Contemporary Engineering Sciences, Vol. 5, 2012, no. 3, 107 - 110

Some Exact Values of Shannon Capacity for Evolving Systems

Marcin Jurkiewicz

Gdansk University of Technology
Department of Algorithms and System Modelling
Gabriela Narutowicza 11/12, 80-233 Gdansk, Poland
marcin.jurkiewicz@eti.pg.gda.pl

Abstract

In the paper we describe the notion of Shannon capacity for evolving channels. Furthermore, using a computer search together with some theoretical results we establish some exact values of the measure.

Keywords: Shannon capacity, information theory

1 Introduction

There are many reasons to consider channels with a reliable transmission, i.e. a transmission with exactly zero probability of error [2]. In 1956, Claude Shannon introduced the notion of a capacity of a noisy channel for a reliable transmission [3]. In the paper we discuss a generalization of the problem, i.e. the Shannon capacity of evolving channels [1] and present some exact values of the Shannon capacity of reducing channels.

2 Preliminary notes

In the following, we will need the following definition

Definition 2.1 *A discrete channel* $Q = (A_X, A_Y, M_{XY})$ *consists of three parts:* A_X *and* A_Y *are input and output alphabets, respectively,* M_{XY} *is the transition matrix with* $p(y|x)$ *elements, which correspond to conditional probabilities.*

We say that a channel Q is noisy, if there are elements $y_1, y_2 \in A_Y$ and an element $x \in A_X$ such that $p(y_1|x)p(y_2|x) > 0$. Given a channel Q and element $x \in A_X$, we define

$$S_x = \{y \in A_Y : p(y|x) > 0\}. \tag{1}$$

S_x is the set of letters attainable on output, when there is x on input.

For a discrete channel Q, there exists the characteristic graph defined as follows

Definition 2.2 *The characteristic graph of a channel Q is a graph $G = (V, E)$ such that the vertex set $V = A_X$ and $\{x, y\} \in E(G)$ iff $S_x \cap S_y \neq \emptyset$.*

There are many operations acting on graphs. In the paper we will use the following operation

Definition 2.3 *Given two graphs G and H, the strong product $G \cdot H$ is defined as follows. The vertices of $G \cdot H$ are all pairs of the Cartesian product $V(G) \times V(H)$. There is an edge between (x, x') and (y, y') iff $\{x, y\} \in E(G)$ and $\{x', y'\} \in E(H)$ or $x = y$ and $\{x', y'\} \in E(H)$ or $x' = y'$ and $\{x, y\} \in E(G)$.*

We write G^n to denote $G \cdot G \cdot \ldots \cdot G$, where G occurs n times.

Let G be a graph. A set of vertices S of G is said to be an independent set of vertices if they are pairwise nonadjacent. The independence number of G, denoted by $\alpha(G)$, is defined to be the size of a largest independent set of G. For m arbitrary graphs $G_1, \ldots, G_m$, we have

$$\alpha(G_1 \cdot \ldots \cdot G_m) \geq \alpha(G_1) \cdot \ldots \cdot \alpha(G_m). \tag{2}$$

The notion of reliable capacity was introduced by Shannon [3]

Definition 2.4 *The Shannon capacity is defined as*

$$C_0 = sup_n \sqrt[n]{\alpha(G^n)}. \tag{3}$$

We derived a generalization of this measure in [1]

Definition 2.5 *Given a sequence $U_1, U_2, \ldots$ of operations acting on graphs. The Shannon capacity of evolving channel is defined as*

$$C_0 = sup_n \sqrt[n]{\alpha(\mathcal{U}_n(G))}. \tag{4}$$

where $\mathcal{U}_n(G) = U_n(U_{n-1}(\ldots(U_1(G))))$.

Notice that $U_1, U_2, \ldots$ represent arbitrary operations on a channel, but sometimes the supremum does not exist, so the measure does not work. However, we get the Shannon capacity, if $\mathcal{U}_n(G) = G^n$.

Let Q be a noisy channel, G be its characteristic graph and $L = (v_1, \ldots, v_l)$ be a sequence of vertices of G such that $v_i \neq v_j$, for $i, j = 1, \ldots, l$ and $i \neq j$. Then we denote a sequence of reductions as

$$G^k[L] = G - v_1 \oplus (\{v_1\}, \emptyset) - v_2 \oplus (\{v_2\}, \emptyset) - \ldots - v_k \oplus (\{v_k\}, \emptyset), \tag{5}$$

where $k \leq l \leq |V(G)|$ and the symbol $-$ means a vertex deletion and the symbol $\oplus$ means a graph join.

In the next section, we will show some exact values for a special case of the Shannon capacity of evolving channel defined below

Definition 2.6 *Given a graph G with n vertices and a permutation π of $\{1,\ldots,n\}$. We define the Shannon capacity of reducing channels as*

$$C_0^R = \max_{k\in[n]} \sqrt[k]{\alpha(G^{\underline{k}}[\pi])}, \tag{6}$$

where $G^{\underline{k}}[\pi] = G^0[\pi]\cdot G^1[\pi]\cdot G^2[\pi]\cdot\ \ldots\ \cdot G^{k-1}[\pi]$ and $G^0[\pi] = G$.

For more details of the section, see [1].

3 Some exact values

It is interesting that we could facilitate the problem of calculating the Shannon capacity of reducing channels.

Lemma 3.1 *Given graphs H and $K_n - e$, i.e. complete graph without an edge, then*

$$\alpha\Big((K_n - e)\cdot H\Big) = 2\alpha(H). \tag{7}$$

Proof. Given a graph H with $V(H) = \{1', 2', \ldots, l'\}$ and an edge $e = \{v_1, v_2\}$ of K_n, with $V(K_n) = \{1, 2, \ldots, n\}$, there are two copies H_1, H_2 of H in $(K_n - e)\cdot H$ on vertex sets $\{(v_1, i') : i' \in V(H)\}$ and $\{(v_2, i') : i' \in V(H)\}$, respectively, which are not linked. Hence, $2\alpha(H) \le \alpha\Big((K_n - e)\cdot H\Big)$. On the other hand, let S be a largest independent set of $(K_n - e)\cdot H$. If $S \subset V(H_1)\cup V(H_2)$, then $|S| \le 2\alpha(H)$. Otherwise, for an vertex $(s_1, s_2) \in S$, which is outside $V(H_1)\cup V(H_2)$, we can replace (s_1, s_2) to vertices (v_1, s_2) and (v_2, s_2), without breaking the independent set condition. Therefore, $\alpha\Big((K_n - e)\cdot H\Big) \le 2\alpha(H)$.

The reduction Shannon capacity has the following property

Theorem 3.2 *Given a graph G and a permutation $\pi = (p_1, \ldots, p_n)$ of all vertices of G. If $G^i[\pi], \ldots, G^n[\pi]$ are complete graphs, then:*

$$C_0^R = \begin{cases} i+1, & \text{if } i = 0, 1 \\ \max_{k\in[i-1]} \sqrt[k]{\alpha(G^{\underline{k}}[\pi])}, & \text{if } 2 \le i < n \end{cases} \tag{8}$$

Proof. It is easy to see that the theorem is true for $i = 0$. If $G^i[\pi], \ldots, G^n[\pi]$ are complete graphs for some $i \in \{1, \ldots, n-1\}$, then $G^{i-1}[\pi]$ is a complete graph without an edge. Using the well known formula $\alpha(H\cdot K_j) = \alpha(H)$, where j is an integer number, we conclude that for $k \ge i$

$$\alpha\Big(G^{\underline{k}}[\pi]\Big) = \alpha\Big(G^{\underline{i}}[\pi]\Big). \tag{9}$$

For $i = 1$, we have $\alpha(G^{\underline{1}}[\pi]) = \alpha(K_n - e) = 2$. For $2 \le i < n$, from Lemma 3.1

$$\alpha\Big(G^{\underline{i}}[\pi]\Big) = \alpha\Big(G^{\underline{i-1}}[\pi]\cdot G^{i-1}[\pi]\Big) = 2\alpha\Big(G^{\underline{i-1}}[\pi]\Big). \tag{10}$$

In addition, from (2), we get $\alpha(G^{\underline{i-1}}[\pi]) \geq \alpha(G^0[\pi]) \cdot \ldots \cdot \alpha(G^{i-2}[\pi])$. Because $G^0[\pi] \subset \ldots \subset G^n[\pi]$, hence $\alpha(G^0[\pi]) \geq \ldots \geq \alpha(G^{i-2}[\pi]) \geq \alpha(G^{i-1}[\pi]) = 2$. So, we get $\alpha(G^{\underline{i-1}}[\pi]) \geq 2^{i-1}$ and therefore

$$\sqrt[i-1]{\alpha(G^{\underline{i-1}}[\pi])} \geq \sqrt[i]{2\alpha(G^{\underline{i-1}}[\pi])}. \tag{11}$$

From these considerations, we get the thesis.

Using preceding results and a computer search we have established among others that for all permutations

- $C_0^R = 1$, for K_n, $n > 0$,
- $C_0^R = 2$, for $2P_2$, $K_3 \cup K_1$, P_4, C_4, $K_4 - e$, W_5, $K_5 - e$,
- $C_0^R = 3$, for S_4, $K_2 \cup E_2$,
- $C_0^R = 4$, for E_4.

These computations brought us to $\alpha(G^{\underline{k}}[\pi]) = \alpha(G^0[\pi]) \cdot \ldots \cdot \alpha(G^{k-1}[\pi])$, for all permutations π and graphs on n vertices, where $1 \leq k \leq n \leq 4$. Additionally, we find that $C_0^R = \alpha(G)$, for all graphs on $n \leq 4$.

ACKNOWLEDGEMENTS. I would like to thank my wife, Dorothy Jurkiewicz, for her patience and care.

References

[1] M. Jurkiewicz, M. Kubale, A note on Shannon capacity for invariant and evolving channels, *J. Appl. Comp. Sci.,* **19**, 2, (2011).

[2] J. Körner and A. Orlitsky, Zero-Error Information Theory, *IEEE Trans. Inform. Theory,* **IT-44** (1998).

[3] C. Shannon, The zero-error capacity of a noisy channel, *IEEE Trans. Inform. Theory,* **IT-2** (1956).

Received: November, 2011