

# Some Exact Values of Shannon Capacity for Evolving Systems

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## Abstract

In the paper we describe the notion of Shannon capacity for evolving channels. Furthermore, using a computer search together with some theoretical results we establish some exact values of the measure.

**Keywords:** Shannon capacity, information theory

## 1 Introduction

There are many reasons to consider channels with a reliable transmission, i.e. a transmission with exactly zero probability of error [2]. In 1956, Claude Shannon introduced the notion of a capacity of a noisy channel for a reliable transmission [3]. In the paper we discuss a generalization of the problem, i.e. the Shannon capacity of evolving channels [1] and present some exact values of the Shannon capacity of reducing channels.

## 2 Preliminary notes

In the following, we will need the following definition

**Definition 2.1** *A discrete channel  $Q = (A_X, A_Y, M_{XY})$  consists of three parts:  $A_X$  and  $A_Y$  are input and output alphabets, respectively,  $M_{XY}$  is the transition matrix with  $p(y|x)$  elements, which correspond to conditional probabilities.*

We say that a channel  $Q$  is noisy, if there are elements  $y_1, y_2 \in A_Y$  and an element  $x \in A_X$  such that  $p(y_1|x)p(y_2|x) > 0$ . Given a channel  $Q$  and element  $x \in A_X$ , we define

$$S_x = \{y \in A_Y : p(y|x) > 0\}. \quad (1)$$

$S_x$  is the set of letters attainable on output, when there is  $x$  on input.

For a discrete channel  $Q$ , there exists the characteristic graph defined as follows

**Definition 2.2** *The characteristic graph of a channel  $Q$  is a graph  $G = (V, E)$  such that the vertex set  $V = A_X$  and  $\{x, y\} \in E(G)$  iff  $S_x \cap S_y \neq \emptyset$ .*

There are many operations acting on graphs. In the paper we will use the following operation

**Definition 2.3** *Given two graphs  $G$  and  $H$ , the strong product  $G \cdot H$  is defined as follows. The vertices of  $G \cdot H$  are all pairs of the Cartesian product  $V(G) \times V(H)$ . There is an edge between  $(x, x')$  and  $(y, y')$  iff  $\{x, y\} \in E(G)$  and  $\{x', y'\} \in E(H)$  or  $x = y$  and  $\{x', y'\} \in E(H)$  or  $x' = y'$  and  $\{x, y\} \in E(G)$ .*

We write  $G^n$  to denote  $G \cdot G \cdot \dots \cdot G$ , where  $G$  occurs  $n$  times.

Let  $G$  be a graph. A set of vertices  $S$  of  $G$  is said to be an independent set of vertices if they are pairwise nonadjacent. The independence number of  $G$ , denoted by  $\alpha(G)$ , is defined to be the size of a largest independent set of  $G$ . For  $m$  arbitrary graphs  $G_1, \dots, G_m$ , we have

$$\alpha(G_1 \cdot \dots \cdot G_m) \geq \alpha(G_1) \cdot \dots \cdot \alpha(G_m). \quad (2)$$

The notion of reliable capacity was introduced by Shannon [3]

**Definition 2.4** *The Shannon capacity is defined as*

$$C_0 = \sup_n \sqrt[n]{\alpha(G^n)}. \quad (3)$$

We derived a generalization of this measure in [1]

**Definition 2.5** *Given a sequence  $U_1, U_2, \dots$  of operations acting on graphs. The Shannon capacity of evolving channel is defined as*

$$C_0 = \sup_n \sqrt[n]{\alpha(\mathcal{U}_n(G))}. \quad (4)$$

where  $\mathcal{U}_n(G) = U_n(U_{n-1}(\dots(U_1(G))))$ .

Notice that  $U_1, U_2, \dots$  represent arbitrary operations on a channel, but sometimes the supremum does not exist, so the measure does not work. However, we get the Shannon capacity, if  $\mathcal{U}_n(G) = G^n$ .

Let  $Q$  be a noisy channel,  $G$  be its characteristic graph and  $L = (v_1, \dots, v_l)$  be a sequence of vertices of  $G$  such that  $v_i \neq v_j$ , for  $i, j = 1, \dots, l$  and  $i \neq j$ . Then we denote a sequence of reductions as

$$G^k[L] = G - v_1 \oplus (\{v_1\}, \emptyset) - v_2 \oplus (\{v_2\}, \emptyset) - \dots - v_k \oplus (\{v_k\}, \emptyset), \quad (5)$$

where  $k \leq l \leq |V(G)|$  and the symbol  $-$  means a vertex deletion and the symbol  $\oplus$  means a graph join.

In the next section, we will show some exact values for a special case of the Shannon capacity of evolving channel defined below



**Definition 2.6** Given a graph  $G$  with  $n$  vertices and a permutation  $\pi$  of  $\{1, \dots, n\}$ . We define the Shannon capacity of reducing channels as

$$C_0^R = \max_{k \in [n]} \sqrt[k]{\alpha(G^k[\pi])}, \quad (6)$$

where  $G^k[\pi] = G^0[\pi] \cdot G^1[\pi] \cdot G^2[\pi] \cdot \dots \cdot G^{k-1}[\pi]$  and  $G^0[\pi] = G$ .

For more details of the section, see [1].

### 3 Some exact values

It is interesting that we could facilitate the problem of calculating the Shannon capacity of reducing channels.

**Lemma 3.1** Given graphs  $H$  and  $K_n - e$ , i.e. complete graph without an edge, then

$$\alpha((K_n - e) \cdot H) = 2\alpha(H). \quad (7)$$

**Proof.** Given a graph  $H$  with  $V(H) = \{1', 2', \dots, l'\}$  and an edge  $e = \{v_1, v_2\}$  of  $K_n$ , with  $V(K_n) = \{1, 2, \dots, n\}$ , there are two copies  $H_1, H_2$  of  $H$  in  $(K_n - e) \cdot H$  on vertex sets  $\{(v_1, i') : i' \in V(H)\}$  and  $\{(v_2, i') : i' \in V(H)\}$ , respectively, which are not linked. Hence,  $2\alpha(H) \leq \alpha((K_n - e) \cdot H)$ . On the other hand, let  $S$  be a largest independent set of  $(K_n - e) \cdot H$ . If  $S \subset V(H_1) \cup V(H_2)$ , then  $|S| \leq 2\alpha(H)$ . Otherwise, for an vertex  $(s_1, s_2) \in S$ , which is outside  $V(H_1) \cup V(H_2)$ , we can replace  $(s_1, s_2)$  to vertices  $(v_1, s_2)$  and  $(v_2, s_2)$ , without breaking the independent set condition. Therefore,  $\alpha((K_n - e) \cdot H) \leq 2\alpha(H)$ .

The reduction Shannon capacity has the following property

**Theorem 3.2** Given a graph  $G$  and a permutation  $\pi = (p_1, \dots, p_n)$  of all vertices of  $G$ . If  $G^i[\pi], \dots, G^n[\pi]$  are complete graphs, then:

$$C_0^R = \begin{cases} i + 1, & \text{if } i = 0, 1 \\ \max_{k \in [i-1]} \sqrt[k]{\alpha(G^k[\pi])}, & \text{if } 2 \leq i < n \end{cases} \quad (8)$$

**Proof.** It is easy to see that the theorem is true for  $i = 0$ . If  $G^i[\pi], \dots, G^n[\pi]$  are complete graphs for some  $i \in \{1, \dots, n-1\}$ , then  $G^{i-1}[\pi]$  is a complete graph without an edge. Using the well known formula  $\alpha(H \cdot K_j) = \alpha(H)$ , where  $j$  is an integer number, we conclude that for  $k \geq i$

$$\alpha(G^k[\pi]) = \alpha(G^i[\pi]). \quad (9)$$

For  $i = 1$ , we have  $\alpha(G^1[\pi]) = \alpha(K_n - e) = 2$ . For  $2 \leq i < n$ , from Lemma 3.1

$$\alpha(G^i[\pi]) = \alpha(G^{i-1}[\pi] \cdot G^{i-1}[\pi]) = 2\alpha(G^{i-1}[\pi]). \quad (10)$$



In addition, from (2), we get  $\alpha(G^{i-1}[\pi]) \geq \alpha(G^0[\pi]) \cdot \dots \cdot \alpha(G^{i-2}[\pi])$ . Because  $G^0[\pi] \subset \dots \subset G^n[\pi]$ , hence  $\alpha(G^0[\pi]) \geq \dots \geq \alpha(G^{i-2}[\pi]) \geq \alpha(G^{i-1}[\pi]) = 2$ . So, we get  $\alpha(G^{i-1}[\pi]) \geq 2^{i-1}$  and therefore

$$\sqrt[i-1]{\alpha(G^{i-1}[\pi])} \geq \sqrt[i]{2\alpha(G^{i-1}[\pi])}. \quad (11)$$

From these considerations, we get the thesis.

Using preceding results and a computer search we have established among others that for all permutations

- $C_0^R = 1$ , for  $K_n$ ,  $n > 0$ ,
- $C_0^R = 2$ , for  $2P_2$ ,  $K_3 \cup K_1$ ,  $P_4$ ,  $C_4$ ,  $K_4 - e$ ,  $W_5$ ,  $K_5 - e$ ,
- $C_0^R = 3$ , for  $S_4$ ,  $K_2 \cup E_2$ ,
- $C_0^R = 4$ , for  $E_4$ .

These computations brought us to  $\alpha(G^k[\pi]) = \alpha(G^0[\pi]) \cdot \dots \cdot \alpha(G^{k-1}[\pi])$ , for all permutations  $\pi$  and graphs on  $n$  vertices, where  $1 \leq k \leq n \leq 4$ . Additionally, we find that  $C_0^R = \alpha(G)$ , for all graphs on  $n \leq 4$ .

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