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The Conley index, cup-length and bifurcation

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Abstract. A module structure of the cohomology Conley index is used to define a relative cup-length. This invariant is applied then to prove a multiplicity theorem for periodic solutions to Hamiltonian systems.

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1. Introduction

In this paper, we consider a module structure of the Conley index of smooth flows in $\mathbb{R}^n$. If $\overline{\Omega}$ is an isolating neighbourhood and (P_1, P_2) is a regular index pair in Ω, then the cohomology $H^*(P_1, P_2)$ is a module over $H^*(\overline{\Omega})$. We define a notion of relative cup-length of $H^*(P_1, P_2)$ over $H^*(\overline{\Omega})$. This notion can be used to derive several results on nontrivial structure of invariant sets. As an example we prove a theorem on a minimal number of periodic solutions to Hamiltonian systems. A natural action of the group S^1 on the space of periodic functions is being used. Some other applications of this tool to bifurcation theory are presented in the PhD thesis of the last author [17].

It is worth mentioning that the concept is not completely new. One can find a cup-length applied to Conley theory in [2] and [4]. A variant of a relative version appeared in [16]. We believe that our approach should also be useful for other problems considered in the bifurcation theory.

The paper is organized as follows. Section 2 contains an abstract algebraic definition of the relative cup-length and simple properties. In Section 3 we recall basic concepts from Conley index theory (main source is [13]) and specify the abstract notion to it. In Section 4 we prove an abstract result on a number of critical points for gradient-like flows. Section 5 contains a reduction procedure for bifurcation problems. In the latter sections this procedure is applied to Hamiltonian systems.

2. Relative cup-length

Throughout this section we assume that $A \subset X \subset Y$ are compact metric spaces and we denote by H^* the Alexander–Spanier cohomology with the coefficients in the fixed abelian group G. The cup product (see [15, Sec. 5.6])

$$\smile : H^k(X) \times H^l(X,A) \to H^{k+l}(X,A)$$

endows $H^*(X,A)$ with a structure of an $H^*(X)$-module. If $k : X \to Y$ denotes the inclusion map, then the formula

$$\beta \cdot \alpha := k^*(\beta) \smile \alpha$$

defines on $H^*(X,A)$ a structure of $H^*(Y)$-module. The following remark is a simple consequence of the naturality property of the cup product (see [6, Prop. 3.10]).

Remark 2.1. If $B \subset A$ is compact, then

$$H^*(X,A) \to H^*(X,B) \to H^*(A,B)$$

is an exact sequence of $H^*(Y)$-modules, where the maps are induced by inclusions.

Definition 2.1. Let $\beta \in H^p(Y)$, $p > 0$, $\beta \neq 0$, and let $A \subset X \subset Y$ be CW-complexes. The *relative cup-length* of β with respect to (X,A) is the number $\chi(\beta; X, A) \in \mathbb{N}$ defined as follows:

- $\chi(\beta; X, A) = 0$ if $H^*(X,A) = 0$;
- $\chi(\beta; X, A) = 1$ if $H^*(X,A) \neq 0$ and $\beta \cdot \alpha = 0$ for every $\alpha \in H^*(X,A)$;
- $\chi(\beta; X, A) = k \geq 2$ if there exists $\alpha_0 \in H^*(X,A)$ such that $\beta^{k-1} \cdot \alpha_0 \neq 0$ and $\beta^k \cdot \alpha = 0$ for every $\alpha \in H^*(X,A)$.

Definition 2.2. The *relative cup-length* of the $H^*(Y)$-module $H^*(X,A)$ is the number given by

$$\Upsilon(X,A;Y) := \max\{\chi(\beta; X, A);\ 0 \neq \beta \in H^k(Y),\ k > 0\}.$$

If $H^k(Y) = \{0\}$ for $k > 0$ but $H^*(X,A)$ is nonzero, we set $\Upsilon(X,A;Y) = 1$, and if $H^l(X,A)$ are trivial for all $l \geq 0$, then $\Upsilon(X,A;Y) := 0$.

Lemma 2.2. *If $B \subset A \subset X \subset Y$, then*

$$\Upsilon(X,B;Y) \leq \Upsilon(X,A;Y) + \Upsilon(A,B;Y).$$

Proof. Let $k_1 = \Upsilon(X,A;Y)$, $k_2 = \Upsilon(A,B;Y)$, $0 \neq \alpha \in H^p(X,B)$, $p \geq 0$, $0 \neq \beta \in H^q(Y)$, $q > 0$. Consider the following inclusions:

$$i : (X,B) \to (X,A), \qquad j : (A,B) \to (X,B).$$

Since $k_2 = \Upsilon(A, B; Y)$, $j^*(\beta^{k_2} \cdot \alpha) = 0$. By Remark 2.1, there exists $\gamma \in H^*(X, A)$ such that $\beta^{k_2} \cdot \alpha = i^*(\gamma)$. Therefore,

$$\beta^{k_1+k_2} \cdot \alpha = i^*(\beta^{k_1} \cdot \gamma).$$

But $\beta^{k_1} \cdot \gamma = 0$ by definition of k_1, and thus $\beta^{k_1+k_2} \cdot \alpha = 0$. This means that

$$\Upsilon(X, B; Y) \leq k_1 + k_2,$$

which ends the proof. □

Lemma 2.3. *If $A \subset X \subset Y_1 \subset Y_2$, then*

$$\Upsilon(X, A; Y_2) \leq \Upsilon(X, A; Y_1).$$

Proof. Consider the following inclusions:

$$s : X \hookrightarrow Y, \qquad k : A \hookrightarrow X, \qquad t : A \hookrightarrow Y.$$

If $\beta \in H^{>0}(Y_2)$, $\alpha \in H^*(X, A)$, then $\beta\alpha = t^*(\beta) \smile \alpha = k^*(s^*(\beta)) \smile \alpha$. Hence $\chi(X, A; \beta) = \chi(X, A; s^*(\beta))$ for all $\beta \in H^{>0}(Y_2)$. Since $t = k \circ s$, the condition $t^*(\beta) \smile \alpha \neq 0$ implies $s^*(\beta) \smile \alpha \neq 0$, and our inequality follows. □

Recall that the *cross product* is defined by the formula

$$a \times b := p_1^*(a) \smile p_2^*(b),$$

where p_1, p_2 denote projections $(X, A) \times (Y, B)$ onto (X, A) and (Y, B). For algebraic properties of the maps

$$\times : H^k(X; R) \times H^l(Y; R) \to H^{k+l}(X \times Y; R),$$
$$\times : H^k(X, A; R) \times H^l(Y, B; R) \to H^{k+l}(X \times Y, X \times B \cup A \times Y; R)$$

see, e.g., [6] or [1, pp. 240–242].

Let $\sigma :=$ generator $H^1(I, \partial I)$, $I := [-1, 1]$.

The formula

$$\mathfrak{S}(a) := a \times \sigma$$

defines a mapping

$$\mathfrak{S} : H^k(X, A) \to H^{k+1}((X, A) \times (I, \partial I)) = H^{k+1}(X \times I, X \times \partial I \cup A \times I).$$

The following lemma holds (cf. [6, Thm. 3.21] for more general version).

Lemma 2.4. *If $X \subset Y$, then $\mathfrak{S}$ is an isomorphism of $H^*(Y)$-modules. More exactly,*

$$\mathfrak{S}(b \cdot a) = p^*(b) \cdot \mathfrak{S}(a),$$

where p denotes the projection $Y \times I$ onto Y.

Proof. Let $b \in H^*(Y)$, $a \in H^*(X, A)$. Consider the following projections:

$$p_1 : (X \times I, A \times I) \to (X, A),$$
$$p_2 : (X \times I, X \times \partial I) \to (I, \partial I),$$
$$\overline{p}_1 : X \times I \to X.$$

The following diagram is commutative, where $i_1(x,t) = (i(x), t)$:

$$\begin{tikzcd} X \times I \arrow[r, "i_1"] \arrow[d, "\overline{p}_1"] & Y \times I \arrow[d, "p"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Using this diagram and the naturality and associativity properties of the cup product (see [1, p. 239]), we obtain

$$\begin{aligned} \mathfrak{S}(b \cdot a) &= (b \cdot a) \times \sigma = p_1^*(i^*(b) \smile a) \smile p_2^*(\sigma) = \overline{p}_1^*(i^*(b)) \smile p_1^*(a) \smile p_2^*(\sigma) \\ &= \overline{p}_1^*(i^*(b)) \smile \mathfrak{S}(a) = i_1^*(p^*(b)) \smile \mathfrak{S}(a) = p^*(b) \cdot \mathfrak{S}(a). \end{aligned}$$ □

Theorem 2.5. *The following formula holds:*

$$\Upsilon((X,A) \times (I, \partial I); Y) = \Upsilon(X, A; Y).$$

Proof. Let us notice that formally $X \times I \subset Y \times I$ and thus $H^*(X \times I, X \times \partial I \cup A \times I)$ is an $H^*(Y \times I)$-module, but $p^* : H^*(Y) \to H^*(Y \times I)$ is an isomorphism which gives the naturally isomorphic $H^*(Y)$-module structure: $b \odot a := p^*(b) \cdot a$ for $b \in H^*(Y)$ and $a \in H^*(X \times I, X \times \partial I \cup A \times I)$. Taking this into account, the desired equality follows directly from Lemma 2.4. □

3. Conley index and the relative cup-length

In this section, we recall the basic notions of the Conley index theory; the reader can refer to [9] and [13] for details. Let X be a locally compact metric space. A continuous map $\eta : X \times \mathbb{R} \to X$ is a *flow* if it satisfies the conditions

$$\eta(x, 0) = x,$$
$$\eta(x, t+s) = \eta(\eta(x,t), s).$$

A set $S \subset X$ is an *invariant set* for the flow η if

$$\eta(S, \mathbb{R}) := \bigcup_{t \in \mathbb{R}} \eta(S, t) = S.$$

For an arbitrary set $N \subset X$ one can define its invariant part

$$\mathrm{Inv}(N, \eta) := \{x \in N \mid \eta(x, \mathbb{R}) \subset N\}.$$

A compact set $N \subset X$ is an *isolating neighbourhood* if $\mathrm{Inv}(N, \eta) \subset \mathrm{int}\, N$. A set S is called an *isolated invariant set* if there is an isolating neighbourhood N such that $S = \mathrm{Inv}(N, \varphi)$. A flow $\eta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is *generated* by a smooth vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\eta(x,t)$ is the solution of the Cauchy problem $\dot{u} = -F(u), u(0) = x$ evaluated at time t. Such a flow is a *gradient flow* if $F = \nabla f$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

Let S be an isolated invariant set for the flow η. A compact pair $N_0 \subset N_1$ of subsets of X is called an *index pair* for S if the following hold:

(a) $\overline{\mathrm{int}(N_1 \setminus N_0)}$ is an isolating neighbourhood for S;
(b) N_0 is *positively invariant* relative to N_1; i.e., if $x \in N_0$ and $\eta(x, [0,t]) \subset N_1$, then $\eta(x, [0,t]) \subset N_0$;

(c) N_0 is an *exit set* for N_1; i.e., if $x \in N_1$ and $t_1 > 0$ such that $\eta(x, t_1) \notin N_1$, then there exists $t_0 \in [0, t_1]$ for which $\eta([0, t_0], x) \subset N_1$ and $\eta(x, t_0) \in N_0$.

The following result implies the correctness of the definition of the *homotopy Conley index* (cf. [9, Thms. 2.2.1 and 2.2.2] or [13, Thms. 23.7 and 23.12]).

Theorem 3.1. *Let S be an isolated invariant set for the flow η. Then there exists an index pair for S. Moreover, if (N_1, N_0) and (N_1', N_0') are index pairs for S, then the pointed topological spaces*

$$(N_1/N_0, [N_0]) \qquad \text{and} \qquad (N_1'/N_0', [N_0'])$$

are homotopically equivalent.

Definition 3.1. Let S be an isolated invariant set for the flow η. The *homotopy Conley index* of S is the homotopy type of the pointed space

$$h(S) = h(S, \eta) := [N_1/N_0, [N_0]],$$

where (N_1, N_0) is an index pair for S.

It is useful to consider the *cohomology Conley index* defined by

$$CH^*(S) := H^*(N, L) = H^*(N/L),$$

where H^* denotes the Alexander–Spanier cohomology and (N, L) is an index pair for S. The last equality means that we identify $H^*(N, L)$ and $H^*(N/L)$ via the isomorphism induced by the quotient map.

It is convenient to extend the index to an index of isolating neighbourhood: if N is an isolating neighbourhood for η, then the *homotopy* (resp., *cohomology*) *Conley index* of N is defined as

$$h(N) = h(N, \eta) := h(\mathrm{Inv}(N, \eta)),$$
$$\big(\text{resp., } CH^*(N) = CH^*(N, \eta) := CH^*(\mathrm{Inv}(N, \eta))\big).$$

Before giving the definition of the relative cup-length of Conley index, we need some useful lemmas. If (N_0, N_1) is an index pair and $t \geq 0$, then, following [13], we set

$$N_1^t := \{x \in N_1;\ \eta(x, [-t, 0]) \subset N_1\},$$
$$N_0^{-t} := \big\{x \in N_1;\ \text{there are } x' \in N_0 \text{ and } t' \in [0, t] \text{ with } \eta(x', [-t', 0]) \subset N_1 \text{ and } \eta(x', t) = x\big\}.$$

For $t \geq 0$, define a map

$$g : N_1/N_0^{-t} \to N_1^t/(N_0 \cap N_1^t)$$

by

$$g([x]) := \begin{cases} [\eta(x, t)] & \text{if } \eta(x, [0, t]) \subset N_1 \setminus N_0; \\ * & \text{otherwise.} \end{cases}$$

It is known (see [13, Lem. 23.14]) that g is a homeomorphism. Therefore, it induces an isomorphism

$$g^* : H^*(N_1^t, N_0 \cap N_1^t) \to H^*(N_1, N_0^{-t}).$$

Lemma 3.2. *Assume that N is an isolating neighbourhood for η and (N_1, N_0) is an index pair for $S \subset N$. If $N_1 \subset N$, then the inclusion $i : (N_1, N_0 \cap N_1^t) \to (N_1, N_0^{-t})$ induces an isomorphism*

$$i^* = (g^*)^{-1} : H^*(N_1, N_0^{-t}) \to H^*(N_1, N_0 \cap N_1^t).$$

Proof. Consider the following diagram, where the vertical arrows denote the quotient maps.

$$\begin{tikzcd}
(N_1, N_0^{-t}) \arrow[d] & (N_1, N_0 \cap N_1^t) \arrow[l, "i"'] \arrow[d] \\
N_1/N_0^{-t} \arrow[r, "g"] & N_1/(N_0 \cap N_1^t)
\end{tikzcd}$$

From the definition of g, it is obvious that the diagram is homotopy commutative and the conclusion follows. $\square$

Definition 3.2. Let N be an isolating neighbourhood for the flow η. We define the *relative cup-length* of η with respect to N as

$$\Upsilon(\eta, N) := \Upsilon(N_1, N_0; N),$$

where (N_1, N_0) is an index pair for S.

The following lemma states that $\Upsilon(\eta, N)$ is well defined.

Lemma 3.3. *Let N be an isolating neighbourhood for η and let $S \subset N$ be an isolated invariant set. If (N_1, N_0) and $(\overline{N}_1, \overline{N}_0)$ are index pairs for S such that $N_1, \overline{N}_1 \subset N$, then*

$$\Upsilon(\overline{N}_1, \overline{N}_0; N) = \Upsilon(N_1, N_0; N).$$

Proof. As in the proof of [13, Lem. 23.17], we consider the following sequence of maps, where j, $\hat{i}$, $\hat{i}_1$ are defined by inclusion maps of pairs of spaces and g, $\hat{g}$ are as above. All of them are homotopy equivalences of pointed spaces, as it is proved in detail in [13]:

$$\begin{tikzcd}
N_1/N_0 \arrow[r, "j"] & N_1/N_0^{-t} \arrow[r, "g"] & N_1^t/\left(N_0 \cap N_1^{-t}\right) \arrow[d, "\hat{i}_1"] \\
\overline{N}_1/\overline{N}_0 & \overline{N}_1^t/\left(\overline{N}_0 \cap \overline{N}_1^t\right) \arrow[l, "\hat{i}"'] & \overline{N}_1/\overline{N}_0^{-t} \arrow[l, "\hat{g}"']
\end{tikzcd}$$

By Lemma 3.2 and definition of j, it follows that the following sequence of isomorphisms

$$\begin{tikzcd} H^*(N_1,N_0) & H^*\left(N_1,N_0^{-t}\right) \arrow[l, "\approx"'] & H^*\left(N_1^t,N_0\cap N_1^{-t}\right) \arrow[l, "\approx"'] \\ H^*\left(\overline{N}_1,\overline{N}_0\right) \arrow[r, "\approx"] & H^*\left(\overline{N}_1^t,\overline{N}_0\cap\overline{N}_1^{\,t}\right) \arrow[r, "\approx"] & H^*\left(\overline{N}_1,\overline{N}_0^{\,-t}\right) \arrow[u, "\approx"] \end{tikzcd}$$

are all induced by inclusions. Therefore, they all are isomorphisms of $H^*(N)$-modules and the conclusion follows. $\square$

One of the main properties of the Conley index is the continuation. The same holds true for the relative cup-length.

Lemma 3.4. *Consider a continuous family of flows $\eta_\lambda : X\times\mathbb{R}\to X$; $\lambda\in[0,1]$. Let $N\subset X$ be an isolating neighbourhood for all flows η_λ. Then*

$$\Upsilon(\eta_0,N)=\Upsilon(\eta_1,N).$$

Proof. Similarly as in the proof of Lemma 3.3 we shall use parts of the proof of [13, Thm. 23.31]. Given $\mu\in[0,1]$, there exists a neighbourhood W of μ in $[0,1]$ with the property that for all $\lambda\in W$, we can find pairs $(N_1,N_0)\subset(P_1^\lambda,P_0^\lambda)\subset(\overline{N_1},\overline{N_0})$ such that (N_1,N_0), $(\overline{N_1},\overline{N_0})$ are index pairs for η_μ in N, and $(P_1^\lambda,P_0^\lambda)$ is an index pair for η_λ in N (see [13, Lem. 23.28]). Then it is shown in the proof of [13, Thm. 23.31] that the inclusion $i:(N_1,N_0)\to(P_1^\lambda,P_0^\lambda)$ induces a homotopy equivalence of pointed spaces N_1/N_0 and P_1^λ/P_0^λ. The same argument applies to show that $i^*: H^*(P_1^\lambda,P_0^\lambda)\approx H^*(N_1,N_0)$ is an isomorphism of $H^*(N)$-modules. Therefore, $\Upsilon(\eta_\lambda,N)=\Upsilon(\eta_\mu,N)$. Since $[0,1]$ is compact and connected, this completes the proof. $\square$

One easily sees that the continuation holds for more general parameter space Λ as in [13].

4. Gradient-like flows

Throughout this section, as before, η denotes a flow on a locally compact metric space X.

Let N be an isolating neighbourhood for η and let $\varphi:\operatorname{int}N\to\mathbb{R}$ be continuous. The flow η is called *gradient-like* with respect to φ if $\eta(x,[0,t])\subset\operatorname{int}N$ and $\eta(x,t)\neq x$ imply $\varphi(\eta(x,t))>\varphi(x)$. We define the *critical level set* of φ with respect to η as

$$\operatorname{Crit}(\varphi,\eta):=\varphi(\{x\in U;\eta(x,t)=x\text{ for all }t\in\mathbb{R}\}).$$

In other words, $c\in\operatorname{Crit}(\varphi,\eta)$ if and only if there is $x\in N$ which is a rest point of the flow and $\varphi(x)=c$.

The aim of this section is to give a proof of the following theorem.

Theorem 4.1. *Assume that X is locally contractible and N is an isolating neighbourhood for η. If η is gradient-like with respect to $\varphi : \operatorname{int} N \to \mathbb{R}$ and* $\operatorname{Crit}(\varphi, \eta)$ *is finite, then*

$$\#\operatorname{Crit}(\varphi, \eta) \geq \Upsilon(\eta, N).$$

Before giving the proof of the theorem we shall recall some definitions and results concerning Morse decompositions.

Recall that the *omega limit set* of $x \in X$ is given by

$$\omega(x) := \bigcap_{t>0} cl\big(\eta(x, [t, \infty))\big)$$

and the *alpha limit set* is

$$\alpha(x) := \bigcap_{t<0} cl\big(\eta(x, (-\infty, t])\big).$$

Assume that S is an isolated invariant set for η. A *Morse decomposition* of S is a finite collection, $\{M_i : 1 \leq i \leq n\}$, of disjoint compact invariant subsets of S which can be ordered $(M_1, M_2, \ldots, M_n)$ in such a way that if $x \in S \setminus \bigcup\{M_i : 1 \leq i \leq n\}$, then there are indices $i < j$ such that $\omega(x) \subset M_i$ and $\alpha(x) \subset M_j$. Such an ordering will be called admissible. The elements M_i of the Morse decomposition of S will be called *Morse sets* of S. For an admissible ordering $(M_1, \ldots, M_n)$ of a Morse decomposition S, define subsets M_{ij}, $i < j$, by

$$M_{ij} := \{x \in S : \omega(x) \cup \alpha(x) \subset M_i \cup M_{i+1} \cup \cdots \cup M_j\}.$$

The proof of the following existence theorem can be found in [13, Thm. 23.7] or in [12, Cor. 4.4].

Theorem 4.2. *Let S be an isolated invariant set for η and $(M_1, M_2, \ldots, M_n)$ an admissible ordering of a Morse decomposition of S. Then there exists an increasing sequence of compact sets (a (Morse) filtration of S),*

$$N_0 \subset N_1 \subset \cdots \subset N_n$$

such that for any $i < j$, the pair (N_j, N_{i-1}) is an index pair for M_{ij}. In particular, (N_n, N_0) is an index pair for S, and (N_j, N_{j-1}) is an index pair for M_j.

Furthermore, given any isolating neighbourhood N of S, and any neighbourhood U of S, the sets N_j can be chosen so that $cl(N_n \setminus N_0) \subset U$ and each N_j is positively invariant relative to N.

Proof of Theorem 4.1. Let

- $S := \operatorname{Inv} N$;
- $\operatorname{Crit}(\varphi, \eta) = \{c_1 < c_2 < \cdots < c_k\}$;
- $M_i := \operatorname{Crit}(\varphi, \eta) \cap \varphi^{-1}(c_i)$.

Choose

$$N_0 \subset N_1 \subset \cdots \subset N_n$$

satisfying the conditions of Theorem 4.2. Lemma 2.2 implies

$$\Upsilon(N_i, N_0; N) \leq \Upsilon(N_{i-1}, N_0; N) + \Upsilon(N_i, N_{i-1}; N) \tag{1}$$

for $i = 1, 2, \ldots, k$. Since M_i is finite and X is locally contractible, we can find a neighbourhood $U \subset N$ of M_i consisting of pairwise disjoint contractible sets. Then we find an index pair (N_i', N_{i-1}') in U. Therefore, $H^*(N_i', N_{i-1}')$ has a trivial structure as an $H^*(N)$-module. Thus by Lemma 3.3, we obtain

$$\Upsilon(N_i, N_{i-1}; N) \leq 1.$$

Therefore,

$$\Upsilon(\eta, N) = \Upsilon(N_k, N_0; N) \leq k. \qquad \square$$

5. Bifurcation

Throughout this section we let E_1, E_0 be Banach spaces, H a Hilbert space and we assume that $E_1 \subset E_0 \subset H$, where the embeddings are continuous.

We assume also that a compact Lie group G acts orthogonally on H, and the action on E_1, E_0 is by isometries (i.e., the norms on E_1, E_0 are G-invariant).

Definition 5.1. Given an open $\Omega \subset E$ and a continuous $f : \Omega \to E_0$, we say that f is a *generalized gradient map* if there is an open $\Omega_0 \subset E_0$, with $\Omega \subset \Omega_0$, and a C^1-function $\varphi : \Omega_0 \to \mathbb{R}$ such that

$$D\varphi(x)(y) = \langle f(x), y\rangle \quad \text{for all } x \in \Omega,\ y \in E_0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in H. Similarly, in the case of an open $\Omega \subset E \times \mathbb{R}$ and a continuous $f : \Omega \to E_0$, we say that f is a *generalized gradient map* if $f_\lambda : \Omega_\lambda \to E_0$ is a generalized gradient map. Here $\Omega_\lambda = \{x \in E; (x, \lambda) \in \Omega\}$.

If X is a Banach space, we denote the open ϵ-ball in X by $B_X(\epsilon) := \{x \in X; \|x\| < \epsilon\}$ and $B_X(x_0, \epsilon) := \{x \in X; \|x - x_0\| < \epsilon\}$.

If $V \subset E$ is a finite-dimensional linear subspace, then there is the *orthogonal decomposition determined by V*

$$E_1 = W_1 \oplus V, \qquad E_0 = W_0 \oplus V, \tag{2}$$

where $W_0 := \{x \in E_0; \langle x, y\rangle = 0 \text{ for all } y \in V\}$, $W_1 := E_1 \cap W_0$.

Definition 5.2. Let $[\lambda_1, \lambda_2] \subset \mathbb{R}$. We say that a C^1-gradient equivariant map

$$f : \Omega_f \to E_0,$$

where $\Omega_f \subset E \oplus \mathbb{R}$ is open G-invariant, $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega_f$, defines a *bifurcation problem* on $[\lambda_1, \lambda_2]$ if

$$f(0, \lambda) = 0 \quad \text{for } (0, \lambda) \in \Omega_f$$

and

$$D_x f(0, \lambda_i) : E \approx E_0, \quad i = 1, 2.$$

We shall also simply say that f is a *bifurcation problem.*

Definition 5.3. Let $f_i : \Omega_i \to E_0$, $i = 1, 2$, be two bifurcation problems on $[\lambda_1, \lambda_2]$. We say that f_1 and f_2 are *equivalent* if there exists an equivariant diffeomorphism

$$\Psi : \Omega_1 \to \Omega_2$$

such that

$$f_2 = f_1 \circ \Psi.$$

Theorem 5.1. *Let $f : \Omega_f \to E_0$ be a bifurcation problem on $[\lambda_1, \lambda_2]$. If there exist decompositions*

$$E_1 = V \oplus W_1, \quad E_0 = V \oplus W_0, \quad f(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)),$$

such that

$$Df_2(0, \lambda)_{|W_1} : W_1 \approx W_0 \quad \textit{for } \lambda \in [\lambda_1, \lambda_2],$$

then there exist

(1) *an open invariant $\Omega \subset \Omega_f$, $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega$;*
(2) *$g : \Omega_g \to E_0$—a bifurcation problem on $[\lambda_1, \lambda_2]$;*

such that

(a) *$f_{|\Omega}$ is a bifurcation problem on $[\lambda_1, \lambda_2]$ equivalent to g;*
(b) *$g(V \cap \Omega_g) \subset V$ and $g^{-1}(0) \subset V$;*
(c) *if $D_1 f_2(0, 0, \lambda) = 0$, then $D_1 g(0, 0, \lambda) = D_1 f_1(0, 0, \lambda)$.*

The proof is based on the following two theorems.

Theorem 5.2 (Equivariant implicit function theorem). *Let V_1, V_2, W be Banach G-spaces, $\Omega \subset V_1 \times V_2$ a G-invariant open set, $(x_0, 0) \in \Omega$ and $F : \Omega \to W$ be continuously differentiable G-map. Assume that $F(x_0, 0) = 0$ and*

$$D_2 F(x_0, 0) : V_2 \to W$$

is a G-equivariant Banach space isomorphism. Then there exist $\epsilon_1, \epsilon_2 > 0$, $B_{V_1}(x_0, \epsilon_1) \times B_{V_2}(\epsilon_2) \subset \Omega$, and a continuously differentiable G-equivariant map $\psi : B_{V_1}(x_0, \epsilon_1) \to B_{V_2}(\epsilon_2)$ such that

$$F(x, \psi(x)) = 0 \tag{3}$$

and

$$D\psi(x) = -\big(D_2 F(x, \psi(x))\big)^{-1} D_1 F(x, \psi(x)) \tag{4}$$

for all $x \in B_{V_1}(x_0, \epsilon_1)$. Furthermore, for every $x \in B_{V_1}(x_0, \epsilon_1)$, $\psi(x)$ is the only solution of (3) *in $B_{V_2}(\epsilon_2)$.*

Proof. The theorem is an equivariant reformulation of [7, Thm. 10.1]. Since the mapping

$$G(x, y) := y - L_0^{-1} F(x, y), \quad L_0 := D_2 F(x_0, 0),$$

defined in [7, p. 134], is in our case equivariant, the proof carries over directly. □

Theorem 5.3 (Parametrized equivariant implicit function theorem). *Let V_1, V_2, W be Banach G-spaces, $\Omega \subset V_1 \times V_2 \times \mathbb{R}$ a G-invariant open set, $(0,0,\lambda) \in \Omega$ for $\lambda \in [\lambda_1, \lambda_2]$. Assume that $F : \Omega \to W$ is a continuously differentiable G-map, $F(0,0,\lambda) = 0$ if $(0,0,\lambda) \in \Omega$ and*

$$D_2F(0,0,\lambda) : V_2 \to W$$

is a G-equivariant Banach space isomorphism if $(0,0,\lambda) \in \Omega$. Then there exist $\epsilon_1, \epsilon_2 > 0$, $B_{V_1}(\epsilon_1) \times B_{V_2}(\epsilon_2) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \subset \Omega$, and a continuously differentiable G-equivariant map $\psi : B_{V_1}(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \to B_{V_2}(\epsilon_2)$ such that

$$F(x, \psi(x,\lambda), \lambda) = 0 \tag{5}$$

and

$$D\psi(x,\lambda) = -\big(D_2F(x,\psi(x,\lambda))\big)^{-1} D_1F(x,\psi(x,\lambda)) \tag{6}$$

for all $x \in B_{V_1}(\epsilon_1) \times [\lambda_1, \lambda_2]$. Furthermore, for every $(x,\lambda) \in B_{V_1}(\epsilon_1) \times [\lambda_1, \lambda_2]$, $\psi(x,\lambda)$ is the only solution of (5) in $B_{V_2}(\epsilon_2)$.

Proof. The theorem follows from Theorem 5.2. One should consider $V_1 \oplus R$ instead of V_1 and then use the compactness of $[\lambda_1, \lambda_2]$. □

Proof of Theorem 5.1. We apply Theorem 5.3 to the map f_2 and obtain a G-equivariant mapping

$$\psi : B_V(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \to B_{W_0}(\epsilon_2).$$

Observe that for each $\lambda \in [\lambda_1, \lambda_2]$, the following holds true:

$$\text{if } x \in B_V(\epsilon_1),\ y \in B_{W_0}(\epsilon_2) \quad \text{then} \quad f_2(x,y,\lambda) = 0 \iff y = \psi(x,\lambda).$$

Taking ϵ_2 smaller if necessary, we define a G-equivariant diffeomorphism

$$\Psi : B_V(\epsilon_1) \times (\lambda_1 - \epsilon_1, \lambda_2 + \epsilon_1) \to \Omega_f$$

by the following formula:

$$\Psi(x,y,\lambda) := (x, y + \psi(x,\lambda), \lambda).$$

The desired map g is given by

$$g := f \circ \Psi.$$

Since V is finite dimensional, for $\epsilon > 0$ small enough, we have

$$g^{-1}(0) \cap (B_V(\epsilon) \times B_{W_1}(\epsilon) \times [\lambda_1, \lambda_2]) \subset B_V(\epsilon) \times [\lambda_1, \lambda_2].$$

Considering $Dg(0,0)$ in a block form, we obtain the last assertion. □

6. Bifurcation in $\mathbb{R}^n$

In this section to simplify the notation, we consider a finite-dimensional bifurcation problem on $I = [-1,1]$ defined by a map f. More precisely, we assume that $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a C^1-map, $f(0,\lambda) = 0$ for $\lambda \in \mathbb{R}$ and $Df(0,\pm 1) : \mathbb{R}^n \approx \mathbb{R}^n$.

Let $A_\lambda := D_x f(0,\lambda)$. Then $f(x,\lambda) = A_\lambda(x) + f_0(x,\lambda)$. For $\tau \in [0,1]$, we set

$$f_\tau(x,\lambda) := A_\lambda(x) + \tau f_0(x,\lambda).$$

Assume further that there exist $\rho, C > 0$ such that

$$\langle f_\tau(x,1), x\rangle \geq C|x|^2 \quad \text{for } |x| \leq 2\rho \tag{7}$$

and

$$\langle f_\tau(x,-1), x\rangle \leq -C|x|^2 \quad \text{for } |x| \leq 2\rho. \tag{8}$$

For $\alpha > 0$ and $0 < \epsilon < \rho$, define

$$F_\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

by $F_\tau(x,\lambda) := (f_\tau(x,\lambda), \alpha(|x| - \epsilon))$. Let

$$\Omega = \{x \in \mathbb{R}^n; |x| \leq 2\rho\} \times [-1,1]$$

and $M := \sup\{|f_\tau(x,\lambda)|; (x,\lambda) \in \Omega,\ \tau \in [0,1]\}$.

Lemma 6.1. *If*

$$\alpha \geq \frac{2M}{\rho(\rho-\epsilon)},$$

then there exists $\delta > 0$ such that $\delta < \epsilon$ and for all $\tau \in [0,1]$, the set $N := \{(x,\lambda) \in \Omega;\ |x| \geq \delta\}$ is an isolating neighbourhood for the flow generated by F_τ.

Proof. First we prove that Ω is an isolating neighbourhood. We fix τ and let $\eta(x,\lambda,t) = (\eta_1(x,\lambda,t), \eta_2(x,\lambda,t)) \in \mathbb{R}^n \times \mathbb{R}$ denote the flow generated by F_τ. It is enough to show that for all $(x,\lambda) \in \partial\overline{\Omega}$,

(a) there exists $T > 0$ such that either $\eta(x,\lambda,T) \notin \overline{\Omega}$ or $\eta(x,\lambda,-T) \notin \overline{\Omega}$.

Let $K := \{(x,\lambda) \in \overline{\Omega}; |x| = 2\rho, \lambda \in [-1,1]\}$. If $(x,\lambda) \in \partial\Omega \setminus K$, then (a) follows immediately from the definition of F_τ.

To complete the proof of our first claim we start from the following simple observations:

(b) if $\eta(x,\lambda,t) \in \overline{\Omega}$ for all $t \in [0,T]$, then

$$|\eta_1(x,\lambda,t) - \eta_1(x,\lambda,0)| \leq TM \quad \text{for } t \in [0,T];$$

(c) if $\eta(x,\lambda,t) \in A := \{(x,\lambda) \in \overline{\Omega}; |x| \geq \rho\}$ for all $t \in [0,T]$, then

$$\eta_2(x,\lambda,t) - \eta_2(x,\lambda,0) \geq (\rho-\epsilon)\alpha t \quad \text{for } t \in [0,T].$$

Let $(x_0, \lambda_0) \in K$ and

$$T_1 := \inf\{t \in (0, \infty); \eta(x_0, \lambda_0, t) \notin A\}.$$

Since every point of A leaves A in a finite time, $T_1 < \infty$. (One can call T_1 the *exit time* of (x_0, λ_0) from A.) Let $(x_1, \lambda_1) := \eta(x_0, \lambda_0, T_1)$. If $(x_1, \lambda_1) \in \partial\Omega$, then (a) holds. Suppose that $(x_1, \lambda_1) \in \Omega$. Then $|x_1| = \rho$, $\lambda_1 \in (-1, 1)$ and (b) implies $\rho \leq MT_1$. Applying (c), one obtains

$$\lambda_1 \geq \lambda_0 + \alpha(\rho - \epsilon)\frac{\rho}{M} > \lambda_0 + 2 > 1.$$

We have obtained a contradiction. Therefore, Ω is an isolating neighbourhood for all η_τ and thus the invariant part

$$\mathrm{Inv}(\Omega, \eta) = \bigcup_{\tau \in [0,1]} \mathrm{Inv}(\Omega, \eta_\tau) \subset \mathrm{int}(\Omega)$$

is compact. Moreover, one easily checks that it is disjoint with $\{0\} \times [-1, 1]$. Thus there exists $\epsilon > \delta \geq 0$ such that $\mathrm{Inv}(\Omega) \in \mathrm{int}(N)$. This proves that N is an isolating neighbourhood for all η_τ. □

Assume now that $V = (\mathbb{R}^n, \varphi)$ is an orthogonal representation of a compact Lie group G; i.e., $\varphi : G \to O(n)$ is a group homomorphism. Let $S(V) := \{x \in V; |x| = 1\}$. The use of V instead of $\mathbb{R}^n$ is a bit of notation abuse—we try to emphasize that $S(V)$ is a G-space.

Lemma 6.2. *Let $f : \Omega_f \to \mathbb{R}^n$ be a gradient equivariant bifurcation problem on $[-1, 1]$ and $A_\lambda := D_x f(0, \lambda)$, $\lambda \in [-1, 1]$. Assume that there is $C > 0$ such that*

$$\langle A_1(x), x\rangle \geq C|x|^2 \quad \textit{for } x \in \mathbb{R}^n \tag{9}$$

and

$$\langle A_{-1}(x), x\rangle \leq -C|x|^2 \quad \textit{for } x \in \mathbb{R}^n. \tag{10}$$

Then for sufficiently small ϵ, the number of zero G-orbits of f in $S(\mathbb{R}^n, \epsilon) \times (-1, 1)$ is not less than the cup-length of $S(V)/G$.

Proof. We keep the notation from the beginning of this section. From (9) and (10), it follows that there exists $\rho > 0$ such that assumptions (7) and (9) are satisfied. Now for $\epsilon < \rho$, we find α and δ as in Lemma 6.1 and obtain an isolating neighbourhood $N = \{(x, \lambda); \delta \leq |x| \leq 2\rho, -1 \leq \lambda \leq 1\}$ which is clearly an invariant set with respect to the action of G (trivial on the parameter space). By Lemma 3.4, it is enough to calculate the equivariant Conley index (and the relative cup-length) for the flow generated by the map $g(x, \lambda) := (D_x f(0, \lambda)(x), \alpha(|x| - \epsilon)) = (A_\lambda x, \alpha(|x| - \epsilon))$.

Now we can make another simplification. Consider a map $B : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $B(x, \lambda) = \lambda x$ and a family of flows generated by vector fields $F_\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $F_\tau(x, \lambda) = (\tau A_\lambda x + (1 - \tau)B(x, \lambda), \alpha(|x| - \epsilon))$ with $\tau \in [0, 1]$. It is easy to verify that N is an isolating neighbourhood for this family of flows. Thus we can do all the calculations for $\tau = 0$. We can easily find an index pair: $N_1 := N$ and

$$\begin{aligned} N_0 := \{(x,1) : 1 \geq |x| \geq \epsilon\} &\cup \{(x,\lambda); |x| = 2\rho, \lambda \in [0,1]\} \\ &\cup \{(x,\lambda); |x| = \delta, \lambda \in [-1,0]\} \\ &\cup \{(x,-1); \delta \leq |x| \leq \epsilon\}. \end{aligned}$$

Since all the sets are G-invariant, their quotient sets constitute an index pair for the flow generated on the orbit space. N is equivariantly homotopy equivalent to $S(V) \times [-1,1] \times [-1,1]$ and $N_0 \approx S(V) \times L$, where $L : \{(t,s) \in \partial([-1,1] \times [-1,1]); ts \geq 0\}$. Therefore, $\overline{N_1} := N_1/G \approx S(V)/G \times [-1,1] \times [-1,1]$, $\overline{N_0} := N_0/G \approx S(V)/G \times L$. Their quotient $\overline{N_1}/\overline{N_0} \approx S(V)/G \wedge S^1$. Thus, by Theorem 2.5, $\Upsilon(\overline{N_1}, \overline{N_0}; \overline{N_1})$ is equal to the cup-length of $S(V)/G$.

Now we can apply Theorem 4.1, since the gradient flow generated by f gives rise to a gradient-like flow on the orbit space and the critical points of this flow are images of the zero G-orbits of f. $\square$

7. Bifurcations of periodic solutions to Hamiltonian systems

By

$$J : \mathbb{R}^{2N} = \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{R}^{2N}$$

we denote a linear automorphism given by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Throughout this section we assume that $H : \mathbb{R}^{2N} \to \mathbb{R}$ is a C^2-function (Hamiltonian) such that

(H1) $H(0) = 0$, $\nabla H(0) = 0$;

(H2) the Hessian $\nabla^2 H(0)$ is nondegenerate.

The main object of our investigation is periodic solutions to the following equation:

$$\dot{u}(t) = J\nabla H(u(t)). \tag{11}$$

We shall use the following Banach spaces:

(1) $\mathcal{E}_0 := C(S^1, \mathbb{R}^{2N})$. The elements of $\mathcal{E}_0$ are identified with continuous functions

$$u : \mathbb{R} \to \mathbb{R}^{2N}, \quad u(t + 2\pi) = u(t), \quad \|u\| := \sup\{|u(t)|; t \in \mathbb{R}\}.$$

(2) $\mathcal{E} := C^1(S^1, \mathbb{R}^{2N})$. As a linear space $\mathcal{E}$ is a subspace of $\mathcal{E}_0$. The norm in $\mathcal{E}$ is defined by a formula

$$\|u\|_1 := \|u\| + \|\dot{u}\|.$$

The above automorphism J defines also automorphisms of our Banach spaces

$$J : \mathcal{E} \to \mathcal{E}, \qquad J : \mathcal{E}_0 \to \mathcal{E}_0.$$

More precisely,

$$J\left(\sum_{i=1}^{2N} u_i \mathbf{e}_i\right) := \sum_{i=1}^{2N} u_i J(\mathbf{e}_i),$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2N}\}$ is the standard basis of $\mathbb{R}^{2N}$.

In the space $\mathcal{E}_0$ we have a continuous inner product

$$\langle u, v\rangle := \sum_{j=1}^{2N} \int_0^{2\pi} u_j(t) v_j(t) dt, \tag{12}$$

where

$$u = \sum_{j=1}^{2N} u_j \mathbf{e}_j, \qquad v = \sum_{j=1}^{2N} v_j \mathbf{e}_j.$$

(In other words, we consider $\mathcal{E}_0$ as a subspace of $\mathcal{L}^2(S^1, \mathbb{R}^{2N})$.)

The formula

$$\mathcal{L}(u) := J(\dot{u})$$

defines a bounded linear operator

$$\mathcal{L} : \mathcal{E} \to \mathcal{E}_0.$$

Denote by

$$\mathcal{H} : \mathcal{E} \to \mathcal{E}_0$$

a map (nonlinear in general) given by a formula

$$(\mathcal{H}(u))(t) := \nabla H(u(t)).$$

Our further considerations are based on the following well-known remark.

Define a map

$$f : \mathcal{E} \times (0, \infty) \to \mathcal{E}_0, \quad f(u, \lambda) := \mathcal{L}(u) + \lambda\mathcal{H}(u). \tag{13}$$

Remark 7.1. A function $u \in \mathcal{E}$ is a periodic solution to equation (11) of period $\frac{2\pi}{\lambda}$ if and only if $f(u, \lambda) = 0$. The map f is (generalized) gradient in the sense introduced in Definition 5.1 with respect to the potential $\chi(u) := \int_0^{2\pi} u(t)dt$.

A change of variables $t \mapsto \lambda t$ gives the first part of the remark. The second part is well known.

Let $A := \nabla^2 H(0)$. The map JA is a *Hamiltonian* (i.e., $(JA)^T J + J(JA) = 0$). Observe that in [8] the notion *Hamiltonian matrix* is used.

Now we describe briefly the spectral decomposition of JA. We try to follow the notation of [8, Sec. 3.3], where further details can be found.

The eigenvalues of JA fall into three groups (because of (H2)):

(1) the pure imaginary $\pm i\omega_1, \ldots, \pm i\omega_r$;
(2) the real eigenvalues $\alpha_1, \ldots, \alpha_s$;
(3) the truly complex $\pm\beta_1 \pm i\gamma_1, \ldots, \pm\beta_t \pm i\gamma_t$.

This defines a direct sum decomposition

$$\mathbb{R}^{2N} = \mathbb{V} \oplus \mathbb{X} \oplus \mathbb{Y}, \tag{14}$$

where their complexifications are composed of generalized eigenspaces as follows:

$$\mathbb{V}^c = \bigoplus_{j=1}^{r} \left(\eta^\dagger(i\omega_j) \oplus \eta^\dagger(-i\omega_j)\right),$$

$$\mathbb{X}^c = \bigoplus_{j=1}^{s} \left(\eta^\dagger(\alpha_j) \oplus \eta^\dagger(-\alpha_j)\right),$$

$$\mathbb{Y}^c = \bigoplus_{j=1}^{t} \left(\eta^\dagger(\beta_j + i\gamma_j) \oplus \eta^\dagger(\beta_j - i\gamma_j) \oplus \eta^\dagger(-\beta_j + i\gamma_j) \oplus \eta^\dagger(-\beta_j - i\gamma_j)\right).$$

We are especially interested in part (1):

$$\begin{aligned} \sigma_0(JA) &= \sigma(JA) \cap \{i\mathbb{R}\} \\ &= \{\pm i\omega_1, \pm i\omega_2, \dots, \pm i\omega_r\}, \quad 0 < \omega_1 < \omega_2 < \cdots < \omega_r. \end{aligned} \tag{15}$$

Denote by $\mathbb{V}_j$, $\mathbb{U}_j$ the subspaces of $\mathbb{R}^{2N}$ such that

$$\mathbb{V}_j^c := \eta^\dagger(i\omega_j) \oplus \eta^\dagger(-i\omega_j), \qquad \mathbb{U}_j^c := \eta(i\omega_j) \oplus \eta(-i\omega_j).$$

Obviously,

$$\mathbb{V} = \bigoplus_{j=1}^{r} \mathbb{V}_j \tag{16}$$

and each summand is A-invariant (and so are $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{U}_j$).

Denote by $A^c : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ the complexification of A and let $\mathbb{U}_j \subset \mathbb{R}^{2N}$, $j = 1, \dots, r$, denote the subspace such that

$$\mathbb{U}_j^c = \operatorname{Ker}(A + i\omega_j) \oplus \operatorname{Ker}(A - i\omega_j).$$

Let $d_j := \frac{1}{2} \dim \mathbb{U}_j$. Clearly d_j is an integer. Let

$$d = d(A) := d_1 + d_2 + \cdots + d_r. \tag{17}$$

In order to make our setup precise, we introduce the following terminology. If $u : \mathbb{R} \to \mathbb{R}^{2N}$ is a periodic C^1-solution to (11) and $\tau \in \mathbb{R}$, then we let u_τ denote the periodic solution to (11) defined by $u_\tau(t) := u(t + \tau)$, $t \in \mathbb{R}$. We say that two periodic solutions u, v to (11) are *geometrically distinct* if $u_\tau \neq v$ for all $\tau \in \mathbb{R}$.

Now we can formulate the main result of this section.

Theorem 7.2. *If H satisfies* (H1) *and* (H2), *then there exists an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ implies the existence of at least d geometrically distinct periodic solutions to* (11) *in* $\{u \in C(\mathbb{R}, \mathbb{R}^{2N}); \|u\| = \epsilon\}$.

8. Proof of Theorem 7.2

Define the operators $\mathcal{A}, \mathcal{D}_\lambda : \mathcal{E} \to \mathcal{E}_0$ by

$$\mathcal{A}(u)(t) := A(u(t)) = (\nabla^2 H(0))(u(t)), \qquad \mathcal{D}_\lambda u := J\dot{u} + \lambda A(u). \tag{18}$$

Note that $\mathcal{D}_\lambda = Df(0,\lambda)$. For any subspace $\mathbb{Z} \subset \mathbb{R}^{2N}$, we denote

$$\mathcal{E}(\mathbb{Z}) := C^1(S^1, \mathbb{Z}), \qquad \mathcal{E}_0(\mathbb{Z}) := C^0(S^1, \mathbb{Z}).$$

We consider the above function spaces together with the S^1-action defined by

$$(\gamma u)(t) := u(t-\theta) \quad \text{for } \gamma := e^{i\theta}.$$

Notation. Throughout this section we tacitly assume that the considered maps are S^1-equivariant and gradient (in the generalized sense, see Definition 5.1). The gradient structure should be clear from the context.

Define an equivalence relation in the set $\mathfrak{S} := \{\omega_1, \omega_2, \dots, \omega_q\}$ by

$$\omega_j \sim \omega_k \iff n\omega_j = m\omega_k, \quad n, m \in \mathbb{N}.$$

This relation divides $\mathfrak{S}$ into pairwise disjoint classes

$$\mathfrak{S} = \bigcup_{k=1}^{p} \mathfrak{S}_k.$$

For $k \in \{1, 2, \dots, q\}$, set $\mathcal{J}_k := \{j \in \{1, \dots, r\}; \omega_j \in \mathfrak{S}_k\}$, $\mathfrak{D}_k := \mathcal{D}_{\lambda_k}$,

$$\mathbb{W}_k := \bigoplus_{j \in \mathcal{J}_k} \mathbb{V}_j, \quad b_k := \sum_{j \in \mathcal{J}_k} d_j,$$

$\mathcal{W}_k := \mathcal{E}(\mathbb{W}_k) \cap \operatorname{Ker} \mathfrak{D}_k$.

For each k, let ν_k denote the greatest real number such that for every $\omega \in \mathfrak{S}_k$ there is $n \in \mathbb{N}$ such that $\omega = n\nu_k$ and let $\lambda_k := \nu_k^{-1}$.

Suppose $\omega_j \in \mathfrak{S}_k$ and let $n_j := \frac{\omega_j}{\nu_k} \in \mathbb{N}$.

If $z = x + iy \in \mathbb{C}^{2N}$, $x, y \in \mathbb{R}^{2N}$, is an eigenvector corresponding to the eigenvalue $i\omega_j$ of $(JA)^c$, then $(JA)^c(x) = -\omega_j y$, $(JA)^c(y) = \omega_j x$ and thus $x - iy$ is an eigenvector corresponding to the eigenvalue $-i\omega_j$. Therefore, vectors x, y span a subspace of $\mathbb{R}^{2N}$ which is invariant for A. Let $\mathbf{z}_p = \mathbf{x}_p + i\mathbf{y}_p$, $p := 1, \dots, d_j$, be a basis of $\operatorname{Ker}((JA)^c + i\omega_j I)$. Then the vectors

$$\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \dots, \mathbf{x}_{d_j}, \mathbf{y}_{d_j} \tag{19}$$

form a basis of $\mathbb{U}_j$. Let $\mathbf{c}_j(t) := \cos(n_j t)$, $\mathbf{s}_j(t) := \sin(n_j t)$. Denote by $\mathcal{U}_j$ the $2d_j$-dimensional subspace of $C^1(S^1, \mathbb{U}_j) \subset \mathcal{E}$ spanned by

$$\mathbf{c}_j \mathbf{x}_p + \mathbf{s}_j \mathbf{y}_p, \quad \mathbf{s}_j \mathbf{x}_p - \mathbf{c}_j \mathbf{y}_p, \quad p = 1, \dots, d_j.$$

Then

$$\mathcal{U}_j = \mathcal{E}(\mathbb{V}_j) \cap \operatorname{Ker} \mathfrak{D}_k \qquad \text{and} \qquad \mathcal{W}_k = \bigoplus_{j \in \mathcal{J}_k} \mathcal{U}_j.$$

Remark 8.1. The assignments

$$\mathbf{c}_j\mathbf{x}_p + \mathbf{s}_j\mathbf{y}_p \mapsto e_p, \quad \mathbf{s}_j\mathbf{x}_p - \mathbf{c}_j\mathbf{y}_p \mapsto ie_p, \quad p = 1, \dots, d_j,$$

where $\{e_p\}$ denotes the standard basis of $\mathbb{C}^{d_j}$, define an isomorphism of real linear spaces

$$\mathcal{A}_j : \mathcal{U}_j \to \mathbb{C}^{d_j}.$$

Lemma 8.2. *The cup-length of $S(\mathcal{W}_k)$ equals b_k.*

Proof. Consider the complex linear space

$$\mathbf{V} := \bigoplus_{j \in \mathcal{J}_k} \mathbb{C}^{d_j}$$

whose points we write as $z = (z_1, \dots, z_q), z_j \in \mathbb{C}^{d_j}$. Let $\mathbf{Y}$ and $\mathbf{Z}$ denote, respectively, the representations of S^1 on $\mathbf{V}$ determined by

$$\gamma(z_1, \dots, z_q) := (\gamma^{d_1} z_1, \dots, \gamma^{d_q} z_q),$$
$$\gamma(z_1, \dots, z_q) := (\gamma^{b_k} z_1, \dots, \gamma^{b_k} z_q).$$

To avoid misunderstandings we denote by $\mathbf{X}$ the standard representation of S^1 on $\mathbf{V}$. Let $\alpha : S(\mathbf{X}) \to S(\mathbf{Y})$, $\beta : S(\mathbf{Y}) \to S(\mathbf{Z})$ denote the S^1-equivariant maps between unit spheres in corresponding representations defined by

$$\alpha(z_1, \dots, z_q) := (z_1^{d_1}, \dots, z_q^{d_q}),$$
$$\beta(z_1, \dots, z_q) := (z_1^{b_k - d_1}, \dots, z_q^{b_k - d_q}).$$

Obviously $S(\mathbf{X})/S^1 = CP^{d-1}$. A slight modification of the arguments given in [6, Sec. 3.2] permits to prove that $S(\mathbf{Z})$ is diffeomorphic to CP^{a-1} and $\beta \circ \alpha$ induces a monomorphism of cohomology rings

$$(\beta \circ \alpha)^* : H^*(S(\mathbf{Z})/S^1) \to H^*(S(\mathbf{X})/S^1).$$

Therefore,

$$\alpha^* : H^*(S(\mathbf{Y})/S^1) \to H^*(S(\mathbf{X})/S^1)$$

is also a monomorphism. Thus the cup-length of $S(\mathbf{Y})$ equals b_k. Since

$$\bigoplus_{j \in \mathcal{J}_k} \mathcal{A}_j : \mathcal{W}_k \to \mathbf{Y}$$

is an isomorphism of real representations of S^1, the proof is completed. □

Let

$$\mathcal{W}_k^{\perp,0} := \{w \in \mathcal{E}_0(W_k); \langle w, v\rangle = 0 \text{ for } v \in \mathcal{W}_k\},$$
$$\mathcal{W}_k^{\perp} := \mathcal{E}(\mathbb{W}_k) \cap \mathcal{W}_k^{\perp,0}.$$

From (12) and (18), we have

$$\langle u, \mathfrak{D}_k(v)\rangle = \langle \mathfrak{D}_k(u), v\rangle = 0 \quad \text{for } u \in \mathcal{W}_k, v \in \mathcal{E}(\mathbb{W}_k).$$

Therefore, $\mathfrak{D}_k(\mathcal{W}_k^{\perp}) \subset \mathcal{W}_k^{\perp,0}$. Since $\mathfrak{D}_k$, as an operator from $\mathcal{E}(\mathbb{W}_k)$ into $\mathcal{E}_0(\mathbb{W}_k)$, is Fredholm of index 0, it maps isomorphically $\mathcal{W}_k^{\perp}$ onto $\mathcal{W}_k^{\perp,0}$. Applying Theorem 5.1, we obtain an $\epsilon > 0$ and a mapping

$$g : B(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta] \to \mathcal{E}_0,$$

where $B(\mathcal{E}, \epsilon) := \{u \in \mathcal{E}; \|u\| < \epsilon\}$, such that

- f and g determine equivalent bifurcation problems on $[\lambda_k - \delta, \lambda_k + \delta]$;
- $g(B(\mathcal{W}_k, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta]) \subset \mathcal{W}_k$;
- $Dg(0, \lambda) = Df(0, \lambda)$ for $\lambda \in [\lambda_k - \delta, \lambda_k + \delta]$.

Setting $\varphi(w, \lambda) := g(w, \lambda)$, $w \in \mathcal{W}_k$, we obtain

$$\varphi : \mathcal{W}_k \times [\lambda_k - \delta, \lambda_k + \delta] \to \mathcal{W}_k,$$

which determines a finite-dimensional bifurcation problem on $[\lambda_k - \delta, \lambda_k + \delta]$ (one may call it a *reduction* of f to $\mathcal{W}_k$). Applying Lemmas 8.2 and 6.2, we obtain the following conclusion.

Conclusion 8.3. *For each $k \in \{1, \dots, q\}$, there exist $\delta, \epsilon > 0$ such that*

(a) *the mapping φ defines a bifurcation problem on $[\lambda_k - \delta, \lambda_k + \delta]$;*
(b) *$f^{-1}(0) \cap (S(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta])$ contains at least b_k different S^1-orbits.*

Now, to complete the proof of Theorem 7.2, it is enough to observe that, for sufficiently small δ and ϵ, different S^1 orbits in

$$f^{-1}(0) \cap \left(\bigcup_{k=1}^{q} S(\mathcal{E}, \epsilon) \times [\lambda_k - \delta, \lambda_k + \delta] \right)$$

correspond to geometrically distinct solutions to (11).

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