Journal of Computational and Applied Mathematics 236 (2011) 1576–1590

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Solving boundary value problems for delay differential equations by a fixed-point method

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ARTICLE INFO

Article history:
Received 21 February 2010
Received in revised form 18 September 2011

MSC:
65L03
65L10
65L20

Keywords:
Boundary value problems for differential equations with deviated arguments
Fixed-point problems
Numerical methods for boundary value problems for differential equations with deviated arguments

ABSTRACT

A general linear boundary value problem for a nonlinear system of delay differential equations (DDE in short) is reduced to a fixed-point problem $v = Av$ with a properly chosen (generally nonlinear) operator A. The unknown fixed-point v is approximated by piecewise linear function v_h defined by its values $v_i = v_h(t_i)$ at grid points t_i, $i = 0, 1, \ldots, N$, where N is a given positive integer and $h = \max_{1 \le i \le N}(t_i - t_{i-1})$. Under suitable assumptions, the existence of a fixed-point of A is equivalent to existence of so called $\varepsilon(h)$-approximate fixed-points of $v_h = Av_h$, which can be found by minimization of $L_2^{(n)}$ norm of residuum $v_h - Av_h$ with respect to the variables v_i. These $\varepsilon(h)$-approximate fixed-points are used for obtaining approximate solutions of the original boundary value problem for a system of DDE. Numerical experiments with the boundary value problem for a system of delay differential equations of population dynamics as well as with two periodic boundary value problems: one for the periodic distributed delay Lotka–Volterra competition system and the second one modeling two coupled identical neurons with time-delayed connections show an efficiency of this kind of approach.

1. Introduction

In [1] the following boundary value problem for a system of delay differential equations of population dynamics is investigated

$$\begin{cases} \dfrac{dN_1(t)}{dt} = r_1 N_1(t)\left[\dfrac{K_1 + A_1 N_2(t-\tau_2)}{1 + N_2(t-\tau_2)} - N_1(t)\right] \\[2ex] \dfrac{dN_2(t)}{dt} = r_2 N_2(t)\left[\dfrac{K_2 + A_2 N_1(t-\tau_1)}{1 + N_1(t-\tau_1)} - N_2(t)\right] \end{cases} \tag{1.1}$$

for $0 < t \le \ell$, subject to the boundary condition

$$N_1(t) = \mu_1 N_1(\ell + t) + \varphi_1(t), \qquad N_2(t) = \mu_2 N_2(\ell + t) + \varphi_2(t), \tag{1.2}$$

where $-\tau = -\max\{\tau_1, \tau_2\} \le t \le 0$ (for the problem itself also see [2]).

The authors of the above-mentioned paper assume that the coefficients r_i, A_i, K_i and the delays τ_i, $i = 1, 2$, of system (1.1) are constant, positive and satisfy $A_i > K_i$, $i = 1, 2$. For the same i, N_i is the amount of a single species, r_i is the growth rate of each species, A_i is the ratio of the mutualism (or cooperation) between different species, K_i is the amount of environmental accommodations (carrying capacity). For obtaining an approximate solution to problem (1.1)–(1.2) they employ Chebyshev polynomial series. They assume that the unknown solution $(N_1, N_2)^T$ and an auxiliary function $(\psi_1, \psi_2)^T$

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doi:10.1016/j.cam.2011.09.021

have finite expansions into $(m+1)$-term Chebyshev polynomial series and reduce the task of finding this approximate solution to determining the coefficients of these expansions.

In this paper we propose the approach which was used in [3] and proved to be more effective than many other known methods as far as the accuracy of the approximated solutions is concerned. As it also works for more general problems than the boundary value problem (1.1)–(1.2) we use the last one as a model problem and generalize it to the following boundary value problem for systems of delay differential equations

$$\frac{dy(t)}{dt} = f(t, y(t), y(t-\tau_1), \ldots, y(t-\tau_k)), \quad t \in (a, b], \tag{1.3}$$

$$P_1 y(t) + P_2 y(b-a+t) = \varphi(t), \quad t \in [a-\tau, a], \tag{1.4}$$

where we assume that: $f = (f_1, \ldots, f_n)^T : [a, b] \times \mathbb{R}^{k+1} \to \mathbb{R}^n$ is continuous on some compact subset $\bar{D}$ of the set $[a, b] \times \mathbb{R}^{k+1}$, $y(t) = (y_1(t), \ldots, y_n(t))^T$ is the unknown function defined on $[a-\tau, b]$, $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^T$ is continuous on $[a-\tau, a]$, $P_k = [p_{ij}^{(k)}]_{i,j=1,\ldots,n}$, $k = 1, 2$, are constant matrices and P_1 is nonsingular, $\tau_i > 0$ for $i = 1, \ldots, k$ and $\tau = \min_{1 \le i \le k}\{\tau_i\}$. In the sequel, we also assume that problem (1.3)–(1.4) has a unique continuous on $[a, b]$ solution.

Let us notice that the boundary condition (1.2), for $a = 0$ and $b = \ell$, can be written in the form (1.4) with

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix} \quad \text{and} \quad \varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}. \tag{1.5}$$

The nonsingularity of P_1 implies that for any $s = (s_1, \ldots, s_n)^T$ the system $P_1 r = -P_2 s - \varphi(a)$ has a unique solution $r = (r_1, \ldots, r_n)^T$. Then defining for $t \in [a-\tau, b]$ the function $q(t) = (q_1(t), \ldots, q_n(t))^T$ with $q_i(t) = r_i + ((s_i - r_i)/(b-a))(t-a)$ and making the substitution $y(t) = u(t) - q(t)$ we can reduce problem (1.3)–(1.4) to an equivalent problem

$$\frac{du(t)}{dt} = \tilde{f}(t, u(t), u(t-\tau_1), \ldots, u(t-\tau_k)), \quad t \in (a, b], \tag{1.6}$$

$$P_1 u(t) + P_2 u(b-a+t) = \tilde{\varphi}(t), \quad t \in [a-\tau, a] \tag{1.7}$$

with $\tilde{\varphi}(t) = \varphi(t) + P_1 q(t) + P_2 q(b-a+t),\ t \in [a-\tau, a]$ and with the property

$$P_1 u(a) + P_2 u(b) = (0, \ldots, 0)^T. \tag{1.8}$$

For this reason, we can assume without loss of generality, that in (1.4) $\varphi(a) = (0, \ldots, 0)^T$.

The equivalence of (1.3)–(1.4) and (1.6)–(1.7) consists in the following: if $y(t)$, $t \in [a-\tau, b]$ is a solution to (1.3)–(1.4) then $u(t) = y(t) + q(t)$, $t \in [a-\tau, b]$ is a solution to (1.6)–(1.7) and vice versa, if $u(t)$, $t \in [a-\tau, b]$ is a solution to (1.6)–(1.7) then $y(t) = u(t) - q(t)$, $t \in [a-\tau, b]$ is a solution to (1.3)–(1.4).

Let us also remark that condition (1.7) includes periodic boundary conditions if $P_1 = -P_2 = I_{n\times n}$ and $\tilde{\varphi}(t) = (0, \ldots, 0)^T$, where $I_{n\times n}$ is the $n \times n$ identity matrix.

The organization of the paper is as follows. In Section 2 we show how to reduce problem (1.3)–(1.4) to a fixed-point problem. In order to make the paper self-contained in Section 3 we recall the necessary definitions, the main theorem and corollary as well as the formulation of the $\boldsymbol{\varepsilon}(\boldsymbol{h})$ *approximate method*, which were introduced in [3]. In Section 4 we prove three lemmas which show the continuity of corresponding operators and the fourth lemma showing the compactness of the corresponding family of functions. These properties guarantee that the $\varepsilon(h)$ approximate method is well defined and convergent. Section 5 is devoted to numerical experiments. Using three examples of boundary value problems: a boundary value problem for a system of delay differential equations of population dynamics, a periodic boundary value problem for the periodic distributed delay Lotka–Volterra competition system and a periodic boundary value problem for a system of delay differential equations modeling two coupled identical neurons with time-delayed connections, we show how to reduce these problems to fixed-point problems and provide the results of numerical experiments. We end the paper with concluding remarks.

2. Reduction of the problem (1.3)–(1.4) to a fixed-point problem

For the sake of shortness let us denote the right-hand side of (1.3) by

$$f(t, y(t), y(t-\tau_1), \ldots, y(t-\tau_k)) = F(t, y(t), y(\cdot))$$

and rewrite (1.3) as

$$y'(t) = F(t, y(t), y(\cdot)), \quad t \in (a, b]. \tag{2.1}$$

Assume that there exists a continuous matrix $B = B(t),\ t \in [a, b]$, such that for the fundamental matrix $\mathcal{Y}$ of the system $y' = By$, satisfying $\mathcal{Y}(a) = I_{n\times n}$, the matrix $Q = P_1 + P_2\mathcal{Y}(b)$ is nonsingular. Let us consider the following auxiliary BVP for the system of differential equations

$$y'(t) = By(t) + v(t), \quad t \in [a, b], \tag{2.2}$$

$$P_1 y(a) + P_2 y(b) = (0, \ldots, 0)^T. \tag{2.3}$$

The solution to (2.2)–(2.3) can be written

$$y(t) = \mathcal{Y}(t)\left[-Q^{-1}P_2\mathcal{Y}(b)\int_a^b \mathcal{Y}^{-1}(s)v(s)\,ds + \int_a^t \mathcal{Y}^{-1}(s)v(s)\,ds\right]. \tag{2.4}$$

Denoting the right-hand side of (2.4) by $(Gv)(t)$ we can write the solution to (2.2)–(2.3) in the form

$$y(t) = (Gv)(t) = \begin{pmatrix} (g_1 v)(t) \\ \vdots \\ (g_n v)(t) \end{pmatrix} \tag{2.5}$$

and reduce the problem (2.1) with boundary condition (1.4) to the following fixed-point problem

$$v(t) = F(t, (Gv)(t), (Gv)(\cdot)) - B(Gv)(t). \tag{2.6}$$

3. An approximate method for a fixed-point problem

3.1. Notation and basic definitions

In [3] there were introduced the following notions and definitions. Let: (L, d) be a metric space with a metric d, M be an interval $(0, h_0]$, $h_0 > 0$, or a sequence of positive real numbers convergent to zero, $\{S_h : h \in M\}$ be a family of spaces of grid functions defined for each h on a finite set of grid points from a compact set K, $\{r_h : h \in M,\ r_h : L \longrightarrow S_h\}$ be a family of operators mapping L onto S_h and $\{p_h : h \in M,\ p_h : S_h \longrightarrow L\}$ be a family of operators mapping S_h into L.

Remark 1. In the sequel, speaking of an arbitrary family of sets Q_h or transformations q_h we shall mean the families $\{Q_h \mid h \in M\}$ or $\{q_h \mid h \in M\}$ respectively.

Definition 3.1. We shall say that the family of spaces S_h and the families of operators r_h and p_h define a convergent approximation of L if for any $x \in L$ the condition

$$d(p_h r_h x, x) \underset{h\to 0}{\longrightarrow} 0 \tag{3.1}$$

holds.

Definition 3.2. We shall say that the family x^h of functions in L is compact in L with $h \to 0$ if for any sequence of functions belonging to this family corresponding to the sequence of values of the parameter h convergent to zero a subsequence convergent in L can be chosen.

Now, let $\epsilon \geq 0$ be a given real number.

Definition 3.3. We shall say that $x \in L$ is an ϵ-fixed-point of $A : L \to L$ if

$$d(x, Ax) \leq \epsilon. \tag{3.2}$$

3.2. The main theorem and corollaries, an approximate method for a fixed-point problem

In [3] the following theorem and corollary have been proved.

Theorem 3.1. *If*

(a) *operator A is continuous in L,*
(b) *the family of spaces S_h and the families of operators r_h and p_h define a convergent approximation of L,*

then A possesses at least one fixed-point if and only if there exists a non-negative function $\varepsilon(h)$, $\varepsilon(h) \to 0$ for $h \to 0$ such that the operator A possesses $\varepsilon(h)$-fixed-points $p_h x_h$ and the family $\{p_h x_h \mid h \in M\}$ is compact with $h \to 0$.

Corollary 3.1. *If*

(i) *the operator $A : L \longrightarrow L$ is continuous and has a unique fixed-point x,*
(ii) *the family of spaces S_h and the families of operators r_h and p_h define a convergent approximation of L,*

(iii) *for* $c_h = d(0, p_h r_h x)$ *a sequence* $(p_h x_h)$ *satisfying the condition*

$$d(p_h x_h, Ap_h x_h) = \varepsilon_c(h) \overset{\text{def}}{=} \min_{\{y_h : d(0, p_h y_h) \le c_h\}} d(p_h y_h, Ap_h y_h), \tag{3.3}$$

is compact with $h \to 0$,

then

$$\lim_{h \to 0} p_h x_h = x. \tag{3.4}$$

Corollary 3.1 suggests the following approximate method for a fixed-point problem with a continuous operator $A : L \longrightarrow L$, which we will call the $\boldsymbol{\varepsilon}(\boldsymbol{h})$ *approximate method*:

1. choose such families of spaces S_h of grid functions $u_h = (u_1, u_2, \ldots, u_{n(h)})$ defined on finite sets of grid points $(x_1, x_2, \ldots, x_{n(h)})$ in a given compact set K and operators p_h and r_h that
   (a) they define a convergent approximation of L,
   (b) the family of $p_h x_h$ satisfying $d(p_h x_h, Ap_h x_h) \longrightarrow 0$ if $h \to 0$ is compact with $h \to 0$,
   (c) the functions
   $$q_h(u_1, u_2, \ldots, u_{n(h)}) = d(p_h u_h, Ap_h u_h) \tag{3.5}$$
   are continuous with respect to the variables $u_1, u_2, \ldots, u_{n(h)}$;
2. choose large enough constant c and small enough parameter h and find an $\varepsilon_c(h)$ fixed-point of A by minimization of q_h on the (compact in S_h) closed ball $B(0_h, c)$ of radius c centered at 0_h, where $0_h = (0, \ldots, 0)$ is the zero function in S_h.

Then $p_h x_h$ approximates the fixed-point x of A because $p_h x_h \to x$ if $h \to 0$.

Remark 2. It follows from the proof of Theorem 3.1 (see [3]) that this method can be applied to the problems with a unique fixed-point if operator A is continuous only in some neighborhood of the fixed-point if the initial approximations to an $\varepsilon_c(h)$ fixed-point is taken close enough to the fixed-point of A.

3.3. An example of a convergent approximation and other auxiliary theorems

Also in [3] there was given the following example of a convergent approximation of the space

$$L_n^2[a, b] = \underbrace{L^2[a, b] \times \cdots \times L^2[a, b]}_{n \text{ times}}$$

with the metric d induced by the norm $\| \cdot \|_2^{(n)}$ defined for $v \in L_n^2[a, b]$ by putting:

$$\|v\|_2^{(n)} = \left(\sum_{i=1}^{n} \|v^i\|_2^2 \right)^{\frac{1}{2}}, \quad \text{where } \|v^i\|_2 = \left(\int_a^b |v^i(s)|^2 \, ds \right)^{\frac{1}{2}}. \tag{3.6}$$

Now, let on $[a, b]$ be defined a family of grids N_H $\left(H = 2h,\ h = \frac{b-a}{2N},\ N = 1, 2, \ldots\right)$ with grid points $x_k = a + kH,\ k = 0, 1, \ldots, N$. Denote the set of h by M and for $h \in M$ let

$$S_h^n = S_h^n[a, b] = \underbrace{S_h[a, b] \times \cdots \times S_h[a, b]}_{n \text{ times}} \tag{3.7}$$

be a linear space (over $\mathbb{R}$) of grid functions defined on the grid N_H.

We define the extension operators $p_h : S_h^n \to L_n^2[a, b]$ by the formula

$$p_h v_h = (\bar{p}_h v_h^1, \ldots, \bar{p}_h v_h^n), \tag{3.8}$$

where $\bar{p}_h v_h^i$, for a fixed h, is a piecewise linear function with values in the grid points N_H equal to the values of the grid function v_h^i in these grid points.

We also extend the domain of the function $v \in L_n^2[a, b]$ by putting

$$v(s) = 0 \quad \text{if } s \notin [a, b]. \tag{3.9}$$

Now, for a fixed h define the restriction operator $r_h : L_n^2[a, b] \to S_h^n$ by putting

$$r_h v = (\bar{r}_h v^1, \ldots, \bar{r}_h v^n), \tag{3.10}$$

where

$$\bar{r}_h v^i(x_k) = \tilde{v}^1(x_k) = \frac{1}{2h} \int_{-\infty}^{\infty} \omega_h(x_k, s) v^i(s) \, ds.$$

The integral appearing in this formula is a Lebesgue integral and the function $\omega_h(x, s)$ is an averaging kernel (for details on averaging kernels see [4]). Next, there were proved the following theorems.

Theorem 3.2 ([3]). *There exists a constant K (independent of $h \in M$ and $u, v \in L_n^2[a,b]$) such that for all h, u, v*

$$d(p_h r_h u, p_h r_h v) \le K d(u, v). \tag{3.11}$$

Theorem 3.3 ([3]). *For each $v \in L_n^2[a,b]$*

$$d(p_h r_h v, v) \underset{h\to 0}{\longrightarrow} 0,$$

i.e. the family of spaces S_h^n and the families of operators p_h and r_h define a convergent approximation of $L_n^2[a,b]$.

4. Solution of fixed-point problem (2.6) by the $\varepsilon(h)$ approximate method

It is clear that operator G defined by formula (2.5) is linear with respect to the variable v.

Denote by $C_0^{(n)}[a,b]$ the space of continuous functions $y = (y_1, \dots, y_n)^T$, $y : [a,b] \to \mathbb{R}^n$ satisfying condition (2.3) with norm defined by the formula $\|y\|_0 = \left(\sum_{i=1}^n (\max_{a\le t\le b} \|y_i(t)\|)^2\right)^{1/2}$.

Using the Hölder inequality it is easy to verify the following lemma:

Lemma 4.1. *The operator $G : L_n^2[a,b] \to C_0^{(n)}[a,b]$ defined by formula (2.5) is continuous.*

Now, for $i = 1, \dots, k$, let us define the operators $Q_i : C_0^{(n)}[a,b] \to C^{(n)}[a,b]$,

$$(S_i y)(t) = \begin{cases} P_1^{-1}(\tilde{\varphi}(t-\tau_i) - P_1^{-1}P_2 y(b-a+t-\tau_i)) & \text{if } t-\tau_i < a, \\ y(t-\tau_i) & \text{if } a \le t-\tau_i \le b. \end{cases}$$

It is also easy to verify the following lemma:

Lemma 4.2. *The operators $S_i : C_0^{(n)}[a,b] \to C^{(n)}[a,b]$ are continuous.*

Denote

$$(Sy)(t) = ((S_1 y)(t), \dots, (S_k y)(t)). \tag{4.1}$$

Next, observe that the righthand side of (2.6) defines the operator
$A : L_n^2[a,b] \to C^{(n)}[a,b] \subset L_n^2[a,b]$, i.e.

$$(Av)(t) = F(t, (Gv)(t), (SGv)(t)) - B(Gv)(t). \tag{4.2}$$

We have the following lemma:

Lemma 4.3. *The operator $A : L_n^2[a,b] \to L_n^2[a,b]$ defined by (4.2) is continuous.*

Proof. For a given $v \in L_n^2[a,b]$ and $c > 0$ let $U(v,c)$ denote an open ball in $L_n^2[a,b]$ with radius c centered at v, for the vectors $y(t) = (y_1(t), \dots, y_n(t))$, $w = (w_1, \dots, w_n)$ and $z = (z_1, \dots, z_n)$ let

$$\max_{a\le t\le b} y(t) = (\max_{a\le t\le b} y_1(t), \dots, \max_{a\le t\le b} y_n(t)),$$
$$|w| = (|w_1|, \dots, |w_n|) \quad \text{and } w \le z \quad \text{if and only if} \quad w_i \le z_i \quad \forall i = 1, \dots, n.$$

Then for $u \in U(v,c)$ the definition of Q implies

$$\max_{a\le t\le b} |SG(u-v)(t)| \le \|P_1^{-1}P_2\|_\infty \max_{a\le t\le b} |G(u-v)(t)|. \tag{4.3}$$

Now, let us return to formula (2.4) and denote by m_{ij} the entries of $-Q^{-1}P_2\mathcal{Y}(b)$, by $\bar{y}_{ij}(t)$ the entries of $\mathcal{Y}(t)$ and by $\tilde{y}_{ij}(s)$ the entries of $\mathcal{Y}^{-1}(s)$. We also put

$$\bar{m} = \max_{1\le i,j\le n} |m_{ij}|, \qquad \bar{r} = \max_{1\le i,j\le n} \left(\int_a^b \tilde{y}_{ij}^2(s)ds\right)^{1/2}.$$

Then from the Hölder inequality we derive the following inequality for the i-th component $[G(u-v)(t)]_i$ of $G(u-v)(t)$

$$|[G(u-v)(t)]_i| \le \bar{r}\left(n\bar{m}\sum_{j=1}^n |\bar{y}_{ij}(t)| + 1\right)\sum_{j=1}^n \left(\int_a^b [u(s)-v(s)]_j^2 ds\right)^{1/2}, \tag{4.4}$$

where $[u(s)-v(s)]_j$, $j = 1, \dots, n$, is the j-th component of $u(s)-v(s)$.

From (4.3)–(4.4) it follows that for c small enough there exists a compact subset of $\bar{D}$, say D_c, such that $(t, (Gu)(t), (SGu)(t)) \in D_c$ whenever $t \in [a, b]$ and $u \in U(v, c)$. Define $\bar{F}$ by the formula

$$\bar{F}(t, (Gv)(t), (SGv)(t)) = F(t, (Gv)(t), (SGv)(t)) - B(Gv)(t) \tag{4.5}$$

and let $\mu_i(\eta),\ i = 1, \ldots, n$, denote the modulus of continuity of the i-th component of $\bar{F}$ on D_c. Then for any $u \in U(v, c)$ and the norm $\|\cdot\|_2^{(n)}$ defined by (3.6) the following inequality holds

$$\|Au - Av\|_2^{(n)} \le (b-a)^{1/2}\mu(\eta), \tag{4.6}$$

where

$$\eta = \max(\max_{1\le i\le n}\{\max_{t\in[a,b]} |G(u-v)_i(t)|\}, \max_{1\le i\le n}\{\max_{t\in[a,b]} |SG(u-v)_i(t)|\}),$$

$$\mu(\eta) = \left(\sum_{i=1}^{n}(\mu_i(\eta))^2\right)^{1/2}.$$

Inequality (4.6) implies the continuity of A in $L_n^2[a, b]$. $\square$

The process of minimization of the functions q_h defined by (3.5) on a ball $B(0, c)$ in S_h produces a family $p_h v_h$. Now, it remains to prove that this family is compact with $h \to 0$. Put $(p_h v_h)_s(t) = p_h v_h(t+s)$, $\|[(p_h v_h)_s - p_h v_h]_i\|_2^2 = \int_a^b |[(p_h v_h)_s(t) - p_h v_h(t)]_i|^2 dt$, where $[(p_h v_h)_s(t) - p_h v_h(t)]_i$ is the i-th component of $(p_h v_h)_s(t) - p_h v_h(t)$.

Then according to Riesz's theorem (see, e.g. [4]), the family $p_h v_h$ is compact with $h \to 0$ if it is bounded in $L_n^2[a, b]$ and

$$(\|[(p_h v_h)_s - p_h v_h]_1\|_2^2, \ldots, \|[(p_h v_h)_s - p_h v_h]_n\|_2^2) \xrightarrow[s\to 0]{} (0, \ldots, 0) \tag{4.7}$$

uniformly with respect to all $p_h v_h$.

It is clear, that condition (4.7) holds if for $i = 1, \ldots, n$, $\|[(p_h v_h)_s - p_h v_h]_i\|_2 \to 0$ with $s \to 0$ uniformly with respect to all $p_h v_h$. Introduce the notation $\tau^h(t) = (\tau_1^h(t), \ldots, \tau_n^h(t)) = p_h v_h(t) - A p_h v_h(t)$. To prove the compactness of (4.9) we will consider the difference $[(p_h v_h)_s - p_h v_h]_i = [(A p_h v_h)_s - A p_h v_h]_i + [(\tau_i^h)_s - \tau_i^h]_i$.

Remark 3. We should bear in mind that assuming (3.9) we obtain for $s > 0$

$$\|[(Ap_h v_h)_s - Ap_h v_h]_i\|_2^2 = \int_a^{b-s}([(Ap_h v_h)_s(\theta) - Ap_h v_h(\theta)]_i)^2 d\theta + \int_{b-s}^{b}([Ap_h v_h(\theta)]_i)^2 d\theta \tag{4.8}$$

and we have to consider the two integrals appearing in (4.8) separately. However, due to the uniform continuity of $\mathcal{Y}$ and $\mathcal{Y}^{-1}$ on $[a, b]$ and the boundedness of $\|[p_h v_h]_i\|_2$, which results in the boundedness of the integrand $([Ap_h v_h(\theta)]_i)^2$, the second integral can be made arbitrarily small by choosing small enough s. A similar problem appears for $s < 0$, where we have to consider one integral over the interval $[a, a+|s|]$ and the second integral on $[a+|s|, b]$. The integral over $[a, a+|s|]$ can also be made arbitrarily small by choosing small enough s.

We have the following lemma

Lemma 4.4. *The family $p_h v_h$ satisfying*

$$\|p_h v_h - Ap_h v_h\|_2^{(n)} = \min_{u_h \in B(0,c)} \|p_h u_h - Ap_h u_h\|_2^{(n)} \tag{4.9}$$

is compact with $h \to 0$.

Proof. Let $p_h v_h$ satisfy (4.9). Obviously, there exists a $\bar{c}$ for which we have $\|p_h v_h\|_2^{(n)} \le \bar{c}(b-a)^{1/2}$, i.e. the family $p_h v_h$ is bounded. We will show that for $i = 1, \ldots, n$ and any $\varepsilon > 0$ there exist h_ε and s_ε such that if $h < h_\varepsilon$ and $|s| < s_\varepsilon$ then

$$\|[(p_h v_h)_s - p_h v_h]_i\|_2 < \varepsilon \tag{4.10}$$

for all $p_h v_h$ satisfying (4.9). For $(\tau_i^h)_s$ the following inequality holds

$$\|(\tau_i^h)_s\|_2 \le \|\tau^h\|_2^{(n)} \le \varepsilon_c(h), \quad \varepsilon_c(h) \xrightarrow[h\to 0]{} 0. \tag{4.11}$$

For $\|[(p_h v_h)_s - p_h v_h]_i\|_2$ we have

$$\|[(p_h v_h)_s - p_h v_h]_i\|_2 \le \|[(Ap_h v_h)_s - Ap_h v_h]_i\|_2 + \|(\tau_i^h)_s\|_2 + \|\tau_i^h\|_2. \tag{4.12}$$

Assume that $s > 0$ and put $M = -Q^{-1}P_2\mathcal{Y}(b)$. Now, consider the difference $|[(Gp_hv_h)_s(t) - Gp_hv_h(t)]_i|$ for $t+s \le b$ (related to the first integral in (4.8))

$$\begin{aligned}&|[(Gp_hv_h)_s(t) - Gp_hv_h(t)]_i|\\ &= \left|\left[(\mathcal{Y}_s(t) - \mathcal{Y}(t))\left(M\int_a^b \mathcal{Y}^{-1}(\theta)p_hv_h(\theta)d\theta + \int_t^{t+s}\mathcal{Y}^{-1}(\theta)p_hv_h(\theta)d\theta\right)\right]_i\right|. \end{aligned} \tag{4.13}$$

The uniform continuity of $\mathcal{Y}$ and $\mathcal{Y}^{-1}$ on $[a, b]$ and the boundedness of $\|[p_hv_h]_i\|_2$ imply the existence of a function $M_{i1}^{(h,c)}$ of argument $s \in \mathbb{R}$, $M_{i1}^{(h,c)}(s) \xrightarrow[s\to 0]{} 0$, such that

$$\max_{a\le t\le b-s} |[(Gp_hv_h)_s(t) - Gp_hv_h(t)]_i| \le M_{i1}^{(h,c)}(s). \tag{4.14}$$

Using a similar reasoning, we can conclude the existence of a function $M_{i2}^{(h,c)}$ of argument $s \in \mathbb{R}$, $M_{i2}^{(h,c)}(s) \xrightarrow[s\to 0]{} 0$, such that

$$\max_{a\le t\le b} |[(SGp_hv_h)_s(t) - SGp_hv_h(t)]_i| \le M_{i2}^{(h,c)}(s). \tag{4.15}$$

Denote by D^c such a compact subset of D that

$$\begin{aligned}&(t, (Gp_hv_h)(t), (SGp_hv_h)(t)) \in D^c,\\ &(t+s, (Gp_hv_h)_s(t), (SGp_hv_h)_s(t)) \in D^c,\end{aligned}$$

whenever $t \in [a, b]$ and $v_h \in B(0, c)$. Let $\bar{\mu}_i(\eta)$ be the modulus of continuity of the i-th component $\bar{F}_i$ of $\bar{F}$ on D^c, where

$$\eta = \eta_s^{(h,c)} = \max(|s|, M_{11}^{(h,c)}(s), \ldots, M_{n1}^{(h,c)}(s), M_{12}^{(h,c)}(s), \ldots, M_{n2}^{(h,c)}(s)).$$

Put $\delta_i^2(s) = \int_{b-s}^b ([Ap_hv_h(\theta)]_i)^2\, d\theta$. Then (cp. Remark 3) $\delta_i(s) \xrightarrow[s\to 0]{} 0$. We have

$$\|[(Ap_hv_h)_s - Ap_hv_h]_i\|_2 \le (b-a)^{1/2}\bar{\mu}_i(\eta) + \delta_i(s). \tag{4.16}$$

It follows from (4.11) and (4.16) that the righthand side of inequality (4.12) can be made arbitrarily small by choosing h and s small enough. A similar reasoning can be employed for $s < 0$ to arrive at a similar to (4.16) inequality. This means that there exist h_ε and s_ε such that if $h < h_\varepsilon$ and $|s| < s_\varepsilon$ then inequality (4.10) is fulfilled for all p_hv_h satisfying (4.9), which implies that p_hv_h is compact with $h \to 0$. $\square$

5. Numerical experiments

Example 5.1. As the first example for numerical experiments we take the problem (1.1)–(1.2) with $r_1 = 1.2$, $r_2 = 1.3$, $K_1 = 2.0$, $K_2 = 1.0$, $A_1 = 2.2$, $A_2 = 2.0$, $\tau_1 = 1.0$, $\tau_2 = 0.5$, $\mu_1 = 0.006$, $\mu_2 = 0.004$, $\ell = 2.0$, $\varphi_1(t) = \varphi_1 = 2.12$, $\varphi_2(t) = \varphi_2 = 1.68$. To reduce it to the form (1.6)–(1.7) with the property (1.8) we put $N_1(t) = \tilde{N}_1(t) - q_1(t)$, $N_2(t) = \tilde{N}_2(t) - q_2(t)$, where

$$q_1(t) = \frac{1-\mu_1+\varphi_1}{2}t + \mu_1 - \varphi_1, \qquad q_2(t) = \frac{1-\mu_2+\varphi_2}{2}t + \mu_2 - \varphi_2.$$

For the given above numerical values of $\mu_1, \mu_2, \varphi_1, \varphi_2$ we get

$$\begin{aligned}&q_1(t) = 1.557t - 2.114, \qquad q_2(t) = 1.338t - 1.676, \qquad \text{and}\\ &\tilde{\varphi}_1(t) = 1.547658t, \qquad \tilde{\varphi}_2(t) = 1.332648t.\end{aligned}$$

Then from (1.1)–(1.2) we obtain the following system for $\tilde{N}_1, \tilde{N}_2$

$$\begin{cases} \dfrac{d\tilde{N}_1(t)}{dt} = q_1'(t) + r_1(\tilde{N}_1(t) - q_1(t))\left[\dfrac{K_1 + A_1(\tilde{N}_2(t-\tau_2) - q_2(t-\tau_2))}{1+\tilde{N}_2(t-\tau_2) - q_2(t-\tau_2)} - \tilde{N}_1(t) + q_1(t)\right] \\ \dfrac{d\tilde{N}_2(t)}{dt} = q_2'(t) + r_2(\tilde{N}_2(t) - q_2(t))\left[\dfrac{K_2 + A_2(\tilde{N}_1(t-\tau_1) - q_1(t-\tau_1))}{1+\tilde{N}_1(t-\tau_1) - q_1(t-\tau_1)} - \tilde{N}_2(t) + q_2(t)\right] \end{cases} \tag{5.1}$$

for $0 < t \le \ell$, subject to the boundary condition

$$P_1\begin{bmatrix}\tilde{N}_1(t)\\ \tilde{N}_2(t)\end{bmatrix} + P_2\begin{bmatrix}\tilde{N}_1(\ell+t)\\ \tilde{N}_2(\ell+t)\end{bmatrix} = \begin{bmatrix}\tilde{\varphi}_1(t)\\ \tilde{\varphi}_2(t)\end{bmatrix}. \tag{5.2}$$

Now, associate with (5.1)–(5.2) the following problem of the form (2.2)–(2.3)

$$\tilde{N}'(t) = B\tilde{N}(t) + v(t), \quad t \in [0, \ell], \tag{5.3}$$

$$P_1\tilde{N}(0) + P_2\tilde{N}(\ell) = (0, 0)^T, \tag{5.4}$$

where

$$\tilde{N}(t) = \begin{bmatrix} \tilde{N}_1(t) \\ \tilde{N}_2(t) \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad P_1 = I_{2\times 2}, \qquad P_2 = \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix}.$$

It is easy to check that the fundamental matrix $\mathcal{Y}(t)$ of the corresponding to (5.3) homogenous system satisfying the condition $\mathcal{Y}(0) = I$ is of the form $\mathcal{Y}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $\det(Q) = \det \begin{bmatrix} 1-\mu_1 & -\ell\mu_1 \\ 0 & 1-\mu_2 \end{bmatrix} = (1-\mu_1)(1-\mu_2) \neq 0$. The solution to the problem (5.3)–(5.4), according to the formulas (2.4), takes the form

$$\tilde{N}_2(t) = \frac{\mu_2}{1-\mu_2}\int_0^\ell v_2(s)ds + \int_0^t v_2(s)ds \tag{5.5}$$

$$\tilde{N}_1(t) = \frac{\mu_1}{1-\mu_1}\left(\frac{\ell}{1-\mu_2}\int_0^\ell v_2(s)ds + \int_0^\ell (v_1(s) - sv_2(s))ds\right) + t\tilde{N}_2(t) + \int_0^t (v_1(s) - sv_2(s))ds. \tag{5.6}$$

Denoting the righthand side of (5.5) by $(g_2v)(t)$ and the righthand side of (5.6) by $(g_1v)(t)$ we can write

$$\tilde{N}(t) = (Gv)(t) \stackrel{\text{def}}{=} \begin{bmatrix} (g_1v)(t) \\ (g_2v)(t) \end{bmatrix}. \tag{5.7}$$

Now, replacing the lefthand side of (5.1) by the righthand side of (5.3) and replacing $\tilde{N}_1$ and $\tilde{N}_2$ at the righthand side of (5.3) by the righthand side of (5.7) we arrive at the following fixed-point problem of the form (2.6)

$$v(x) = F(t, (Gv)(t), (Gv)(\cdot)) - B(Gv)(t) \tag{5.8}$$

where $F(t, (Gv)(t), (Gv)(\cdot)) = \begin{bmatrix} f_1(t, (Gv)(t), (Gv)(\cdot)) \\ f_2(t, (Gv)(t), (Gv)(\cdot)) \end{bmatrix}$ is defined by the formulas

$$\begin{cases} f_1(t, (Gv)(t), (Gv)(\cdot)) = q_1'(t) + r_1((g_1v)(t) - q_1(t)) \\ \qquad \times \left[\dfrac{K_1 + A_1((g_2v)(t-\tau_2) - q_2(t-\tau_2))}{1 + (g_2v)(t-\tau_2) - q_2(t-\tau_2)} - (g_1v)(t) + q_1(t)\right] \\ f_2(t, (Gv)(t), (Gv)(\cdot)) = q_2'(t) + r_2((g_2v)(t) - q_2(t)) \\ \qquad \times \left[\dfrac{K_2 + A_2((g_1v)(t-\tau_1) - q_1(t-\tau_1))}{1 + (g_1v)(t-\tau_1) - q_1(t-\tau_1)} - (g_2v)(t) + q_2(t)\right]. \end{cases} \tag{5.9}$$

Problem (1.1)–(1.2) was solved by minimization of $\|p_hv_h - Ap_hv_h\|_2^{(n)}$, where A is defined by the righthand side of formula (5.8) with F defined by the righthand side of (5.9). v_h are discrete functions defined on a grid $(x_0, x_1, \ldots, x_n)$, where, $x_j = a + jH,\ j = 0, 1, \ldots, n,\ H = (b-a)/n$ and p_h are extension operators assigning to each $[v_h]_i = (v_{0i}, v_{1i}, \ldots, v_{ni})$ a piecewise linear function $[p_hv_h]_j$ satisfying $[p_hv_h(x_j)]_i = v_{ji},\ j = 0, 1, \ldots, n$. The exact formulas for the integrals appearing in formulas (5.5)–(5.6) were used. The integral defining the norm $\|\cdot\|_2$ was always evaluated by applying the composite Simpson formula obtained by applying the 'simple' Simpson formula to each interval $[x_j, x_{j+1}],\ j = 0, 1, \ldots, n-1$. Figs. 1 and 2 represent the graphs of approximate solution to the fixed point of A obtained for $h = 0.05$ while Figs. 3 and 4 represent the corresponding to it graphs of approximate solution to problem (1.1)–(1.2). We denoted by w the minimum of $\|p_hv_h - Ap_hv_h\|_2^{(n)}$ obtained for a given h.

It was relatively easy to obtain these approximations for a coarse grid with equidistance gridpoints for $h = 1/10$, $h = 1/20$, $h = 1/25$ and $h = 1/30$ using a gradient method of minimization and a PC with a C++ compiler. The authors of [1] write: 'we recommend the use of Chebyshev series of degree $m = 20$–30 from the viewpoint of computational efficiency'. Recall, that for their method the number of unknowns to be found is $4m + 4$ and for the method considered in this paper with $h = 1/N$ the number of variables of the minimized function is $4N + 2$. There was no problem to carry on computations with the gradient method of minimization for $h = 1/40$, $h = 1/80$, $1/160$ and $h = 1/320$ if as an initial approximation to the grid values of $\varepsilon(h/2)$-fixed-point (to start the gradient method) the obtained approximation to the grid values of $\varepsilon(h)$-fixed-point (with linear approximations of the missing values) were used.

To estimate the accuracy of approximate solutions to problem (1.1)–(1.2) the authors of [1] use the maximum on $[0, \ell]$ of absolute values of residuals obtained by replacing N_1 and N_2 in (1.1) with the approximate solutions and denote it by ε_1. The maximum on $[-\tau, 0]$ of absolute values of residuals obtained by replacing N_1 and N_2 in (1.2) with the approximate solutions they denote by ε_2. A linear combination of ε_1 and ε_2 serves then as an estimate for the error of the approximate solution. To compare our results with the results reported in [1] we placed in Table 1 the residuals ε_1 for our approximate solutions corresponding to different $h = 1/N$ and the residuals $\bar{\varepsilon}_1$ reported in [1] corresponding to approximated solutions obtained for different m. We do not report our residuals ε_2 as they all are zero up to the rounding errors. We can see that our results (residuals ε_1) are slightly better than those reported in [1].

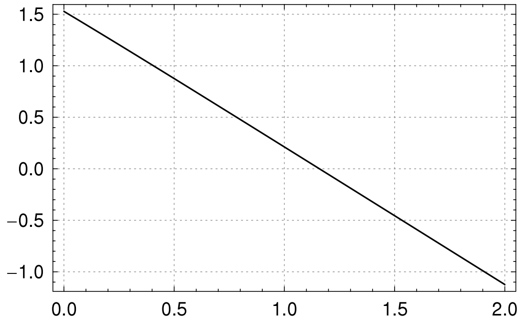

Fig. 1. v_1 ($h = 0.05$).

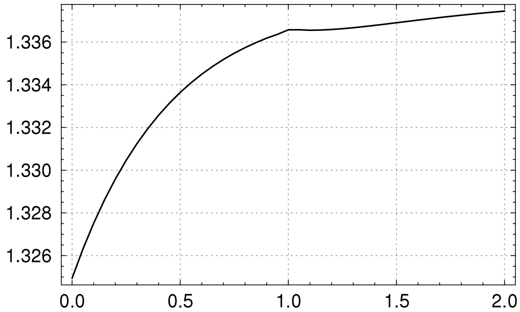

Fig. 2. v_2 ($h = 0.05$).

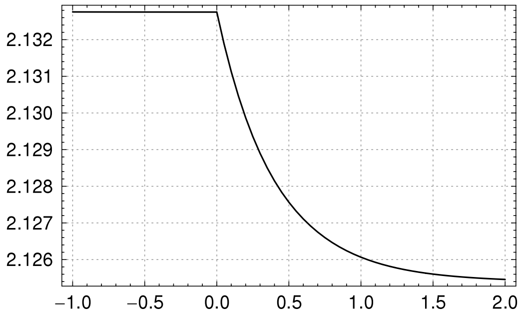

Fig. 3. N_1 ($h = 0.05$).

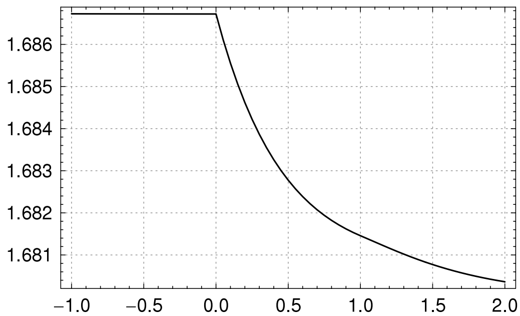

Fig. 4. N_2 ($h = 0.05$).

Remark 4. For applying the proposed approximate method to problem (1.1)–(1.2) it is not necessary to transform it to an equivalent system of the form (1.6)–(1.7). It is done for the proof of its convergence and ease of presentation. Namely, it is easy to notice that $N(t),\ t \in [-\tau, \ell]$, is a solution to (1.1)–(1.2) if and only if $\tilde{N}(t) = N(t) + q(t),\ t \in [-\tau, \ell]$, is a solution to (5.1)–(5.2). Similarly, $v(t),\ t \in [0, \ell]$, is a fixed-point of A defined by the righthand side of (5.8) if and only if $w(t) = v(t) + q'(t) - Bq(t)$ is a fixed-point of $\mathcal{A}$ defined by a formula similar to that given by the righthand side of (5.8)

Table 1
Residuals for different values of step $h = 1/N$.

N	w	ε_1	m	$\bar{\varepsilon}_1$	N	w	ε_1
10	$4.30 \cdot 10^{-5}$	$1.73 \cdot 10^{-4}$	10	$1.97 \cdot 10^{-4}$	40	$2.70 \cdot 10^{-6}$	$1.11 \cdot 10^{-5}$
20	$1.08 \cdot 10^{-5}$	$4.40 \cdot 10^{-5}$	20	$9.84 \cdot 10^{-5}$	80	$6.74 \cdot 10^{-7}$	$2.78 \cdot 10^{-6}$
25	$6.38 \cdot 10^{-6}$	$2.61 \cdot 10^{-5}$	25	$7.59 \cdot 10^{-5}$	160	$1.69 \cdot 10^{-7}$	$6.97 \cdot 10^{-7}$
30	$4.79 \cdot 10^{-6}$	$1.97 \cdot 10^{-5}$	30	$6.43 \cdot 10^{-5}$	320	$4.21 \cdot 10^{-8}$	$1.75 \cdot 10^{-7}$

where F is defined with the use of the righthand side of (1.1) and G is defined by the formula

$$\mathcal{Y}(t)\left[Q^{-1}\left(\varphi(0) - P_2\mathcal{Y}(b)\int_a^b \mathcal{Y}^{-1}(s)w(s)\,ds\right) + \int_a^t \mathcal{Y}^{-1}(s)w(s)\,ds\right]$$

instead of formula (2.4), which was used to obtain (5.5)–(5.6). Moreover, $p_h v_h$ satisfies (4.9) if and only if $= p_h w_h$ defined by the formula $p_h w_h = p_h v_h + q' - Bq$ satisfies (4.9).

Example 5.2. As the second example consider the following system of delay equations with distributed delays given in [5].

$$\begin{cases} y_1'(t) = y_1(t)\left[3 - \sin t - (3-\cos t)y_1(t) - (2+\sin t)\displaystyle\int_{-T_{12}}^{0} K_{12}(s)y_2(t+s)ds\right] \\ y_2'(t) = y_2(t)\left[6 - \cos t - (10-\sin t)y_2(t) - (2+\sin t)\displaystyle\int_{-T_{21}}^{0} K_{21}(s)y_1(t+s)ds\right] \end{cases} \tag{5.10}$$

for $0 < t \le 2\pi$, subject to periodic boundary conditions

$$\begin{aligned} y_1(t) &= y_1(2\pi + t) \\ y_2(t) &= y_2(2\pi + t), \end{aligned} \tag{5.11}$$

where $t \in [-T, 0]$, and $T = \max(T_{12}, T_{21})$.

There are many papers devoted to this kind of periodic equations as they can model, for example, plankton allelopathy and population dynamics (see [6–9] respectively and the literature therein). They give conditions for existence periodic (and some of them positive) solutions but do not treat the problem of solving such equations. We will show that we can verify numerically their theoretical results using the proposed in this paper method.

For our numerical experiments we take $T_{12} = T_{21} = 1$ and $K_{12}(s) = K_{21}(s) = s + 3/2$ so that $\int_{-T_{12}}^{0} K_{12}(s)ds = \int_{-T_{21}}^{0} K_{21}(s)ds = 1$. These conditions are required by the corresponding existence theorem in [5].

The first step consists in replacing the integrals appearing on the righthand side of system (5.10) with a numerical quadrature, for example, using a composite Simpson rule. The composite Simpson rule with 5 nodes applied to the integrals $\int_{-T_{12}}^{0} K_{12}(s)u(s)ds$ or $\int_{-T_{21}}^{0} K_{21}(s)u(s)ds$ gives

$$\int_{-T_{12}}^{0} K_{12}(s)u(s)ds \approx \frac{1}{24}(u(-1) + 6u(-0.75) + 4u(-0.5) + 10u(-0.25) + 3u(0)). \tag{5.12}$$

Next, using the notation $y(t) = [y_1(t), y_2(t)]^T$ we rewrite periodic conditions (5.11) in the form

$$P_1 y(t) + P_2 y(2\pi + t) = 0, \quad t \in [-T, 0], \tag{5.13}$$

where $P_1 = I_2$, $P_2 = -I_2$. So, problem (5.10)–(5.11) can be written in the form

$$\begin{cases} y_1'(t) = y_1(t)\Bigg(3 - \sin t - (3-\cos t)y_1(t) - \Bigg[\dfrac{2+\sin t}{24}(y_2(t-1) \\ \qquad +6y_2(t-0.75) + 4y_2(t-0.5) + 10y_2(t-0.25) + 3y_2(t))\Bigg]\Bigg) \\ y_2'(t) = y_2(t)\Bigg(6 - \cos t - (10-\sin t)y_2(t) - \Bigg[\dfrac{(2+\sin t)}{24}(y_1(t-1) \\ \qquad +6y_1(t-0.75) + 4y_1(t-0.5) + 10y_1(t-0.25) + 3y_1(t))\Bigg]\Bigg) \end{cases} \tag{5.14}$$

for $0 < t \le 2\pi$, subject to the boundary condition

$$P_1\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + P_2\begin{bmatrix} y_1(2\pi+t) \\ y_2(2\pi+t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \in [-T, 0]. \tag{5.15}$$

Now, we can easily see that we cannot use the auxiliary system

$$y'(t) = By(t) + v(t), \quad t \in [a, b], \tag{5.16}$$

$$P_1 y(a) + P_2 y(b) = (0, \dots, 0)^T \tag{5.17}$$

with matrix B as in (5.3) because the matrix $Q = P_1 + P_2 \mathcal{Y}(2\pi)$, where $\mathcal{Y}(s)$ is a fundamental matrix of the corresponding to (5.16) homogeneous system satisfying the condition $\mathcal{Y}(0) = I_2$, is singular. To get a nonsingular matrix Q we will consider the auxiliary system (5.16)–(5.17) with matrix B of the form

$$B = \begin{bmatrix} 0 & 1 \\ -9/4 & 0 \end{bmatrix} \tag{5.18}$$

for which a fundamental matrix of the corresponding homogeneous system is

$$\mathcal{Y}(t) = \begin{bmatrix} \cos\left(\frac{3}{2}t\right) & \frac{2}{3}\sin\left(\frac{3}{2}t\right) \\ -\frac{3}{2}\sin\left(\frac{3}{2}t\right) & \cos\left(\frac{3}{2}t\right) \end{bmatrix} \tag{5.19}$$

and the matrix Q takes the form

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then, introducing the following notation

$$\begin{aligned}
I_1 &= \int_0^{2\pi} \left(\cos\left(\frac{3}{2}s\right) v_1(s) - \frac{2}{3}\sin\left(\frac{3}{2}s\right) v_2(s) \right) ds \\
I_2 &= \int_0^{2\pi} \left(\frac{3}{2}\sin\left(\frac{3}{2}s\right) v_1(s) + \cos\left(\frac{3}{2}s\right) v_2(s) \right) ds \\
w_1(t) &= \int_0^{t} \left(\cos\left(\frac{3}{2}s\right) v_1(s) - \frac{2}{3}\sin\left(\frac{3}{2}s\right) v_2(s) \right) ds \\
w_2(t) &= \int_0^{t} \left(\frac{3}{2}\sin\left(\frac{3}{2}s\right) v_1(s) + \cos\left(\frac{3}{2}s\right) v_2(s) \right) ds
\end{aligned} \tag{5.20}$$

the solution to (5.16)–(5.17) can be written as

$$\begin{aligned}
y_1(t) &= \cos\left(\frac{3}{2}t\right)\left(-\frac{1}{2}I_1 + w_1(t)\right) - \frac{1}{3}\sin\left(\frac{3}{2}t\right)(I_2 - 2w_2(t)) \\
y_2(t) &= \cos\left(\frac{3}{2}t\right)\left(-\frac{1}{2}I_2 + w_2(t)\right) + \frac{4}{3}\sin\left(\frac{3}{2}t\right)(I_1 - 2w_1(t)).
\end{aligned} \tag{5.21}$$

Denoting the righthand sides of (5.21) by $(g_1 v)(t)$ and $(g_2 v)(t)$ respectively and defining operator G as $(Gv)(t) = [(g_1 v)(t), (g_2 v)(t)]^T$ we arrive at the following fixed-point problem

$$v(x) = F(t, (Gv)(t), (Gv)(\cdot)) - B(Gv)(t) \tag{5.22}$$

where $F(t, (Gv)(t), (Gv)(\cdot)) = \begin{bmatrix} f_1(t, (Gv)(t), (Gv)(\cdot)) \\ f_2(t, (Gv)(t), (Gv)(\cdot)) \end{bmatrix}$ is defined by

$$\begin{cases}
f_1(t, (Gv)(t), (Gv)(\cdot)) = y_1(t)\Bigg(3 - \sin t - (3 - \cos t)y_1(t) \\
\quad - \left[\frac{2 + \sin t}{24}(y_2(t-1) + 6y_2(t-0.75) + 4y_2(t-0.5) + 10y_2(t-0.25) + 3y_2(t))\right]\Bigg) - (g_2 v)(t) \\
f_2(t, (Gv)(t), (Gv)(\cdot)) = y_2(t)\Bigg(6 - \cos t - (10 - \sin t)y_2(t) \\
\quad - \left[\frac{(2 + \sin t)}{24}(y_1(t-1) + 6y_1(t-0.75) + 4y_1(t-0.5) + 10y_1(t-0.25) + 3y_1(t))\right]\Bigg) + \frac{9}{4}(g_1 v)(t).
\end{cases} \tag{5.23}$$

Problem (5.22) was solved in exactly the same way as in the case of Example 5.1. To be consistent with Example 5.1 in Table 2 we placed the residuals w and ε_1 for corresponding values of h. The graphs of obtained approximations to y_1, y_2 and v_1, v_2 are depicted in Figs. 5 and 6 respectively.

Table 2
Residuals for different values of steps h for Examples 2, 3a and 3b.

Example 2			Example 3a			Example 3b		
h	w	ε_1	h	w	ε_1	h	w	ε_1
0.079	$8.0 \cdot 10^{-4}$	$7.0 \cdot 10^{-4}$	0.125	$3.4 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$	0.829	$4.6 \cdot 10^{-2}$	$9.7 \cdot 10^{-3}$
0.039	$2.0 \cdot 10^{-4}$	$1.7 \cdot 10^{-4}$	0.063	$4.9 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$	0.414	$3.3 \cdot 10^{-2}$	$6.6 \cdot 10^{-3}$
0.020	$5.0 \cdot 10^{-5}$	$4.4 \cdot 10^{-5}$	0.031	$1.9 \cdot 10^{-4}$	$8.8 \cdot 10^{-5}$	0.207	$2.9 \cdot 10^{-2}$	$5.9 \cdot 10^{-3}$
0.010	$1.2 \cdot 10^{-5}$	$1.1 \cdot 10^{-5}$	0.016	$4.7 \cdot 10^{-5}$	$2.1 \cdot 10^{-5}$	0.104	$1.9 \cdot 10^{-2}$	$4.1 \cdot 10^{-3}$

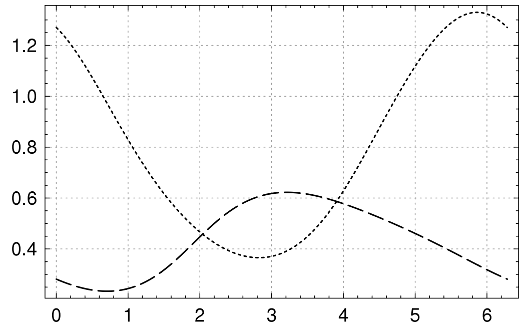

Fig. 5. Example 2: ($h = 0.010$), y_1 (dotted line), y_2 (dashed line).

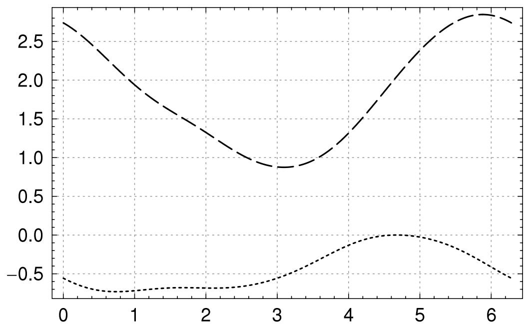

Fig. 6. Example 2: ($h = 0.010$), v_1 (dotted line), v_2 (dashed line).

Example 5.3. As the third example consider the following system of delay equations modeling two coupled identical neurons with time-delayed connections, which was investigated in [10].

$$\begin{cases} y_1'(t) = y_1(t) - \lambda y_1(t) + \beta_0 \tanh(y_1(t-\tau_s)) + \beta_{12} \tanh(y_2(t-\tau_2)) \\ y_2'(t) = y_2(t) - \lambda y_2(t) + \beta_0 \tanh(y_2(t-\tau_s)) + \beta_{21} \tanh(y_1(t-\tau_1)) \end{cases} \tag{5.24}$$

for $t \in [0, t_f]$ subject to periodic boundary conditions

$$\begin{aligned} y_1(t) &= y_1(t_f + t) \\ y_2(t) &= y_2(t_f + t), \end{aligned} \tag{5.25}$$

for $t \in [-1.5, 0]$. We consider system (5.24) for the values of parameters: $\lambda = 0.5$, $\beta_0 = -1$, $\beta_{12} = 1$, $\tau_1 = 0.2$, $\tau_2 = 0.2$, $\tau_s = 1.5$ and:

(a) $t_f = 10.0174$, $\beta_{21} = 1.27406$;
(b) $t_f = 66.3164$, $\beta_{21} = 2.35001$.

To reduce problem (5.24)–(5.25) with t_f, β_{21} from (a) and (5.24)–(5.25) with t_f, β_{21} from (b) to fixed-point problems we can use the auxiliary problem (5.16)–(5.17). However, we have to calculate new matrices Q. For the data given in (a) and (b) they are equal (to six decimal figures) respectively

$$\text{(a)} \quad Q = \begin{bmatrix} 1.776400 & -0.420161 \\ 0.945361 & 1.776400 \end{bmatrix}, \quad \text{(b)} \quad Q = \begin{bmatrix} 0.507960 & 0.580382 \\ -1.305860 & 0.507960 \end{bmatrix}.$$

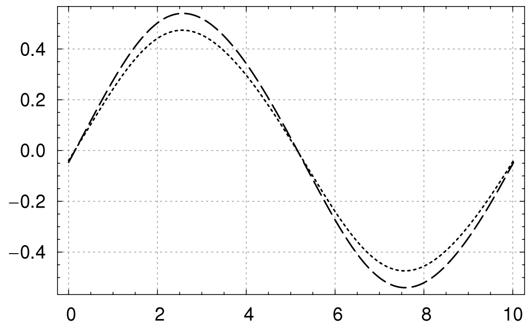

Fig. 7. Example 3a: $(h = 0.010)$, y_1 (dotted line), y_2 (dashed line).

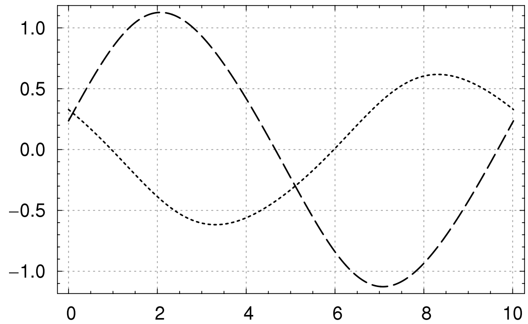

Fig. 8. Example 3a: $(h = 0.010)$, v_1 (dotted line), v_2 (dashed line).

Now, using the notation like in (5.20) with the integral upper limit 2π replaced with 10.0174 for case (a) and with 66.3164 for case (b) we obtain for (a)

$$\begin{aligned} y_1(t) &= \cos\left(\frac{3}{2}t\right)\left(-\frac{1}{2}I_1 + 0.118262 I_2 + w_1(t)\right) + \frac{2}{3}\sin\left(\frac{3}{2}t\right)\left(-0.266089 I_1 - \frac{1}{2}I_2 + w_2(t)\right) \\ y_2(t) &= \cos\left(\frac{3}{2}t\right)\left(-0.266089 I_1 - \frac{1}{2}I_2 + w_2(t)\right) - \frac{3}{2}\sin\left(\frac{3}{2}t\right)\left(-\frac{1}{2}I_1 + 0.118262 I_2 + w_1(t)\right) \end{aligned} \tag{5.26}$$

and for (b)

$$\begin{aligned} y_1(t) &= \cos\left(\frac{3}{2}t\right)\left(-\frac{1}{2}I_1 - 0.571287 I_2 + w_1(t)\right) + \frac{2}{3}\sin\left(\frac{3}{2}t\right)\left(1.285400 I_1 - \frac{1}{2}I_2 + w_2(t)\right) \\ y_2(t) &= \cos\left(\frac{3}{2}t\right)\left(1.285400 I_1 - \frac{1}{2}I_2 + w_2(t)\right) - \frac{3}{2}\sin\left(\frac{3}{2}t\right)\left(-\frac{1}{2}I_1 - 0.571287 I_2 + w_1(t)\right). \end{aligned} \tag{5.27}$$

Using these formulas we arrive at the fixed-point problem (5.22) with

$$\begin{cases} f_1(t, (Gv)(t), (Gv)(\cdot)) = y_1(t) - 0.5y_1(t) - \tanh(y_1(t-1.5)) + \tanh(y_2(t-0.2)) - (g_2 v)(t) \\ f_2(t, (Gv)(t), (Gv)(\cdot)) = y_2(t) - 0.5y_2(t) - \tanh(y_2(t-1.5)) + 1.27406\tanh(y_1(t-0.2)) + \dfrac{9}{4}(g_1 v)(t) \end{cases} \tag{5.28}$$

for case (a) and

$$\begin{cases} f_1(t, (Gv)(t), (Gv)(\cdot)) = y_1(t) - 0.5y_1(t) - \tanh(y_1(t-1.5)) + \tanh(y_2(t-0.2)) - (g_2 v)(t) \\ f_2(t, (Gv)(t), (Gv)(\cdot)) = y_2(t) - 0.5y_2(t) - \tanh(y_2(t-1.5)) + 2.35001\tanh(y_1(t-0.2)) + \dfrac{9}{4}(g_1 v)(t) \end{cases} \tag{5.29}$$

for case (b).

Problem (5.22) for both cases were solved in exactly the same way as in the case of Examples 5.1 and 5.2. In Table 2 we placed the residuals w and ε_1 for corresponding values of h. The graphs of obtained approximations to y_1, y_2 and v_1, v_2 are depicted in Figs. 7–10 respectively.

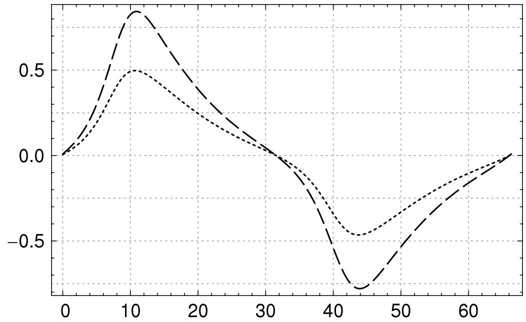

Fig. 9. Example 3b: ($h = 0.010$), y_1 (dotted line), y_2 (dashed line).

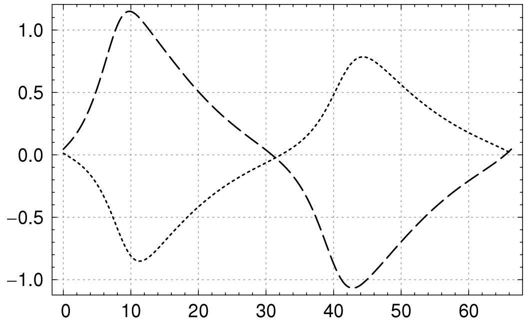

Fig. 10. Example 3b: ($h = 0.010$), v_1 (dotted line), v_2 (dashed line).

6. Concluding remarks

In this paper we showed how the $\boldsymbol{\varepsilon}(\boldsymbol{h})$ *approximate method* introduced in [3] can be employed to solve general linear boundary problem for nonlinear system of delay differential equations. Such linear boundary problems include periodic boundary problems for systems of delay differential equations with distributed and discrete delays and model a variety of biological and physical phenomena.

We see that the same auxiliary linear boundary value problem (2.2)–(2.3) needed to reduce a differential problem to a fixed-point problem can be used for different problems as long as the matrix Q is nonsingular. We also see that the formulas for the solution y of (2.2)–(2.3) do not depend on the original problem (1.3)–(1.4) but depend on the interval $[a, b]$. If the function v appearing on the righthand side of (2.2) is piecewise linear and the matrix B is constant than for each $t \in [a, b]$ the solution y can be evaluated exactly up to the rounding error. If the interval $[a, b]$ is the same and nonsingular matrices Q are the same for different problems under consideration then we can use the same formulas for calculating values $y(t)$ at $t = t_i, t = t_i - \tau_1, \ldots, t = t_i - \tau_k$ for suitable values of i, which are necessary to evaluate the residuum $w = \|p_h v_h - A p_h v_h\|_2^{(n)}$. We remark that if $\max_{1 \le l \le k} \tau_l \le b - a$ then solving the problem under consideration we always evaluate the values of the unknown function y at points $t \in [a, b]$ because if $t_i - \tau_l \notin [a, b]$ for some i and l then we use boundary conditions to express $y(t_i - \tau_l)$ by a formula which involves the value $y(b - a + t_i - \tau_l)$.

Some authors before solving the boundary problem for a system of delay differential equations on the interval $[a, b]$, $b > 1$, transform the interval $[a, b]$ into the standard interval $[0, 1]$. However, this results in multiplying the derivative of unknown function by the factor b and increasing in this way the rate of change of y. Numerical experiments with such an approach employed to the boundary value problem for delay differential equations with known solution and then using the proposed in this paper method show that both residua w and ε_1 are larger and the solution is less precise than when the proposed method is applied to the problem with the original interval $[a, b]$.

It is also easy to see that the proposed method can be employed for solving boundary problems with advanced arguments after proving the corresponding lemmas from Section 4.

Acknowledgments

The authors wish to express their gratitude to anonymous referees for their helpful remarks.

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