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AN UPPER BOUND ON THE TOTAL OUTER-INDEPENDENT DOMINATION NUMBER OF A TREE

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Abstract. A total outer-independent dominating set of a graph $G = (V(G), E(G))$ is a set D of vertices of G such that every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of a graph G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. We prove that for every tree T of order $n \geq 4$, with l leaves and s support vertices we have $\gamma_t^{oi}(T) \leq (2n + s - l)/3$, and we characterize the trees attaining this upper bound.

Keywords: total outer-independent domination, total domination, tree.

Mathematics Subject Classification: 05C05, 05C69.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star we mean a connected graph in which exactly one vertex has degree greater than one. By a double star we mean a graph obtained from a star by joining a positive number of vertices to one of its leaves.

We say that a subset of $V(G)$ is independent if there is no edge between every two its vertices. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D, while it is a total dominating set if every vertex of G has a neighbor in D. The domination (total domination, respectively) number of G, denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating (total dominating, respectively) set of G. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2], and further studied for example in [1]. For a comprehensive survey of domination in graphs, see [3, 4].

A subset $D \subseteq V(G)$ is a total outer-independent dominating set, abbreviated TOIDS, of G if every vertex of G has a neighbor in D, and the set $V(G) \setminus D$ is independent. The total outer-independent domination number of G, denoted by $\gamma_t^{oi}(G)$, is the minimum cardinality of a total outer-independent dominating set of G. A total outer-independent dominating set of G of minimum cardinality is called a $\gamma_t^{oi}(G)$-set. The study of total outer-independent domination in graphs was initiated in [5].

Chellali and Haynes [1] established the following upper bound on the total domination number of a tree. For every nontrivial tree T of order n with s support vertices we have $\gamma_t(T) \leq (n+s)/2$.

We prove the following upper bound on the total outer-independent domination number of a tree. For every tree T of order $n \geq 4$, with l leaves and s support vertices we have $\gamma_t^{oi}(T) \leq (2n+s-l)/3$. Moreover, we characterize the trees attaining this upper bound.

2. RESULTS

Since the one-vertex graph does not have a total outer-independent dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following two straightforward observations.

Observation 2.1. *Every support vertex of a graph G is in every $\gamma_t^{oi}(G)$-set.*

Observation 2.2. *For every connected graph G of diameter at least three there exists a $\gamma_t^{oi}(G)$-set that contains no leaf.*

We show that if T is a tree of order $n \geq 4$, with l leaves and s support vertices, then $\gamma_t^{oi}(T)$ is bounded above by $(2n+s-l)/3$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_6, and let $A(T_1)$ be a set containing all vertices of P_6 which are not leaves. Let H be a path P_3 with one of the leaves labeled u, and the support vertex labeled v. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a copy of H by joining the vertex u to a vertex of T_k adjacent to a path P_3. Let $A(T) = A(T') \cup \{u, v\}$.
- Operation $\mathcal{O}_2$: Attach a copy of H by joining the vertex u to a vertex of T_k which is not a leaf and is adjacent to a support vertex. Let $A(T) = A(T') \cup \{u, v\}$.
- Operation $\mathcal{O}_3$: Attach a copy of H by joining the vertex u to a leaf of T_k adjacent to a weak support vertex. Let $A(T) = A(T') \cup \{u, v\}$.

Now we prove that for every tree T of the family $\mathcal{T}$, the set $A(T)$ defined above is a TOIDS of minimum cardinality equal to $(2n+s-l)/3$.

Lemma 2.3. *If $T \in \mathcal{T}$, then the set $A(T)$ defined above is a $\gamma_t^{oi}(T)$-set of size $(2n+s-l)/3$.*

Proof. We use the terminology of the construction of the trees $T = T_k$, the set $A(T)$, and the graph H defined above. To show that $A(T)$ is a $\gamma_t^{oi}(T)$-set of cardinality $(2n+s-l)/3$ we use induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_6$, then $(2n+s-l)/3 = (12+2-2)/3 = 4 = |A(T)| = \gamma_t^{oi}(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let n' mean the order of the tree T', l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n' + 3$, $s = s' + 1$, and $l = l' + 1$. The vertex of T' to which is attached P_3 we denote by x. Let abc mean a path P_3 adjacent to x, and such that $a \neq u$. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 2.1, we have $v \in D$. Each one of the vertices v and b has to have a neighbor in D, thus $u, a \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we conclude that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 2$. We get $\gamma_t^{oi}(T) = |A(T)| = |A(T')| + 2 = (2n' + s' - l')/3 + 2 = (2n - 6 + s - 1 - l + 1)/3 + 2 = (2n + s - l)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n' + 3$, $s = s' + 1$, and $l = l' + 1$. The vertex of T' to which is attached P_3 we denote by x. Let y mean a support vertex adjacent to x. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let D be a $\gamma_t^{oi}(T)$-set that contains no leaf. By Observation 2.1 we have $v, y \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we conclude that $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 2$. In the same way as in the previous possibility we get $\gamma_t^{oi}(T) = (2n + s - l)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n' + 3$, $s = s'$, and $l = l'$. The leaf to which is attached P_3 we denote by x. Let y mean a neighbor of x other than u. It is easy to see that $A(T) = A(T') \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_t^{oi}(T)$-set that does not contain the vertex x, and does not contain any leaf. Let D be such a set. By Observation 2.1 we have $v \in D$. The vertex v has to have a neighbor in D, thus $u \in D$. The set $V(T) \setminus D$ is independent, thus $y \in D$. Let us observe that $D \setminus \{u, v\}$ is a TOIDS of the tree T' as the vertex x has a neighbor in $D \setminus \{u, v\}$. Therefore $\gamma_t^{oi}(T') \leq \gamma_t^{oi}(T) - 2$. Now we conclude $\gamma_t^{oi}(T) = \gamma_t^{oi}(T') + 2$. We get $\gamma_t^{oi}(T) = |A(T)| = |A(T')| + 2 = (2n' + s' - l')/3 + 2 = (2n - 6 + s - l)/3 + 2 = (2n + s - l)/3$. □

Now we establish the main result, an upper bound on the total outer-independent domination number of a tree together with the characterization of the extremal trees.

Theorem 2.4. *If T is a tree of order $n \ge 4$, with l leaves and s support vertices, then $\gamma_t^{oi}(T) \le (2n+s-l)/3$ with equality if and only if $T = K_{1,3}$ or $T \in \mathcal{T}$.*

Proof. First assume that $\mathrm{diam}(T) = 2$. Thus T is a star $K_{1,m}$ with $m \ge 3$. If $m = 3$, then $T = K_{1,3}$. We have $\gamma_t^{oi}(T) = 2 = (8+1-3)/3 = (2n+s-l)/3$. If $m \ge 4$, then $(2n+s-l)/3 = (2m+2+1-m)/3 = (m+3)/3 \ge (4+3)/3 > 2 = \gamma_t^{oi}(T)$. Now let us assume that $\mathrm{diam}(T) = 3$. Thus T is a double star. We have $(2n+s-l)/3 = (2n+2-n+2)/3 = (n+4)/3 \ge (4+4)/3 > 2 = \gamma_t^{oi}(T)$. Now assume that $\mathrm{diam}(T) = 4$. Let $v_1v_2v_3v_4v_5$ mean a longest path in T. If v_3 is adjacent to a leaf, then all support vertices of T form a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le s$. Now we get $\gamma_t^{oi}(T) \le s = s/3+2s/3 = s/3+2(n-l)/3 = (2n+s-2l)/3 < (2n+s-l)/3$. Now assume that T is not adjacent to any leaf. It is easy to observe that all support vertices of T together with the vertex v_3 form a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le s+1$. We have $n = l+s+1$. Now we get $\gamma_t^{oi}(T) \le s+1 = s/3+2s/3+1 = s/3+2(n-l-1)/3+1 = (2n+s-2l-2)/3+1 = (2n+s-l)/3+(1-l)/3 < (2n+s-l)/3$. Now let us assume that $\mathrm{diam}(T) = 5$. Let $v_1v_2v_3v_4v_5v_6$ mean a longest path in T. If both vertices v_3 and v_4 are adjacent to a leaf, then all support vertices of T form a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le s$. Now we get $\gamma_t^{oi}(T) \le s = s/3+2s/3 = s/3+2(n-l)/3 = (2n+s-2l)/3 < (2n+s-l)/3$. Now assume that exactly one of the vertices v_3 and v_4 is adjacent to a leaf. Without loss of generality we assume that v_3 is adjacent to a leaf. It is easy to observe that all support vertices of T together with the vertex v_4 form a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le s+1$. We have $n = l+s+1$. Now we get $\gamma_t^{oi}(T) \le s+1 = s/3+2s/3+1 = s/3+2(n-l-1)/3+1 = (2n+s-2l-2)/3+1 = (2n+s-l)/3+(1-l)/3 < (2n+s-l)/3$. Now assume that neither v_3 nor v_4 is adjacent to a leaf. It is easy to observe that all support vertices of T together with the vertices v_3 and v_4 form a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le s+2$. We have $n = l+s+2$. Now we get $\gamma_t^{oi}(T) \le s+2 = s/3+2s/3+2 = s/3+2(n-l-2)/3+2 = (2n+s-2l-4)/3+2 = (2n+s-l)/3+(2-l)/3$. If T has exactly two leaves, then $T = P_6 = T_1 \in \mathcal{T}$. By Lemma 2.3 we have $\gamma_t^{oi}(T) = (2n+s-l)/3$. Now assume that T has at least three leaves. We have $\gamma_t^{oi}(T) \le (2n+s-l)/3+(2-l)/3 < (2n+s-l)/3$.

Now assume that $\mathrm{diam}(T) \ge 6$. Thus the order of the tree T is an integer $n \ge 7$. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$, with l' leaves and s' support vertices.

First assume that some support vertex of T, say x, is strong. Let y mean a leaf adjacent to x. Let $T' = T - y$. We have $n' = n-1$, $s' = s$, and $l' = l-1$. Let D' be any $\gamma_t^{oi}(T')$-set. By Observation 2.1 we have $x \in D'$. Of course, D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T')$. Now we get $\gamma_t^{oi}(T) \le \gamma_t^{oi}(T') = (2n'+s'-l')/3 = (2n-2+s-l+1)/3 = (2n+s-l)/3-1/3 < (2n+s-l)/3$. Therefore every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v, w be the parent of u, and d be the parent of w in the rooted tree. By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) \ge 3$. Assume that among the descendants of u there is a support vertex, say x, different than v. Let $T' = T - T_v$. We have $n' = n-2$, $s' = s-1$, and $l' = l - 1$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex x has to have a neighbor in D', thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1 \leq (2n' + s' - l')/3 + 1 = (2n - 4 + s - 1 - l + 1)/3 + 1 = (2n + s - l)/3 - 1/3 < (2n + s - l)/3$.

Now assume that some descendant of u, say x, is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $s' = s - 1$, and $l' = l - 1$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. The vertex v has to have a neighbor in D', thus $u \in D'$. It is easy to see that D' is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T')$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') \leq (2n' + s' - l')/3 = (2n - 2 + s - 1 - l + 1)/3 = (2n + s - l)/3 - 2/3 < (2n + s - l)/3$.

Now assume that $d_T(u) = 2$. First assume that there is a descendant of w, say k, such that the distance of w to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3, say klm. Let $T' = T - T_u$. We have $n' = n - 3$, $s' = s - 1$, and $l' = l - 1$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 \leq (2n' + s' - l')/3 + 2 = (2n - 6 + s - 1 - l + 1)/3 + 2 = (2n + s - l)/3$. If $\gamma_t^{oi}(T) = (2n + s - l)/3$, then obviously $\gamma_t^{oi}(T') = (2n' + s' - l')/3$. The tree T' has at least seven vertices. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Now assume that there is a descendant of w, say k, such that the distance of w to the most distant vertex of T_k is two. Thus k is a support vertex. Let $T' = T - T_u$. In the same way as in the previous possibility we get $\gamma_t^{oi}(T) \leq (2n + s - l)/3$. If $\gamma_t^{oi}(T) = (2n + s - l)/3$, then $\gamma_t^{oi}(T') = (2n' + s' - l')/3$. The tree T' has at least six vertices. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Now assume that some descendant of w, say k, is a leaf. Let $T' = T - t - k$. We have $n' = n - 2$, $s' = s - 1$, and $l' = l - 1$. Let D' be a $\gamma_t^{oi}(T')$-set that contains no leaf. By Observation 2.1 we have $u \in D'$. The vertex u has to have a neighbor in D', thus $w \in D'$. It is easy to observe that $D' \cup \{v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 1 \leq (2n' + s' - l')/3 + 1 = (2n - 4 + s - 1 - l + 1)/3 + 1 = (2n + s - l)/3 - 1/3 < (2n + s - l)/3$.

Now assume that $d_T(w) = 2$. First assume that d is adjacent to a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $s' = s - 1$, and $l' = l$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 \leq (2n' + s' - l')/3 + 2 = (2n - 6 + s - 1 - l)/3 + 2 = (2n + s - l)/3 - 1/3 < (2n + s - l)/3$.

Now assume that d is not adjacent to any leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $s' = s$, and $l' = l$. Let D' be any $\gamma_t^{oi}(T')$-set. It is easy to see that $D' \cup \{u, v\}$ is a TOIDS of the tree T. Thus $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2$. Now we get $\gamma_t^{oi}(T) \leq \gamma_t^{oi}(T') + 2 \leq (2n' + s' - l')/3 + 2 = (2n - 6 + s - l)/3 + 2 = (2n + s - l)/3$. If $\gamma_t^{oi}(T) = (2n + s - l)/3$, then $\gamma_t^{oi}(T') = (2n' + s' - l')/3$. The tree T' has at least four vertices and is different from $K_{1,3}$ as T' has no strong support vertex. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$. $\square$

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