Topology and its Applications 159 (2012) 3752–3759

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Estimation of the minimal number of periodic points for smooth self-maps of odd dimensional real projective spaces ☆

Grzegorz Graff [a,*], Jerzy Jezierski [b]

[a] Faculty of Applied Physics and Mathematics, Gdansk University of Technology, Narutowicza 11/12, 80-233 Gdansk, Poland
[b] Faculty of Applied Informatics and Mathematics, Warsaw University of Life Sciences (SGGW), Nowoursynowska 159, 00-757 Warsaw, Poland

ARTICLE INFO

MSC:
primary 37C25, 55M20
secondary 37C05

Keywords:
Nielsen number
Indices of iterates
Smooth maps
Minimal number of periodic points

ABSTRACT

Let f be a smooth self-map of a closed connected manifold of dimension $m \geqslant 3$. The authors introduced in [G. Graff, J. Jezierski, Minimizing the number of periodic points for smooth maps. Non-simply connected case, Topology Appl. 158 (3) (2011) 276–290] the topological invariant $NJD_r[f]$, where r is a fixed natural number, which is equal to the minimal number of r-periodic points in the smooth homotopy class of f. In this paper smooth self-maps of real projective space $\mathbb{R}P^m$, where $m > 3$ is odd, are considered and the estimations from below and above for $NJD_r[f]$ are given.

1. Introduction

Let f be a smooth self-map of a compact manifold M. The central question in the smooth branch of Nielsen periodic point theory is the following: *what is the minimal number of r-periodic points in the smooth homotopy class of f?* In other words, one seeks for the invariant that determines the number

$$MF_r^{diff}(f) = \min\{\#\mathrm{Fix}(g^r)\colon g \overset{s}{\sim} f\}, \tag{1.1}$$

where $\overset{s}{\sim}$ means that the maps g and f are C^1-homotopic.

We will consider a smooth closed connected manifold of dimension at least 3. It is remarkable that for $r = 1$, i.e. for fixed points, the classical (continuous) and smooth Nielsen theories coincide [21]. However, for $r > 1$ these theories are much different. Namely, if the minimum in (1.1) is taken over continuous homotopies, then the respective number, $MF_r(f)$, is given by Jiang invariant $NF_r(f)$ (cf. [17,20]). In the smooth case $MF_r^{diff}(f) = NJD_r[f]$, the invariant introduced by the authors in [8]. For smooth f, $NJD_r[f] \geqslant NF_r(f)$ and the equality holds only in some exceptional situations [16].

In the definition of $NJD_r[f]$ in addition to Reidemeister relations fixed points indices of iterations are involved. There are strong restrictions for local indices of iterations of smooth maps [1], in contrast to continuous maps, which result in the inequality $NJD_r[f] \geqslant NF_r(f)$. For example, for self-maps of simply-connected manifolds $NF_r(f) \in \{0, 1\}$, while $NJD_r[f]$ is usually greater than 1. In the simply-connected case $NJD_r[f]$ (denoted then by $D_r[f]$) has been found by the authors for some special kinds of manifolds [3–7].

The computations of the invariants $NF_r(f)$ and $NJD_r[f]$ are in general very challenging tasks, nevertheless $NF_r(f)$ was found in many particular cases [11–15,18,22–25]. The determination of the invariants simplifies a little for self-maps of manifolds with simple Reidemeister relations. In [9] we found $NJD_r[f]$ for all self-maps of 3-dimensional real projective

☆ The research supported by MSHE grant No. N N201 373236.
* Corresponding author.
E-mail addresses: graff@mif.pg.gda.pl (G. Graff), jezierski@acn.waw.pl (J. Jezierski).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.08.027

space $\mathbb{R}P^3$. The recent finding of all forms of local indices of iterations in arbitrary dimension [10], make it possible to try to calculate $NJD_r[f]$ also for $\mathbb{R}P^m$, where $m > 3$. However, the precise determination of the invariant for higher-dimensional manifolds is a very complicated combinatorial task. In this paper we give an estimate for $NJD_r[f]$ from below and from above for self-maps of $\mathbb{R}P^m$, where m is odd (the case of even m is more difficult, see Remark 5.7). The obtained estimates provide some valuable information concerning periodic points. Namely, if $a \leqslant NJD_r[f] \leqslant b$, then

(1) every smooth map g smoothly homotopic to f has at least a r-periodic points,
(2) there exists a smooth map g smoothly homotopic to f having at most b r-periodic points.

2. Invariant $D_r^m[f]$

The topological invariant $D_r^m[f]$ was introduced in [5] and is equal to the minimal number of r-periodic points in smooth homotopy class of f, a self-map of a simply-connected manifold:

Theorem 2.1. *([5]) Let M be a closed smooth connected and simply-connected manifold of dimension $m \geqslant 3$ and $r \in \mathbb{N}$ be a fixed number. Then, for a smooth map $f : M \to M$ we have*

$$D_r^m[f] = MF_r^{diff}(f).$$

In the final sections we will make use of this invariant to estimate $NJD_r[f]$ for f being a self-map of $\mathbb{R}P^m$. Now, we give the definition of $D_r^m[f]$ and describe its basic properties.

Definition 2.2. A sequence of integers $\{c_n\}_{n=1}^{\infty}$ is called $DD^m(p)$ sequence if there are: a C^1 map $\phi : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ is open; and P, an isolated p-orbit of ϕ, such that $c_n = \operatorname{ind}(\phi^n, P)$ (notice that $c_n = 0$ if n is not a multiple of p). The finite sequence $\{c_n\}_{n|r}$ will be called $DD^m(p \mid r)$ sequence if this equality holds for $n \mid r$, where r is fixed.

For a fixed integer $r \geqslant 1$ the invariant $D_r^m[f]$ is defined as the minimal number of $DD^m(p \mid r)$ sequences which in sum give the sequence of Lefschetz numbers of iterations.

Definition 2.3. Let $\{L(f^n)\}_{n|r}$ be a finite sequence of Lefschetz numbers. We decompose $\{L(f^n)\}_{n|r}$ into the sum:

$$L\big(f^n\big) = c_1(n) + \cdots + c_s(n), \tag{2.1}$$

where c_i is a $DD^m(l_i \mid r)$ sequence for $i = 1, \ldots, s$. Each such decomposition determines the number $l = l_1 + \cdots + l_s$. We define the number $D_r^m[f]$ as the smallest l which can be obtained in this way.

Remark 2.4. The combinatorial procedure described in Definition 2.3 has a clear geometrical interpretation. Namely, let f be a smooth self-map of a manifold M of dimension at least 3 and r be a fixed natural number. By the strong result (so-called Canceling and Creating Procedures proved in [17]) one can create any periodic orbit in the smooth homotopy class of f (and thus its sequence of indices of iterations is $DD^m(p \mid r)$ sequence). What is more, one can also remove in the smooth homotopy class any set of periodic points provided their indices of iterations are equal in total to 0. As a consequence, every decomposition of $\{L(f^n)\}_{n|r}$ into $DD^m(p \mid r)$ sequences gives the associated orbit structure for some map in the smooth homotopy class.

Thus, $MF_r^{diff}(f)$ i.e. the minimal number of r-periodic points in the smooth homotopy class of f is given by $D_r^m[f]$.

Any sequence of indices of iterations can be written down in the convenient form of integral combination of some basic periodic sequences $\{\operatorname{reg}_k(n)\}_n$.

Definition 2.5. For a given k we define the *basic sequence*:

$$\operatorname{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

It turns out that any sequence of indices of iterations (as well as Lefschetz numbers of iterations) can be uniquely represented in the form of *periodic expansion* (cf. [19]) i.e.

$$\operatorname{ind}\big(f^n, x_0\big) = \sum_{k=1}^{\infty} a_k \operatorname{reg}_k(n), \tag{2.2}$$

where $a_n = \frac{1}{n}\sum_{k|n} \mu(k) \operatorname{ind}(f^{(n/k)}, x_0)$, μ is the Möbius function, i.e. $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^s$ if k is a product of s different primes, $\mu(k) = 0$ if $p^2 \mid k$ for some prime p.

Remark 2.6. The coefficients a_n in the formula (2.2) must be integers, which was proved by Dold [2].

For manifolds of dimension $m \geqslant 4$, the computations of $D_r^m[f]$ become easier due to the following:

Theorem 2.7. *([4]) For $m \geqslant 4$, in Definition 2.3 of $D_r^m[f]$, one may equivalently use only $DD^m(1 \mid r)$ sequences.*

Both sides of the equality (2.1) can be represented in the form of periodic expansions, as a consequence for the effective computation of $D_r^m[f]$ for $m \geqslant 4$ one needs:

(1) periodic expansion of $L(f^n) = \sum_{k|r} b_k \operatorname{reg}_k(n)$,
(2) all possible forms of periodic expansions of local fixed point indices of iterations of a smooth map $\{\operatorname{ind}(g^n, x)\}_{n=1}^\infty$ at a fixed point.

The information necessary in item (2), i.e. the complete list of all $DD^m(1)$ sequences, has been recently provided in [10].

Before we give that list (Theorem 2.9 below), we first introduce some notation. By $\operatorname{LCM}(H)$ we mean the least common multiple of all elements in H with the convention that $\operatorname{LCM}(\emptyset) = 1$. We define the set $\overline{H}$ by: $\overline{H} = \{\operatorname{LCM}(Q){:}\ Q \subset H\}$.

Next, for natural s we denote by $L(s)$ any set of natural numbers of the form $\bar{L}$, where $\#L = s$ and $1, 2 \notin L$.

By $L_2(s)$ we denote any set of natural numbers of the form $\bar{L}$, where $\#L = s + 1$ and $1 \notin L$, $2 \in L$.

Example 2.8. Consider $L_2(1)$. This is any set of the form $\bar{L}$, where L has 2 elements, with $1 \notin L$ and $2 \in L$. Assume that the second element in L is equal to w. Then

$$\begin{aligned}
\bar{L} = \overline{\{2, w\}} &= \bigl\{\operatorname{LCM}(Q){:}\ Q \subset \{2, w\}\bigr\} \\
&= \bigl\{\operatorname{LCM}(\emptyset), \operatorname{LCM}\bigl(\{2\}\bigr), \operatorname{LCM}\bigl(\{w\}\bigr), \operatorname{LCM}\bigl(\{2, w\}\bigr)\bigr\} \\
&= \bigl\{1, 2, w, \operatorname{LCM}\bigl(\{2, w\}\bigr)\bigr\}.
\end{aligned}$$

Theorem 2.9. *([10]) Let g be a C^1 self-map of $\mathbb{R}^m$, where $m > 1$ is odd, and $g(x_0) = x_0$. Then the sequence of local indices of iterations $\{\operatorname{ind}(g^n, x_0)\}_{n=1}^\infty$ has one of the following forms.*

(A^o): $\operatorname{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-3}{2})} a_k \operatorname{reg}_k(n)$.

(B^o), (C^o), (D^o): $\operatorname{ind}(g^n, x_0) = \sum_{k \in L(\frac{m-1}{2})} a_k \operatorname{reg}_k(n)$, *where*

$$a_1 = \begin{cases} 1 & \text{in the case } (B^o), \\ -1 & \text{in the case } (C^o), \\ 0 & \text{in the case } (D^o). \end{cases}$$

(E^o), (F^o): $\operatorname{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-1}{2})} a_k \operatorname{reg}_k(n)$, *where*

$$a_1 = 1 \quad \text{and} \quad a_2 = \begin{cases} 0 & \text{in the case } (E^o), \\ -1 & \text{in the case } (F^o). \end{cases}$$

Let us mention here that there are similar formulas for the case of even m, see [10].

Remark 2.10. Theorem 2.9 could be interpreted in the following way: the geometrical condition of smoothness of g leads to some algebraical restrictions for indices of iterations of g. Namely, the form of $\{\operatorname{ind}(g^n, x_0)\}_{n=1}^\infty$ depends on the derivative of $Dg(x_0)$. More precisely, the possible indices k that can appear in basic sequences $a_k \operatorname{reg}_k$ in the periodic expansion of $\{\operatorname{ind}(g^n, x_0)\}_{n=1}^\infty$ could be expressed in terms of degrees of primitive roots of unity which are contained in the spectrum of $Dg(x_0)$ [1].

3. Reidemeister graph

In order to obtain the bounds for $\#\operatorname{Fix}(f^r)$ we will need the notion of the *Reidemeister graph* $\mathcal{GOR}(f; r)$. Now we recall the scheme of the construction of this graph in general case (see [19] for the details) and then describe the form of $\mathcal{GOR}(f; r)$ for self-maps of $\mathbb{R}P^m$.

The set of vertices of $\mathcal{GOR}(f; r)$ is, by the definition, the disjoint sum of orbits of Reidemeister classes $\bigcup_{k|r} \mathcal{OR}(f^k)$. There are natural maps $i_{k,l} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$ (for $l \mid k$) which introduce the partial order in $\mathcal{GOR}(f; r) = \bigcup_{k|r} \mathcal{OR}(f^k)$ ($A \preccurlyeq B \Leftrightarrow i_{k,l}(A) = B$).

The space $\mathbb{R}P^m$ for odd m is oriented and thus one may associate with each its self-map f its degree $\beta = \deg(f)$. Let us remind that the fundamental group $\pi_1 \mathbb{R}P^m = \mathbb{Z}_2$. By $\mathcal{R}(f^n)$ we will denote the Reidemeister class of f^n. The orbits of Reidemeister classes depend on the parity of β in the following way [15]:

For all $n \in \mathbb{N}$:

- if β is even then the homotopy group homomorphism $f_\# : \pi_1 \mathbb{R}P^m \to \pi_1 \mathbb{R}P^m$ is zero map and $\mathcal{R}(f^n) = \mathcal{OR}(f^n) = \{*\}$, a singleton set,
- if β is odd then $f_\#$ is the isomorphism, thus $\mathcal{R}(f^n) = \mathcal{OR}(f^n) = \mathbb{Z}_2$.

Remark 3.1. In the further part of the paper we will consider only the case of odd β, because in the other case the computation of $NJD_r[f]$ reduces to the simply-connected case. Namely, if β is even, each orbit of Reidemeister classes consists of only one element, and thus $NJD_r[f] = D_r[h]$, where h is a self-map of S^m of degree β.

The aim of the paper is to give an estimation of the invariant $NJD_r[f]$ in the case of self-maps of m-dimensional real projective space $\mathbb{R}P^m$, where $m > 3$ is odd. However, the obtained results remain valid in more general situation described by the following

Standing Assumptions 3.2.

(1) $f : M \to M$ is a self-map of a smooth closed connected manifold of dimension $\geqslant 4$ and r is a given natural number,
(2) $\pi_1 M = \mathbb{Z}_2$, $f_\# = \mathrm{id}$,
(3) all coefficients a_{l^*} in the Reidemeister graph, standing at Reg_{l^*} with l dividing r, are nonzero.

The above assumptions are satisfied for self-maps $f : \mathbb{R}P^m \to \mathbb{R}P^m$ where $m > 3$ is odd and $|\beta| = |\deg f| \geqslant 3$ is also odd. In that case the items (1) and (2) follow from our previous considerations. The item (3) is proved in [9, Lemma 5.5 and Corollary 5.6] for $\mathbb{R}P^3$, but exactly the same arguments act also for higher odd dimensional projective spaces. The definition of the coefficients a_{l^*} is given below by the formula (4.2).

Let us mention here, that the fulfillment of the condition (3) of our Standing Assumptions for self-maps of $\mathbb{R}P^m$ with $|\deg f| \geqslant 3$ results from the fact that $\{|L(f^n)|\}_{n=1}^{\infty}$ grow fast (exponentially) and thus the moduli of the coefficients a_{l^*} also grow fast.

Now we continue the construction of the Reidemeister graph and the invariant $NJD_r[f]$ under our Standing Assumptions. In the set of orbits of the Reidemeister classes we define the natural map induced by inclusion of the respective Nielsen classes. If $N^l \subset \mathrm{Fix}(f^l)$, $N^k \subset \mathrm{Fix}(f^k)$ are Nielsen classes representing the Reidemeister classes $A^l \in \mathcal{OR}(f^l)$ and $A^k \in \mathcal{OR}(f^k)$ respectively, then $N^l \subset N^k$ implies $i_{k,l}(A^l) = A^k$ (cf. [19]).

By Standing Assumptions, $\mathcal{OR}(f^l) = \mathbb{Z}_2$. Let us denote $\mathcal{OR}(f^l) = \{l', l''\}$, $\mathcal{OR}(f^k) = \{k', k''\}$, where l' and k' correspond to the identity element in $\mathbb{Z}_2$.

The map $i_{k,l} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$ has the following form (cf. [8])

$$i_{k,l}(l') = k', \tag{3.1}$$

$$i_{k,l}(l'') = \begin{cases} k'' & \text{if } \frac{k}{l} \text{ is odd,} \\ k' & \text{if } \frac{k}{l} \text{ is even.} \end{cases} \tag{3.2}$$

Definition 3.3. Let us consider the natural number r and the set $\bigcup_{k|r} \mathcal{OR}(f^k) = \bigcup_{k|r} \{k', k''\}$. In this set we introduce the partial order "$\preccurlyeq$" in the following way: $l^* \preccurlyeq k^*$, where $l^* \in \{l', l''\}$, $k^* \in \{k', k''\}$ if and only if

- $l \mid k$,
- $i_{k,l} : \{l', l''\} \to \{k', k''\}$ maps l^* on k^*.

If $l^* \preccurlyeq k^*$ then we say that l^* is preceding k^*. We use the notation $l^* \prec k^*$ if $l^* \preccurlyeq k^*$ but $l^* \neq k^*$.

Now we can give the definition of the Reidemeister graph for f, a self-map of a manifold M which satisfies our Standing Assumptions.

Definition 3.4. Letting r be fixed, the partially ordered set of Reidemeister orbits $\bigcup_{k|r} \{k', k''\}$ can be perceived as a directed graph (and denoted by $\mathcal{GOR}(f; r)$). There is an edge from vertex l^* to k^* if and only if $l^* \preccurlyeq k^*$, with the convention that if $l^* \prec k^* \prec s^*$ then we omit the edge from l^* to s^* (understanding that there is the connection between these two vertices through k^*).

4. $NJD_r[f]$ for a self-map f of M satisfying Standing Assumptions

In this section we give the definition of $NJD_r[f]$ for a self-map f of M satisfying our Standing Assumptions 3.2.

4.1. Index function

We define an index function by the formula: $I(n^*) = \mathrm{ind}(f^n, n^*)$. In this way we obtain a function I defined on the set of the vertices of the graph $\mathcal{GOR}(f;r)$.

Let us recall that $\mathbb{R}P^m$ is a Jiang space for odd m, so both Nielsen classes of the given self-map of $\mathbb{R}P^m$ have equal indices [20]. As $L(f^n) = 1 - \beta^n = I(n') + I(n'')$ we get that

$$I(n') = I(n'') = \frac{1-\beta^n}{2}, \tag{4.1}$$

where β is the degree of the map f.

Now we generalize the notion of periodic expansion onto the maps of $\mathcal{GOR}(f;r)$.

Definition 4.1. For each vertex l^*, where $l^* \in \{l', l''\}$, we define basic integer-valued function on the graph:

$$\mathrm{Reg}_{l^*}(n^*) = \begin{cases} l & \text{if } l^* \preccurlyeq n^*, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.2. $\mathrm{Reg}_{3'}(6') = 3$, $\mathrm{Reg}_{3'}(6'') = 0$.

Function I can be uniquely represented as an integral combination of basic functions Reg_{l^*} (so-called generalized periodic expansion) [8].

$$I(n^*) = \sum_{l^* \preccurlyeq n^*} a_{l^*} \mathrm{Reg}_{l^*}(n^*). \tag{4.2}$$

4.2. Attaching sequences at vertices

Let Γ be one of the sequences (A)–(F) given in Theorem 2.9. It is represented as a combination of reg's: $\Gamma = \sum_{d \in O} a_d \,\mathrm{reg}_d$. We will say that we attach Γ at the vertex l^* if we define the following function Γ_{l^*} on the Reidemeister graph:

$$\Gamma_{l^*}(n^*) = \sum_{l^* \preccurlyeq (dl)^*,\, d \in O} a_d \,\mathrm{Reg}_{(dl)^*}(n^*). \tag{4.3}$$

Definition 4.3. We will say that a sequence Γ, of one of the types (A)–(F), attached at the vertex l^* *realizes* $a_{k^*} \mathrm{Reg}_{k^*}$ (or a_{k^*} for short) if this expression appears in the right-hand side of the formula (4.3).

4.3. Definition of $NJD_r[f]$

The index function I can be expressed as a sum of the sequences (A)–(F) attached at some vertices:

$$I(n^*) = \sum_{l^* \preccurlyeq n^*} a_{l^*} \mathrm{Reg}_{l^*}(n^*) = \Gamma^1_{l_1^*}(n^*) + \cdots + \Gamma^s_{l_s^*}(n^*). \tag{4.4}$$

Each such decomposition determines the sum $l_1 + \cdots + l_s$, which we call *the decomposition number*.

Definition 4.4. $NJD_r[f]$ is defined as the minimal decomposition number under all possible decompositions.

Remark 4.5. In [8] we described more general construction of the invariant $NJD_r[f]$ for any self-map of a closed smooth connected manifold of dimension at least 3. In general case one must take into account that:

- the sequences are attached at the vertices of the Reidemeister graph of f but $\mathcal{GOR}(f;r)$ may be more complex than the one described by relations in Definition 3.3.
- Index function I may take much more complicated form than (4.1).

In any case, the following theorem holds:

Theorem 4.6. *([8]) For any self-map of a closed smooth connected manifold of dimension greater than* 3 *and a fixed integer $r \in \mathbb{N}$ there is*:

$$NJD_r^m[f] = MF_r^{diff}(f).$$

The geometrical interpretation of $NJD_r^m[f]$ the reader may find in [9, Section 4.4].

Remark 4.7. The aim of the paper is to minimize the set $\mathrm{Fix}(f^r)$ in the smooth homotopy class (for a given $r \in \mathbb{N}$). This leads also to the question about the possible orbital structure of such minimal periodic sets. For example, we know that in dimension $m \geqslant 4$ and for f being a self-map of a simply-connected manifold, $\mathrm{Fix}(f^r)$ may consist only of fixed points ([4], cf. also Remark 2.7). When, in the simply-connected case, could $\mathrm{Fix}(f^r)$ contain longer orbits? By Remark 2.4 the promising way to answer such a question is to analyze the decomposition of the sequence of Lefschetz numbers $\{L(f^k)\}_{k|r}$ into $DD^m(p \mid r)$ sequences. In general (non-simply-connected) case one needs to follow such an analysis on Reidemeister graph.

Remark 4.8. Let us recall that for connected simply-connected closed smooth manifold M the invariants $D_r^m[f]$ and $NJD_r^m[f]$ coincide, since then for each iteration there is a single Nielsen class.

5. Estimation of $NJD_r^m[f]$ for maps satisfying Standing Assumptions

For the rest of the paper we assume that f is a map satisfying our Standing Assumptions 3.2.

The computations of the invariant $D_r^m[f]$ in [4,5] (and $NJD_r^m[f]$ in [9]) show that the most troublesome coefficient is the one standing at reg_1. This is because in some forms of sequences of indices listed in Theorem 2.9 the coefficient at reg_1 is not arbitrary, but belongs to the set $\{-1, 0, 1\}$. As a consequence, in some situations one can represent the term $a_1 \,\mathrm{reg}_1$ (in the minimal realization, cf. Definition 2.3) as a sum of the other $DD^m(1 \mid r)$ sequences that appear in the minimal realization. However, it is not easy to describe all these situations. To avoid this difficulty we introduce the *invariant* $D_r^m[f]$ *mod* reg_1. The computation of the last invariant is much simpler and gives the approximate value of $D_r^m[f]$ [4], namely:

$$D_r^m[f] \bmod \mathrm{reg}_1 = D_r^m[f] \text{ or } D_r^m[f] - 1.$$

Definition 5.1. By $(D_r^m[f] \bmod \mathrm{reg}_1)$ we denote the number of sequences in the minimal decomposition of $L(f^n) = \sum_{k|r} b_k \,\mathrm{reg}_k(n)$ into $DD^m(1 \mid r)$ sequences *modulo* reg_1 i.e. we require only that the equality (2.1) holds for all divisors $n \mid r$ different than 1 (thus we ignore the coefficient at reg_1).

Remark 5.2. In the simply-connected case $NJD_r[f] = D_r[f]$, and $D_r[f]$ may be expressed in the language of the Reidemeister graph in the following way. If M is simply connected then $\mathcal{GOR}(f; r)$ constitutes the graph of all divisors of r and the procedure of calculating $D_r[f]$ described in Definition 2.3 can be equivalently expressed as finding minimal number of $DD^m(1)$ sequences attached at 1 realizing in sum $\{L(f^n)\}_{n|r}$.

Remark 5.3. Let us consider a self-map $f : M \to M$ of a connected simply-connected closed manifold M, satisfying the condition:

($*$) all coefficients b_k for $k \neq 1$ in the periodic expansion of $L(f^n) = \sum_{k|r} b_k \,\mathrm{reg}_k(n)$ are nonzero.

We will denote the family of such maps by $\mathcal{B}$. It was proved in [4] that, for a given dimension m, $D_r^m[f] \bmod \mathrm{reg}_1$ has the common value for all maps f in $\mathcal{B}$. Let P be an odd natural number, we will denote for short this common value of $(D_r^m[f] \bmod \mathrm{reg}_1)$ for $r = P$ by h_P, assuming that the dimension $m \geqslant 4$ is fixed.

Remark 5.4. Let us mention that the algorithm of determining h_P was described in [4] and successfully applied for calculating h_P in the case P is a product of different odd primes. Namely, let the dimension of the manifold be equal to m ($m = 2s$ or $m = 2s + 1$) and P be a product of v different odd primes, where $v \geqslant s$. We represent v in the form $v = k \cdot s + R$ where $R = 1, \ldots, s$ and $k \in \mathbb{Z}$. Then

$$h_P = \frac{2^{sk+R} - 2^R}{2^s - 1} + 1. \tag{5.1}$$

Our aim is to estimate $NJD_r[f]$, where $r = P \cdot 2^R$ with P odd, by h_P. The main idea of finding the useful estimation is based on the decomposition of $\mathcal{GOR}(f; r)$ into parts, each of which is isomorphic to the graph of all divisors of the odd number P, and observing that each such part gives the contribution to $NJD_r[f]$ equal to h_P.

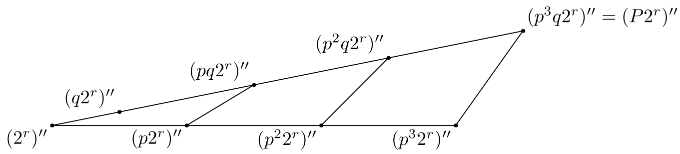

Fig. 1. $\mathcal{GOR}(2^sP)''$ for $P = p^3q$; p, q primes.

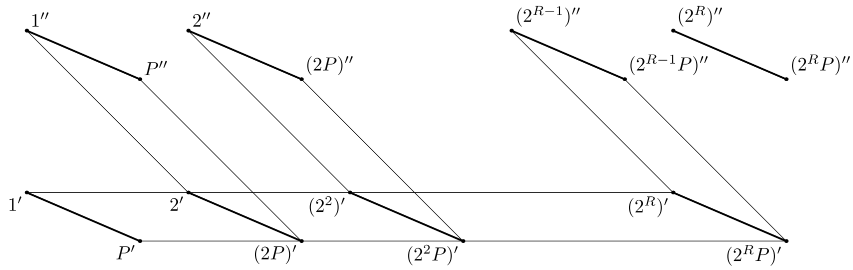

Fig. 2. $\mathcal{GOR}(f;r)$ for self-maps of $\mathbb{R}P^m$.

Now, let f be a map satisfying Standing Assumptions and $\mathcal{GOR}(f;r)$ be the Reidemeister graph for f, $r = P \cdot 2^R$, P is odd. For a fixed $0 \leqslant s \leqslant R$ we consider a part of this graph, defined as:

$$\mathcal{GOR}\big(2^sP\big)'' = \big\{\big(2^sk\big)''\colon\ k \mid P\big\}.$$

Lemma 5.5. *In order to realize all coefficients of $\mathcal{GOR}(2^sP)''$, maybe except for the coefficient at the vertex $(2^s)''$, one needs h_P $DD^m(1)$ sequences attached at $(2^s)''$. They give the contribution $h_P \cdot 2^s$ to $NJD_r[f]$.*

Proof. The equality

$$\mathcal{GOR}\big(2^sP\big)'' = \big\{\big(2^sk\big)''\colon\ k \mid P\big\}, \tag{5.2}$$

shows that $\mathcal{GOR}(2^sP)''$ is isomorphic to the graph of all divisors of P. Thus, by Remark 5.2, to realize (in the sense of Definition 4.3) all coefficients of $\mathcal{GOR}(2^sP)''$ modulo $a_{(2^s)''}$ it is enough to attach ($D^m_P[f]$ mod reg_1) $DD^m(1)$ sequences at $(2^s)''$. Furthermore, due to item (3) of our Standing Assumptions the condition $(*)$ of Remark 5.3 is satisfied for the isomorphic graph of all divisors of P. As a consequence, ($D^m_P[f]$ mod reg_1) does not depend on f and is equal h_P. This ends the proof. □

The example of $\mathcal{GOR}(2^sP)''$ for $P = p^3q$, where p and q are primes, is given in Fig. 1. We are now in a position to formulate the main result of the paper.

Theorem 5.6. *Let f be a map satisfying our Standing Assumptions i.e.*

(1) *$f: M \to M$ is a self-map of a smooth closed connected manifold of dimension $\geqslant 4$ and r is a given natural number of the form $r = 2^P \cdot R$, where P is odd,*
(2) $\pi_1 M = \mathbb{Z}_2$, $f_\# = \mathrm{id}$,
(3) *all coefficients a_{l^*} in the general periodic expansion of index function (4.2) (i.e. the coefficients in the Reidemeister graph standing at Reg_{l^*}) are nonzero.*

Then:

$$2^{R+1} \cdot h_P \leqslant NJD_r[f] \leqslant 2^{R+1} \cdot (h_P + 1).$$

Proof. In Fig. 2 a symbolic representation of $\mathcal{GOR}(f;r)$ is given. This graph can be interpreted literally in the case P is a prime number. In the general case the aslope lines (in bold), joining $(2^s)^*$ and $(2^sP)^*$, represent a graph isomorphic to a graph of all divisors of an odd number P. This means that on the bold edges there are other vertices, which are not specified. For example, the line joining $(2^s)^*$ and $(2^sP)^*$ for $P = p^3q$, where p, q are primes, is in fact the graph given in Fig. 1.

Now, we fix $0 \leqslant s \leqslant R$. Then, by Lemma 5.5, to realize all the coefficients at vertices $\{(l \cdot 2^s)''\colon\ l \mid P,\ l \neq 1\}$ one needs h_P sequences attached at $(2^s)''$, which gives the contribution to $NJD_r[f]$ equal to $2^s \cdot h_P$.

Similarly, to realize the coefficients at $\{l'\colon\ l \mid P,\ l \neq 1\}$ one needs h_P sequences attached at $1'$. In sum this gives the following estimates of $NJD_r[f]$ from below:

$$\begin{aligned} NJD_r[f] &\geqslant \sum_{s=0}^{R} h_P \cdot 2^s + h_P \\ &= h_P\left[1 + \sum_{s=0}^{R} 2^s\right] = h_P \cdot 2^{R+1}. \end{aligned} \tag{5.3}$$

Now we give an upper estimate. Let us notice that realizing the graph $\mathcal{GOR}(2^s P)''$ we can also realize the graph $\mathcal{GOR}(2^{s+1} P)' = \{(2^{s+1} \cdot l)'\colon\ l \mid P\}$ for $s = 0, \dots, R-1$, modulo the coefficient at $(2^{s+1})'$. In fact, let $c(n)$ be a sequence attached at $(2^s)''$ realizing the coefficient at $(2^s l)''$, where $l \mid P$. Then, as P is odd, $c(n)$ is of one of the types (B^o), (C^o) or (D^o) of Theorem 2.9. We can change it for the sequence $c'(n)$ of the type (E^o) or (F^o) of Theorem 2.9 realizing also $(2^{s+1} l)'$ for $0 \leqslant s \leqslant R-1$. This is possible, since the dimension of the manifold M is odd by the assumption.

As the result, the only coefficients which may still remain unrealized are $\{(2^s)^*\colon\ 0 \leqslant s \leqslant R\}$.

The vertices $1'', 2'', \dots, (2^R)''$ are irreducible, so to realize the coefficients at these vertices it is necessary (and sufficient) to attach a single sequence at each of them, which gives the contribution to $NJD_r[f]$ equal to: $1 + 2 + \dots + 2^R = 2^{R+1} - 1$. Furthermore, if we use for that purpose the sequences of the type (A), then they realize also the coefficients at the vertices $2', \dots, (2^R)'$. The remaining coefficient at $1'$ can be realized by one sequence of the type (A) attached at this vertex. Finally, summing the contributions of the three parts of $\mathcal{GOR}(f;r)$ considered above, we obtain:

$$NJD_r[f] \leqslant h_P \cdot 2^{R+1} + \left(2^{R+1} - 1\right) + 1, \tag{5.4}$$

which ends the proof. □

Remark 5.7. For a self-map $f : \mathbb{R}P^m \to \mathbb{R}P^m$, where m is even $\mathcal{GOR}(f;r)$ is the same as in the case of m odd. However, the forms of local fixed point indices of iterations in even dimensions are much different (cf. [10]). This makes it impossible to apply the same trick which allowed us, in the proof of the inequality (5.3), to realize both $\mathcal{GOR}(2^s P)''$ and $\mathcal{GOR}(2^{s+1} P)'$ by one sequence, and consequently it is much more difficult to find the reasonable estimate for $NJD_r[f]$ from above in the case of even m.

References

[1] S.N. Chow, J. Mallet-Parret, J.A. Yorke, A periodic orbit index which is a bifurcation invariant, in: Geometric Dynamics, Rio de Janeiro, 1981, in: Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 109–131.
[2] A. Dold, Fixed point indices of iterated maps, Invent. Math. 74 (1983) 419–435.
[3] G. Graff, Minimal number of periodic points for smooth self-maps of two-holed 3-dimensional closed ball, Topol. Methods Nonlinear Anal. 33 (1) (2009) 121–130.
[4] G. Graff, J. Jezierski, Combinatorial scheme of finding minimal number of periodic points for smooth self-maps of simply-connected manifolds, J. Fixed Point Theory Appl. (2012), http://dx.doi.org/10.1007/s11784-012-0076-1, in press.
[5] G. Graff, J. Jezierski, Minimal number of periodic points for C^1 self-maps of compact simply-connected manifolds, Forum Math. 21 (3) (2009) 491–509.
[6] G. Graff, J. Jezierski, Minimal number of periodic points for self-maps of S^3, Fund. Math. 204 (2009) 127–144.
[7] G. Graff, J. Jezierski, Minimization of the number of periodic points for smooth self-maps of closed simply-connected 4-manifolds, Discrete Contin. Dyn. Syst. Suppl. (2011) 523–532.
[8] G. Graff, J. Jezierski, Minimizing the number of periodic points for smooth maps. Non-simply connected case, Topology Appl. 158 (3) (2011) 276–290.
[9] G. Graff, J. Jezierski, M. Nowak-Przygodzki, Minimal number of periodic points for smooth self-maps of $\mathbb{R}P^3$, Topology Appl. 157 (2010) 1784–1803.
[10] G. Graff, J. Jezierski, P. Nowak-Przygodzki, Fixed point indices of iterated smooth maps in arbitrary dimension, J. Differential Equations 251 (2011) 1526–1548.
[11] E. Hart, P. Heath, E. Keppelmann, Algorithms for Nielsen type periodic numbers of maps with remnant on surfaces with boundary and on bouquets of circles. I, Fund. Math. 200 (2) (2008) 101–132.
[12] P. Heath, E. Keppelmann, Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds I, Topology Appl. 76 (1997) 217–247.
[13] P. Heath, E. Keppelmann, Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds II, Topology Appl. 106 (2000) 149–167.
[14] P. Heath, E. Keppelmann, Fibre techniques in Nielsen periodic point theory on solvmanifolds III. Calculations, Quaestiones Math. 25 (2002) 177–208.
[15] J. Jezierski, Homotopy periodic sets of selfmaps of real projective spaces, Bol. Soc. Mat. Mexicana (3) 11 (2) (2005) 294–302.
[16] J. Jezierski, The least number, of n-periodic points of a self-map of a solvmanifold, can be realised by a smooth map, Topology Appl. 158 (9) (2011) 1113–1120.
[17] J. Jezierski, Wecken's theorem for periodic points in dimension at least 3, Topology Appl. 153 (11) (2006) 1825–1837.
[18] J. Jezierski, E. Keppelmann, W. Marzantowicz, Wecken property for periodic points on the Klein bottle, Topol. Methods Nonlinear Anal. 33 (2009) 51–64.
[19] J. Jezierski, W. Marzantowicz, Homotopy Methods in Topological Fixed and Periodic Points Theory, Topol. Fixed Point Theory Appl., vol. 3, Springer, Dordrecht, 2006.
[20] B.J. Jiang, Lectures on the Nielsen Fixed Point Theory, Contemp. Math., vol. 14, Amer. Math. Soc., Providence, 1983.
[21] B.J. Jiang, Fixed point classes from a differential viewpoint, in: Lecture Notes in Math., vol. 886, Springer, 1981, pp. 163–170.
[22] H.J. Kim, J.B. Lee, W.S. Yoo, Computation of the Nielsen type numbers for maps on the Klein bottle, J. Korean Math. Soc. 45 (5) (2008) 1483–1503.
[23] S.H. Lee, A note on Nielsen type numbers, Commun. Korean Math. Soc. 25 (2) (2010) 263–271.
[24] J.B. Lee, X. Zhao, Nielsen type numbers and homotopy minimal periods for maps on 3-solvmanifolds, Algebr. Geom. Topol. 8 (1) (2008) 563–580.
[25] J.B. Lee, X. Zhao, Nielsen type numbers and homotopy minimal periods for maps on the 3-nilmanifolds, Sci. China Ser. A 51 (3) (2008) 351–360.