}^d$ is usc with compact convex values and of sublinear growth. Hence, the differential inclusion $\dot{x} \in F(x)$ generates a multivalued admissible flow (see Preliminaries). We say that $\dot{x} \in \partial\Phi(x)$ is a *gradient differential inclusion.*

Assume that $F : \mathbb{R}^d \multimap \mathbb{R}^d$ is of the form $F(u) = Lu + \varphi(u)$ for some usc map φ with compact convex values and sublinear growth. We say that F has a *variational structure*, if there exists a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\partial\Phi(u) \subset F(u)$, where Φ is defined in (4). As we will see in the sequel, multivalued maps with a variational structure plays an important role in our investigations. If $\mathbb{E}$ is a Hilbert space, $P^d : \mathbb{H} \to \mathbb{H}^d$ is the ortogonal finite-dimensional projection, and $F : \mathbb{H} \multimap \mathbb{H}$ is of the form $F(u) = Lu + \partial f(u)$ for some linear bounded operator $L : \mathbb{H} \to \mathbb{H}$ with $L(\mathbb{H}^d) \subset \mathbb{H}^d$, and a locally Lipschitz map $f : \mathbb{H} \to \mathbb{R}$, then the map $F_d : \mathbb{H}^d \multimap \mathbb{H}^d$, $F_d(x) := Lx + P_d(\partial f(i_d(x)))$, where $i_d : \mathbb{H}^d \hookrightarrow \mathbb{H}$ is the inclusion map, has a variational structure, since $Lx + \partial(f \circ i_d)(x)) \subset Lx + P_d(\partial f(i_d(x)))$ (see [1],[4])). Moreover, F_d need not be a generalized gradient of any locally Lipschitz function.

Example 4.1 Consider a 1-Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} |x| & \text{for } |x| \le |y|, \\ |y| & \text{otherwise.} \end{cases}$$

Obviously, $\partial(f \circ i)(x)) = \{0\}$, where $i(x) := (x, 0)$, while $F_1(x) = P_1(\partial f(x, 0))) = \{0\}$ for $x \neq 0$ and $F_1(0) = P_1(\partial f(0, 0))) = [-1, 1]$. Hence, F_1 is not a generalized gradient of any locally Lipschitz function.

For multivalued flows generated by differential inclusions with compact convex valued right-hand sides a homotopy index has been constructed (see [13]). Below we repeat the construction and investigate the index in the context of gradient differential inclusions.

Let K be an isolated invariant set, and let N be its isolating neighborhood, i.e., $K = \mathrm{Inv}(N, F) := \mathrm{Inv}(N, \varphi)$, where φ is a multivalued flow generated by the inclusion $\dot{x} \in F(x)$. By Proposition 3.2 it follows that K is a compact subset of $\operatorname{int} N$ (see also [13], Lemma 5.2.3). Now it is easy to prove ([13], Lemmas 5.2.5 and 5.3.1) that for each $\varepsilon > 0$ there exists a smooth ε-approximation of F generating a global flow on $\mathbb{R}^d$, and there is $0\delta > 0$ such that for each two such smooth δ-approximations g_1, g_2 the set N is an isolating neighborhood and they generate global flows with the same Conley homotopy index $h(\mathrm{Inv}(N, g_1), g_1) = h(\mathrm{Inv}(N, g_2), g_2)$.

Definition 4.2 Let K be an isolated compact invariant set, and $K = \mathrm{Inv}(N, F)$. By the *homotopy index* of K we mean the homotopy type

$$H(K, F) := h(\mathrm{Inv}(N, g), g)$$

for sufficiently near smooth approximation g of F.

One can easily check that this definition does not depend on the choice of an isolating neighborhood N of K and the choice of g.

For $F = \partial\Phi$, where Φ is of the form (4), the index can be described using smooth approximations of the given locally Lipschitz map f, as we can see below.

We say that $\tilde{f} : U \to \mathbb{R}$ is a C^∞_ε-*approximation* of a locally Lipschitz function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^d$ is open and $\varepsilon : U \to (0, +\infty)$ is a continuous map, if

(i) $|f(x) - \tilde{f}(x)| < \varepsilon(x)$, for every $x \in U$,

(ii) $\nabla\tilde{f}$ is an ε-approximation[4] of ∂f.

In the sequel we will apply only a simplified version of the following result with a constant function $\varepsilon(x) = \varepsilon > 0$.

Proposition 4.3 ([3], Theorem 3.7) *Suppose $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^d$ is open, is a locally Lipschitz function. Then for any continuous map $\varepsilon : U \to (0, +\infty)$ there exists a C^∞_ε-approximation of f.*

Assume that $F : \mathbb{R}^d \multimap \mathbb{R}^d$, $F(u) = Lu + \varphi(u)$ has a variational structure with a multivalued selection $\partial\Phi = L + \partial f$. If $K = \mathrm{Inv}(N, F)$, then

$$H(K, F) = H(\mathrm{Inv}(N, \partial\Phi), \partial\Phi) = h(\mathrm{Inv}(N, \nabla\tilde{f}), \tilde{f})$$

for every sufficiently near C^∞_ε-approximation $\tilde{f}$ of f, since near approximations of $\partial\Phi$ are near approximations of F.

The index given in Definition 4.2 has standard important properties collected in the following proposition.

Proposition 4.4 *comp.* ([13], Theorems 5.3.1, 5.3.2, 5.3.4)

(Pr1) (Existence)[5] *If $H(K, F) \neq \bar{0}$, then $K \neq \emptyset$, i.e., there is a full trajectory in N, where N is an isolating neighborhood of K.*

(Pr2) (Additivity) *If K_1, K_2 are disjoint isolated invariant sets, then $K = K_1 \cup K_2$ is an isolated invariant set and*

$$H(K, F) = H(K_1, F) \vee H(K_2, F).$$

(Pr3) (Continuation) *Let $F : [0, 1] \times \mathbb{R}^d \multimap \mathbb{R}^d$ be a compact convex valued usc map with $\sup\limits_{y \in F(\lambda,u)} |y| \leq c(1 + |u|)$ for some $c > 0$ and every $(\lambda, u) \in [0, 1] \times \mathbb{R}^d$. If $K_\lambda = \mathrm{Inv}(N, F(\cdot, \lambda)) \subset \mathrm{int}\, N$ for every $\lambda \in [0, 1]$, then $H(K_\lambda, F(\cdot, \lambda))$ is independent of $\lambda \in [0, 1]$.*

Remark. In the continuation property Theorem 5.3.4 in [13] the author assume that $F(\lambda, \cdot)$ is usc, and $F(\cdot, u)$ is continuous for every $u \in \mathbb{R}^d$ and usc uniformly on bounded subsets of $\mathbb{R}^d$. As the author proves in Lemma 5.3.3, under these assumptions the map F is jointly

[4]It means that $d_{F(B(x,\varepsilon(x)))}(f(x)) < \varepsilon(x)$ for every $x \in U$.
[5]This is also called the Ważewski property.

usc in every (λ, u), so the assumption in (Pr3) is weaker than in Theorem 5.3.4 in [13]. Our formulation is suitable for gradient inclusions. Indeed, if $\Phi : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ is of the form

$$\Phi(\lambda, u) = \frac{1}{2}\langle L_\lambda u, u\rangle + f(\lambda, u), \quad \text{for } u \in \mathbb{R}^d, \tag{5}$$

where $\lambda \mapsto L_\lambda(\cdot)$ and f are locally Lipschitz, then the generalized gradient of Φ with respect to the second variable satisfies

$$\partial_u \Phi(\lambda, u) = L_\lambda u + \partial_u f(\lambda, u),$$

and it is jointly usc in every $(\lambda, u) \in [0,1] \times \mathbb{R}^d$. Note also that the continuation property is true if an isolating neighborhood ranges during a homotopy.

Example 4.5 One can check that for a function $\Phi : [0,1] \times \mathbb{R}$, $\Phi(\lambda, u) := f(\lambda, u) = |u|^{1+\lambda}$, i.e., with $L \equiv 0$, the generalized gradient $\partial_u \Phi$ is not continuous with respect to the first variable.

Proof of Proposition 4.4. For (Pr1) and (Pr2) see [13]. We prove (Pr3).

Let $r > 0$ be such that $N \subset B(0,r)$. By the Cellina approximation theorem and a standard mollifiers technique (see [13], Lemma 5.3.4) it follows that for every $\varepsilon > 0$ there exists a continuous bounded map $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ such that $f(\lambda, \cdot) \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and

$$f(\lambda, u) \in F\left(([\lambda - \varepsilon, \lambda + \varepsilon] \cap [0,1]) \times B(u, \varepsilon)\right) + B(0, \varepsilon)$$

for every $(\lambda, u) \in [0,1] \times \overline{B(0,r)}$. Moreover, using Lemma 2.1 in [7] we can choose an approximation f with

$$f(\lambda, u) \in F\left(\{\lambda\} \times B(u, \varepsilon)\right) + B(0, \varepsilon),$$

i.e., such that $f(\lambda, \cdot)$ is an ε-approximation of $F(\lambda, \cdot)$. Since $K_\lambda \subset \operatorname{int} N$ for every $\lambda \in [0,1]$ and the set $K = \bigcup_{\lambda \in [0,1]} K_\lambda$ is closed (see Proposition 3.2), K is a compact subset of $\operatorname{int} N$. Hence, there is $\delta > 0$ such that for every δ-approximation f of F chosen as above one has $\operatorname{Inv}(N, f(\lambda, \cdot)) \subset \operatorname{int} N$ for every $\lambda \in [0,1]$ (see [13], Lemma 5.3.5). Continuity of f implies that the corresponding flows π_λ continuously depend on λ. From the continuation property of the homotopy Conley index for flows one obtains

$$H(K_0, F(0, \cdot)) = h(\operatorname{Inv}(N, f(0, \cdot), f(0, \cdot)) = h(\operatorname{Inv}(N, f(1, \cdot), f(1, \cdot)) = H(K_1, F(1, \cdot)),$$

and the proof is finished. □

Corollary 4.6 *If K_1, K_2 are disjoint isolated invariant sets, $K_1 \cup K_2 \subset K$, and $K = \operatorname{Inv}(N, F)$. If $H(K, F) \neq H(K_1, F) \vee H(K_2, F)$, then there exists a full trajectory in K which is not contained in $K_1 \cup K_2$. In particular, it is the case if $H(K, F) = \bar{0}$ and $H(K_i, F) \neq \bar{0}$, for some $i \in \{1, 2\}$.*

Corollary 4.7 *Assume that* $F : \mathbb{R}^d \multimap \mathbb{R}^d$, $F(u) = Lu + \varphi(u)$ *has a variational structure with a multivalued selection* $\partial\Phi = L + \partial f$. *Let* p_1, p_2 *be critical points of* Φ *in an isolating neighborhood* N *for* F, *which are isolated invariant sets for* F. *Assume that* $\langle \partial\Phi(u), \partial\Phi(u)\rangle^- > 0$ *for every* $u \in N \setminus \{p_1, p_2\}$, *where* $\langle \partial\Phi(u), \partial\Phi(u)\rangle^- := \min\{\langle y, y'\rangle \mid y, y' \in \partial\Phi(u)\}$.

If $H(\mathrm{Inv}(N, F), F) \neq H(\{p_1\}, F) \vee H(\{p_2\}, F)$, *then there exists a heteroclinic or homoclinic nontrivial orbit in* N. *In particular, if* $(\{p_1\}, \{p_2\})$ *is an attractor-repeller pair, then there is a trajectory joining the equilibria.*

Proof. Notice that p_1, p_2 are the only critical points of Φ in N. From Lemma 4.5 in [7] it follows that $\{p_1\}$ and $\{p_2\}$ are isolated invariant sets for $\partial\Phi$. Therefore $H(\{p_i\}, F) = H(\{p_i\}, \partial\Phi)$ for $i \in \{1, 2\}$. Since $\partial\Phi$ is a selection of F, we have

$$\begin{aligned} H(\mathrm{Inv}(N, \partial\Phi), \partial\Phi) &= H(\mathrm{Inv}(N, F), F) \neq \\ &\neq H(\{p_1\}, F) \vee H(\{p_2\}, F) = H(\{p_1\}, \partial\Phi) \vee H(\{p_2\}, \partial\Phi). \end{aligned}$$

Now, Corollary 4.6 applies, and there is a full trajectory $x(\cdot)$ for $\partial\Phi$ (so, for F) in N with $x(0) = x_0 \in N$. Lemma 4.4 in [7] shows that $\omega(x_0) \cup \alpha(x_0) \subset \{p_1, p_2\}$. The proof is finished. $\square$

Remark. Note that the only critical points p_1, p_2 of Φ need not be isolated sets for F. For example, we can examine the map $F : \mathbb{R} \multimap \mathbb{R}$,

$$F(x) = \begin{cases} 1 & \text{for } |x| > 1, \\ [-1, 1] & \text{for } |x| \leq 1, \end{cases}$$

with the selection $\partial\Phi$, where

$$\Phi(x) = \begin{cases} x + 2 & \text{for } x < -1, \\ -x & \text{for } |x| \leq 1, \\ x - 2 & \text{for } x > 1. \end{cases}$$

Obviously, Φ has two critical points -1 and 1 which are not isolated sets for F. Notice that $H(\mathrm{Inv}([-2, 2], F), F) = \bar{0}$ and $H(\{-1\}, F) \vee H(\{1\}, F) = \Sigma^0 \vee \Sigma^1 \neq \bar{0}$. Moreover, $\langle \partial\Phi(x), \partial\Phi(x)\rangle^- = 1 > 0$ for $x \notin \{-1, 1\}$. One can easily find a trajectory joining 1 with -1.

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