

# Estimation of the minimal number of periodic points for smooth self-maps of odd dimensional real projective spaces <sup>☆</sup>

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## ABSTRACT

Let  $f$  be a smooth self-map of a closed connected manifold of dimension  $m \geq 3$ . The authors introduced in [G. Graff, J. Jezierski, Minimizing the number of periodic points for smooth maps. Non-simply connected case, *Topology Appl.* 158 (3) (2011) 276–290] the topological invariant  $NJD_r[f]$ , where  $r$  is a fixed natural number, which is equal to the minimal number of  $r$ -periodic points in the smooth homotopy class of  $f$ . In this paper smooth self-maps of real projective space  $\mathbb{R}P^m$ , where  $m > 3$  is odd, are considered and the estimations from below and above for  $NJD_r[f]$  are given.

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## 1. Introduction

Let  $f$  be a smooth self-map of a compact manifold  $M$ . The central question in the smooth branch of Nielsen periodic point theory is the following: *what is the minimal number of  $r$ -periodic points in the smooth homotopy class of  $f$ ?* In other words, one seeks for the invariant that determines the number

$$MF_r^{\text{diff}}(f) = \min\{\#\text{Fix}(g^r) : g \stackrel{\sim}{\sim} f\}, \quad (1.1)$$

where  $\stackrel{\sim}{\sim}$  means that the maps  $g$  and  $f$  are  $C^1$ -homotopic.

We will consider a smooth closed connected manifold of dimension at least 3. It is remarkable that for  $r = 1$ , i.e. for fixed points, the classical (continuous) and smooth Nielsen theories coincide [21]. However, for  $r > 1$  these theories are much different. Namely, if the minimum in (1.1) is taken over continuous homotopies, then the respective number,  $MF_r(f)$ , is given by Jiang invariant  $NF_r(f)$  (cf. [17,20]). In the smooth case  $MF_r^{\text{diff}}(f) = NJD_r[f]$ , the invariant introduced by the authors in [8]. For smooth  $f$ ,  $NJD_r[f] \geq NF_r(f)$  and the equality holds only in some exceptional situations [16].

In the definition of  $NJD_r[f]$  in addition to Reidemeister relations fixed points indices of iterations are involved. There are strong restrictions for local indices of iterations of smooth maps [1], in contrast to continuous maps, which result in the inequality  $NJD_r[f] \geq NF_r(f)$ . For example, for self-maps of simply-connected manifolds  $NF_r(f) \in \{0, 1\}$ , while  $NJD_r[f]$  is usually greater than 1. In the simply-connected case  $NJD_r[f]$  (denoted then by  $D_r[f]$ ) has been found by the authors for some special kinds of manifolds [3–7].

The computations of the invariants  $NF_r(f)$  and  $NJD_r[f]$  are in general very challenging tasks, nevertheless  $NF_r(f)$  was found in many particular cases [11–15,18,22–25]. The determination of the invariants simplifies a little for self-maps of manifolds with simple Reidemeister relations. In [9] we found  $NJD_r[f]$  for all self-maps of 3-dimensional real projective

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space  $\mathbb{R}P^3$ . The recent finding of all forms of local indices of iterations in arbitrary dimension [10], make it possible to try to calculate  $NJD_r[f]$  also for  $\mathbb{R}P^m$ , where  $m > 3$ . However, the precise determination of the invariant for higher-dimensional manifolds is a very complicated combinatorial task. In this paper we give an estimate for  $NJD_r[f]$  from below and from above for self-maps of  $\mathbb{R}P^m$ , where  $m$  is odd (the case of even  $m$  is more difficult, see Remark 5.7). The obtained estimates provide some valuable information concerning periodic points. Namely, if  $a \leq NJD_r[f] \leq b$ , then

- (1) every smooth map  $g$  smoothly homotopic to  $f$  has at least  $a$   $r$ -periodic points,
- (2) there exists a smooth map  $g$  smoothly homotopic to  $f$  having at most  $b$   $r$ -periodic points.

## 2. Invariant $D_r^m[f]$

The topological invariant  $D_r^m[f]$  was introduced in [5] and is equal to the minimal number of  $r$ -periodic points in smooth homotopy class of  $f$ , a self-map of a simply-connected manifold:

**Theorem 2.1.** ([5]) *Let  $M$  be a closed smooth connected and simply-connected manifold of dimension  $m \geq 3$  and  $r \in \mathbb{N}$  be a fixed number. Then, for a smooth map  $f : M \rightarrow M$  we have*

$$D_r^m[f] = MF_r^{\text{diff}}(f).$$

In the final sections we will make use of this invariant to estimate  $NJD_r[f]$  for  $f$  being a self-map of  $\mathbb{R}P^m$ . Now, we give the definition of  $D_r^m[f]$  and describe its basic properties.

**Definition 2.2.** A sequence of integers  $\{c_n\}_{n=1}^{\infty}$  is called  $DD^m(p)$  sequence if there are: a  $C^1$  map  $\phi : U \rightarrow \mathbb{R}^m$ , where  $U \subset \mathbb{R}^m$  is open; and  $P$ , an isolated  $p$ -orbit of  $\phi$ , such that  $c_n = \text{ind}(\phi^n, P)$  (notice that  $c_n = 0$  if  $n$  is not a multiple of  $p$ ). The finite sequence  $\{c_n\}_{n|r}$  will be called  $DD^m(p|r)$  sequence if this equality holds for  $n|r$ , where  $r$  is fixed.

For a fixed integer  $r \geq 1$  the invariant  $D_r^m[f]$  is defined as the minimal number of  $DD^m(p|r)$  sequences which in sum give the sequence of Lefschetz numbers of iterations.

**Definition 2.3.** Let  $\{L(f^n)\}_{n|r}$  be a finite sequence of Lefschetz numbers. We decompose  $\{L(f^n)\}_{n|r}$  into the sum:

$$L(f^n) = c_1(n) + \dots + c_s(n), \quad (2.1)$$

where  $c_i$  is a  $DD^m(l_i|r)$  sequence for  $i = 1, \dots, s$ . Each such decomposition determines the number  $l = l_1 + \dots + l_s$ . We define the number  $D_r^m[f]$  as the smallest  $l$  which can be obtained in this way.

**Remark 2.4.** The combinatorial procedure described in Definition 2.3 has a clear geometrical interpretation. Namely, let  $f$  be a smooth self-map of a manifold  $M$  of dimension at least 3 and  $r$  be a fixed natural number. By the strong result (so-called Canceling and Creating Procedures proved in [17]) one can create any periodic orbit in the smooth homotopy class of  $f$  (and thus its sequence of indices of iterations is  $DD^m(p|r)$  sequence). What is more, one can also remove in the smooth homotopy class any set of periodic points provided their indices of iterations are equal in total to 0. As a consequence, every decomposition of  $\{L(f^n)\}_{n|r}$  into  $DD^m(p|r)$  sequences gives the associated orbit structure for some map in the smooth homotopy class.

Thus,  $MF_r^{\text{diff}}(f)$  i.e. the minimal number of  $r$ -periodic points in the smooth homotopy class of  $f$  is given by  $D_r^m[f]$ .

Any sequence of indices of iterations can be written down in the convenient form of integral combination of some basic periodic sequences  $\{\text{reg}_k(n)\}_n$ .

**Definition 2.5.** For a given  $k$  we define the *basic sequence*:

$$\text{reg}_k(n) = \begin{cases} k & \text{if } k | n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

It turns out that any sequence of indices of iterations (as well as Lefschetz numbers of iterations) can be uniquely represented in the form of *periodic expansion* (cf. [19]) i.e.

$$\text{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \text{reg}_k(n), \quad (2.2)$$

where  $a_n = \frac{1}{n} \sum_{k|n} \mu(k) \text{ind}(f^{n/k}, x_0)$ ,  $\mu$  is the Möbius function, i.e.  $\mu : \mathbb{N} \rightarrow \mathbb{Z}$  is defined by the following three properties:  $\mu(1) = 1$ ,  $\mu(k) = (-1)^s$  if  $k$  is a product of  $s$  different primes,  $\mu(k) = 0$  if  $p^2 | k$  for some prime  $p$ .

**Remark 2.6.** The coefficients  $a_n$  in the formula (2.2) must be integers, which was proved by Dold [2].

For manifolds of dimension  $m \geq 4$ , the computations of  $D_r^m[f]$  become easier due to the following:

**Theorem 2.7.** ([4]) For  $m \geq 4$ , in Definition 2.3 of  $D_r^m[f]$ , one may equivalently use only  $DD^m(1|r)$  sequences.

Both sides of the equality (2.1) can be represented in the form of periodic expansions, as a consequence for the effective computation of  $D_r^m[f]$  for  $m \geq 4$  one needs:

- (1) periodic expansion of  $L(f^n) = \sum_{k|r} b_k \text{reg}_k(n)$ ,
- (2) all possible forms of periodic expansions of local fixed point indices of iterations of a smooth map  $\{\text{ind}(g^n, x)\}_{n=1}^\infty$  at a fixed point.

The information necessary in item (2), i.e. the complete list of all  $DD^m(1)$  sequences, has been recently provided in [10]. Before we give that list (Theorem 2.9 below), we first introduce some notation. By  $\text{LCM}(H)$  we mean the least common multiple of all elements in  $H$  with the convention that  $\text{LCM}(\emptyset) = 1$ . We define the set  $\bar{H}$  by:  $\bar{H} = \{\text{LCM}(Q) : Q \subset H\}$ .

Next, for natural  $s$  we denote by  $L(s)$  any set of natural numbers of the form  $\bar{L}$ , where  $\#L = s$  and  $1, 2 \notin L$ .

By  $L_2(s)$  we denote any set of natural numbers of the form  $\bar{L}$ , where  $\#L = s + 1$  and  $1 \notin L, 2 \in L$ .

**Example 2.8.** Consider  $L_2(1)$ . This is any set of the form  $\bar{L}$ , where  $L$  has 2 elements, with  $1 \notin L$  and  $2 \in L$ . Assume that the second element in  $L$  is equal to  $w$ . Then

$$\begin{aligned} \bar{L} &= \overline{\{2, w\}} = \{\text{LCM}(Q) : Q \subset \{2, w\}\} \\ &= \{\text{LCM}(\emptyset), \text{LCM}(\{2\}), \text{LCM}(\{w\}), \text{LCM}(\{2, w\})\} \\ &= \{1, 2, w, \text{LCM}(\{2, w\})\}. \end{aligned}$$

**Theorem 2.9.** ([10]) Let  $g$  be a  $C^1$  self-map of  $\mathbb{R}^m$ , where  $m > 1$  is odd, and  $g(x_0) = x_0$ . Then the sequence of local indices of iterations  $\{\text{ind}(g^n, x_0)\}_{n=1}^\infty$  has one of the following forms.

- $(A^0)$ :  $\text{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-3}{2})} a_k \text{reg}_k(n)$ .  
 $(B^0), (C^0), (D^0)$ :  $\text{ind}(g^n, x_0) = \sum_{k \in L(\frac{m-1}{2})} a_k \text{reg}_k(n)$ , where

$$a_1 = \begin{cases} 1 & \text{in the case } (B^0), \\ -1 & \text{in the case } (C^0), \\ 0 & \text{in the case } (D^0). \end{cases}$$

- $(E^0), (F^0)$ :  $\text{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-1}{2})} a_k \text{reg}_k(n)$ , where

$$a_1 = 1 \quad \text{and} \quad a_2 = \begin{cases} 0 & \text{in the case } (E^0), \\ -1 & \text{in the case } (F^0). \end{cases}$$

Let us mention here that there are similar formulas for the case of even  $m$ , see [10].

**Remark 2.10.** Theorem 2.9 could be interpreted in the following way: the geometrical condition of smoothness of  $g$  leads to some algebraical restrictions for indices of iterations of  $g$ . Namely, the form of  $\{\text{ind}(g^n, x_0)\}_{n=1}^\infty$  depends on the derivative of  $Dg(x_0)$ . More precisely, the possible indices  $k$  that can appear in basic sequences  $a_k \text{reg}_k$  in the periodic expansion of  $\{\text{ind}(g^n, x_0)\}_{n=1}^\infty$  could be expressed in terms of degrees of primitive roots of unity which are contained in the spectrum of  $Dg(x_0)$  [1].

### 3. Reidemeister graph

In order to obtain the bounds for  $\#\text{Fix}(f^r)$  we will need the notion of the Reidemeister graph  $\mathcal{GOR}(f; r)$ . Now we recall the scheme of the construction of this graph in general case (see [19] for the details) and then describe the form of  $\mathcal{GOR}(f; r)$  for self-maps of  $\mathbb{R}P^m$ .

The set of vertices of  $\mathcal{GOR}(f; r)$  is, by the definition, the disjoint sum of orbits of Reidemeister classes  $\bigcup_{k|r} \mathcal{OR}(f^k)$ . There are natural maps  $i_{k,l} : \mathcal{OR}(f^l) \rightarrow \mathcal{OR}(f^k)$  (for  $l|k$ ) which introduce the partial order in  $\mathcal{GOR}(f; r) = \bigcup_{k|r} \mathcal{OR}(f^k)$  ( $A \preceq B \Leftrightarrow i_{k,l}(A) = B$ ).

The space  $\mathbb{R}P^m$  for odd  $m$  is oriented and thus one may associate with each its self-map  $f$  its degree  $\beta = \deg(f)$ . Let us remind that the fundamental group  $\pi_1 \mathbb{R}P^m = \mathbb{Z}_2$ . By  $\mathcal{R}(f^n)$  we will denote the Reidemeister class of  $f^n$ . The orbits of Reidemeister classes depend on the parity of  $\beta$  in the following way [15]:

For all  $n \in \mathbb{N}$ :

- if  $\beta$  is even then the homotopy group homomorphism  $f_{\#} : \pi_1 \mathbb{R}P^m \rightarrow \pi_1 \mathbb{R}P^m$  is zero map and  $\mathcal{R}(f^n) = \mathcal{OR}(f^n) = \{*\}$ , a singleton set,
- if  $\beta$  is odd then  $f_{\#}$  is the isomorphism, thus  $\mathcal{R}(f^n) = \mathcal{OR}(f^n) = \mathbb{Z}_2$ .

**Remark 3.1.** In the further part of the paper we will consider only the case of odd  $\beta$ , because in the other case the computation of  $NJD_r[f]$  reduces to the simply-connected case. Namely, if  $\beta$  is even, each orbit of Reidemeister classes consists of only one element, and thus  $NJD_r[f] = D_r[h]$ , where  $h$  is a self-map of  $S^m$  of degree  $\beta$ .

The aim of the paper is to give an estimation of the invariant  $NJD_r[f]$  in the case of self-maps of  $m$ -dimensional real projective space  $\mathbb{R}P^m$ , where  $m > 3$  is odd. However, the obtained results remain valid in more general situation described by the following

**Standing Assumptions 3.2.**

- (1)  $f : M \rightarrow M$  is a self-map of a smooth closed connected manifold of dimension  $\geq 4$  and  $r$  is a given natural number,
- (2)  $\pi_1 M = \mathbb{Z}_2$ ,  $f_{\#} = \text{id}$ ,
- (3) all coefficients  $a_{l^*}$  in the Reidemeister graph, standing at  $\text{Reg}_{l^*}$  with  $l$  dividing  $r$ , are nonzero.

The above assumptions are satisfied for self-maps  $f : \mathbb{R}P^m \rightarrow \mathbb{R}P^m$  where  $m > 3$  is odd and  $|\beta| = |\deg f| \geq 3$  is also odd. In that case the items (1) and (2) follow from our previous considerations. The item (3) is proved in [9, Lemma 5.5 and Corollary 5.6] for  $\mathbb{R}P^3$ , but exactly the same arguments act also for higher odd dimensional projective spaces. The definition of the coefficients  $a_{l^*}$  is given below by the formula (4.2).

Let us mention here, that the fulfillment of the condition (3) of our Standing Assumptions for self-maps of  $\mathbb{R}P^m$  with  $|\deg f| \geq 3$  results from the fact that  $\{|L(f^n)|\}_{n=1}^{\infty}$  grow fast (exponentially) and thus the moduli of the coefficients  $a_{l^*}$  also grow fast.

Now we continue the construction of the Reidemeister graph and the invariant  $NJD_r[f]$  under our Standing Assumptions. In the set of orbits of the Reidemeister classes we define the natural map induced by inclusion of the respective Nielsen classes. If  $N^l \subset \text{Fix}(f^l)$ ,  $N^k \subset \text{Fix}(f^k)$  are Nielsen classes representing the Reidemeister classes  $A^l \in \mathcal{OR}(f^l)$  and  $A^k \in \mathcal{OR}(f^k)$  respectively, then  $N^l \subset N^k$  implies  $i_{k,l}(A^l) = A^k$  (cf. [19]).

By Standing Assumptions,  $\mathcal{OR}(f^l) = \mathbb{Z}_2$ . Let us denote  $\mathcal{OR}(f^l) = \{l', l''\}$ ,  $\mathcal{OR}(f^k) = \{k', k''\}$ , where  $l'$  and  $k'$  correspond to the identity element in  $\mathbb{Z}_2$ .

The map  $i_{k,l} : \mathcal{OR}(f^l) \rightarrow \mathcal{OR}(f^k)$  has the following form (cf. [8])

$$i_{k,l}(l') = k', \tag{3.1}$$

$$i_{k,l}(l'') = \begin{cases} k'' & \text{if } \frac{k}{l} \text{ is odd,} \\ k' & \text{if } \frac{k}{l} \text{ is even.} \end{cases} \tag{3.2}$$

**Definition 3.3.** Let us consider the natural number  $r$  and the set  $\bigcup_{k|r} \mathcal{OR}(f^k) = \bigcup_{k|r} \{k', k''\}$ . In this set we introduce the partial order “ $\preceq$ ” in the following way:  $l^* \preceq k^*$ , where  $l^* \in \{l', l''\}$ ,  $k^* \in \{k', k''\}$  if and only if

- $l \mid k$ ,
- $i_{k,l} : \{l', l''\} \rightarrow \{k', k''\}$  maps  $l^*$  on  $k^*$ .

If  $l^* \preceq k^*$  then we say that  $l^*$  is preceding  $k^*$ . We use the notation  $l^* < k^*$  if  $l^* \preceq k^*$  but  $l^* \neq k^*$ .

Now we can give the definition of the Reidemeister graph for  $f$ , a self-map of a manifold  $M$  which satisfies our Standing Assumptions.

**Definition 3.4.** Letting  $r$  be fixed, the partially ordered set of Reidemeister orbits  $\bigcup_{k|r} \{k', k''\}$  can be perceived as a directed graph (and denoted by  $\mathcal{GOR}(f; r)$ ). There is an edge from vertex  $l^*$  to  $k^*$  if and only if  $l^* \preceq k^*$ , with the convention that if  $l^* < k^* < s^*$  then we omit the edge from  $l^*$  to  $s^*$  (understanding that there is the connection between these two vertices through  $k^*$ ).

#### 4. $NJD_r[f]$ for a self-map $f$ of $M$ satisfying Standing Assumptions

In this section we give the definition of  $NJD_r[f]$  for a self-map  $f$  of  $M$  satisfying our Standing Assumptions 3.2.

##### 4.1. Index function

We define an index function by the formula:  $I(n^*) = \text{ind}(f^n, n^*)$ . In this way we obtain a function  $I$  defined on the set of the vertices of the graph  $\mathcal{GOR}(f; r)$ .

Let us recall that  $\mathbb{R}P^m$  is a Jiang space for odd  $m$ , so both Nielsen classes of the given self-map of  $\mathbb{R}P^m$  have equal indices [20]. As  $L(f^n) = 1 - \beta^n = I(n') + I(n'')$  we get that

$$I(n') = I(n'') = \frac{1 - \beta^n}{2}, \quad (4.1)$$

where  $\beta$  is the degree of the map  $f$ .

Now we generalize the notion of periodic expansion onto the maps of  $\mathcal{GOR}(f; r)$ .

**Definition 4.1.** For each vertex  $l^*$ , where  $l^* \in \{l', l''\}$ , we define basic integer-valued function on the graph:

$$\text{Reg}_{l^*}(n^*) = \begin{cases} l & \text{if } l^* \preceq n^*, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 4.2.**  $\text{Reg}_{3'}(6') = 3$ ,  $\text{Reg}_{3'}(6'') = 0$ .

Function  $I$  can be uniquely represented as an integral combination of basic functions  $\text{Reg}_{l^*}$  (so-called generalized periodic expansion) [8].

$$I(n^*) = \sum_{l^* \preceq n^*} a_{l^*} \text{Reg}_{l^*}(n^*). \quad (4.2)$$

##### 4.2. Attaching sequences at vertices

Let  $\Gamma$  be one of the sequences (A)–(F) given in Theorem 2.9. It is represented as a combination of  $\text{reg}$ 's:  $\Gamma = \sum_{d \in \mathbb{O}} a_d \text{reg}_d$ . We will say that we attach  $\Gamma$  at the vertex  $l^*$  if we define the following function  $\Gamma_{l^*}$  on the Reidemeister graph:

$$\Gamma_{l^*}(n^*) = \sum_{l^* \preceq (dl)^*, d \in \mathbb{O}} a_d \text{Reg}_{(dl)^*}(n^*). \quad (4.3)$$

**Definition 4.3.** We will say that a sequence  $\Gamma$ , of one of the types (A)–(F), attached at the vertex  $l^*$  realizes  $a_{k^*} \text{Reg}_{k^*}$  (or  $a_{k^*}$  for short) if this expression appears in the right-hand side of the formula (4.3).

##### 4.3. Definition of $NJD_r[f]$

The index function  $I$  can be expressed as a sum of the sequences (A)–(F) attached at some vertices:

$$I(n^*) = \sum_{l^* \preceq n^*} a_{l^*} \text{Reg}_{l^*}(n^*) = \Gamma_{l_1^*}^1(n^*) + \cdots + \Gamma_{l_s^*}^s(n^*). \quad (4.4)$$

Each such decomposition determines the sum  $l_1 + \cdots + l_s$ , which we call *the decomposition number*.

**Definition 4.4.**  $NJD_r[f]$  is defined as the minimal decomposition number under all possible decompositions.

**Remark 4.5.** In [8] we described more general construction of the invariant  $NJD_r[f]$  for any self-map of a closed smooth connected manifold of dimension at least 3. In general case one must take into account that:

- the sequences are attached at the vertices of the Reidemeister graph of  $f$  but  $\mathcal{GOR}(f; r)$  may be more complex than the one described by relations in Definition 3.3.
- Index function  $I$  may take much more complicated form than (4.1).

In any case, the following theorem holds:

**Theorem 4.6.** ([8]) For any self-map of a closed smooth connected manifold of dimension greater than 3 and a fixed integer  $r \in \mathbb{N}$  there is:

$$NJD_r^m[f] = MF_r^{\text{diff}}(f).$$

The geometrical interpretation of  $NJD_r^m[f]$  the reader may find in [9, Section 4.4].

**Remark 4.7.** The aim of the paper is to minimize the set  $\text{Fix}(f^r)$  in the smooth homotopy class (for a given  $r \in \mathbb{N}$ ). This leads also to the question about the possible orbital structure of such minimal periodic sets. For example, we know that in dimension  $m \geq 4$  and for  $f$  being a self-map of a simply-connected manifold,  $\text{Fix}(f^r)$  may consist only of fixed points ([4], cf. also Remark 2.7). When, in the simply-connected case, could  $\text{Fix}(f^r)$  contain longer orbits? By Remark 2.4 the promising way to answer such a question is to analyze the decomposition of the sequence of Lefschetz numbers  $\{L(f^k)\}_{k|r}$  into  $DD^m(p|r)$  sequences. In general (non-simply-connected) case one needs to follow such an analysis on Reidemeister graph.

**Remark 4.8.** Let us recall that for connected simply-connected closed smooth manifold  $M$  the invariants  $D_r^m[f]$  and  $NJD_r^m[f]$  coincide, since then for each iteration there is a single Nielsen class.

## 5. Estimation of $NJD_r^m[f]$ for maps satisfying Standing Assumptions

For the rest of the paper we assume that  $f$  is a map satisfying our Standing Assumptions 3.2.

The computations of the invariant  $D_r^m[f]$  in [4,5] (and  $NJD_r^m[f]$  in [9]) show that the most troublesome coefficient is the one standing at  $\text{reg}_1$ . This is because in some forms of sequences of indices listed in Theorem 2.9 the coefficient at  $\text{reg}_1$  is not arbitrary, but belongs to the set  $\{-1, 0, 1\}$ . As a consequence, in some situations one can represent the term  $a_1 \text{reg}_1$  (in the minimal realization, cf. Definition 2.3) as a sum of the other  $DD^m(1|r)$  sequences that appear in the minimal realization. However, it is not easy to describe all these situations. To avoid this difficulty we introduce the invariant  $D_r^m[f] \bmod \text{reg}_1$ . The computation of the last invariant is much simpler and gives the approximate value of  $D_r^m[f]$  [4], namely:

$$D_r^m[f] \bmod \text{reg}_1 = D_r^m[f] \text{ or } D_r^m[f] - 1.$$

**Definition 5.1.** By  $(D_r^m[f] \bmod \text{reg}_1)$  we denote the number of sequences in the minimal decomposition of  $L(f^n) = \sum_{k|r} b_k \text{reg}_k(n)$  into  $DD^m(1|r)$  sequences modulo  $\text{reg}_1$  i.e. we require only that the equality (2.1) holds for all divisors  $n|r$  different than 1 (thus we ignore the coefficient at  $\text{reg}_1$ ).

**Remark 5.2.** In the simply-connected case  $NJD_r[f] = D_r[f]$ , and  $D_r[f]$  may be expressed in the language of the Reidemeister graph in the following way. If  $M$  is simply connected then  $\mathcal{GOR}(f; r)$  constitutes the graph of all divisors of  $r$  and the procedure of calculating  $D_r[f]$  described in Definition 2.3 can be equivalently expressed as finding minimal number of  $DD^m(1)$  sequences attached at 1 realizing in sum  $\{L(f^n)\}_{n|r}$ .

**Remark 5.3.** Let us consider a self-map  $f : M \rightarrow M$  of a connected simply-connected closed manifold  $M$ , satisfying the condition:

(\*) all coefficients  $b_k$  for  $k \neq 1$  in the periodic expansion of  $L(f^n) = \sum_{k|r} b_k \text{reg}_k(n)$  are nonzero.

We will denote the family of such maps by  $\mathcal{B}$ . It was proved in [4] that, for a given dimension  $m$ ,  $D_r^m[f] \bmod \text{reg}_1$  has the common value for all maps  $f$  in  $\mathcal{B}$ . Let  $P$  be an odd natural number, we will denote for short this common value of  $(D_r^m[f] \bmod \text{reg}_1)$  for  $r = P$  by  $h_P$ , assuming that the dimension  $m \geq 4$  is fixed.

**Remark 5.4.** Let us mention that the algorithm of determining  $h_P$  was described in [4] and successfully applied for calculating  $h_P$  in the case  $P$  is a product of different odd primes. Namely, let the dimension of the manifold be equal to  $m$  ( $m = 2s$  or  $m = 2s + 1$ ) and  $P$  be a product of  $v$  different odd primes, where  $v \geq s$ . We represent  $v$  in the form  $v = k \cdot s + R$  where  $R = 1, \dots, s$  and  $k \in \mathbb{Z}$ . Then

$$h_P = \frac{2^{sk+R} - 2^R}{2^s - 1} + 1. \quad (5.1)$$

Our aim is to estimate  $NJD_r[f]$ , where  $r = P \cdot 2^R$  with  $P$  odd, by  $h_P$ . The main idea of finding the useful estimation is based on the decomposition of  $\mathcal{GOR}(f; r)$  into parts, each of which is isomorphic to the graph of all divisors of the odd number  $P$ , and observing that each such part gives the contribution to  $NJD_r[f]$  equal to  $h_P$ .

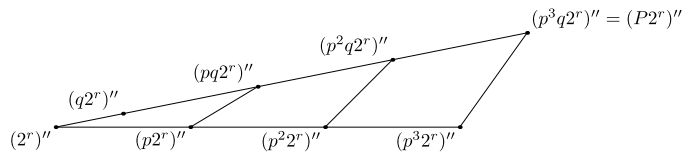


Fig. 1.  $\mathcal{GOR}(2^s P)''$  for  $P = p^3 q$ ;  $p, q$  primes.

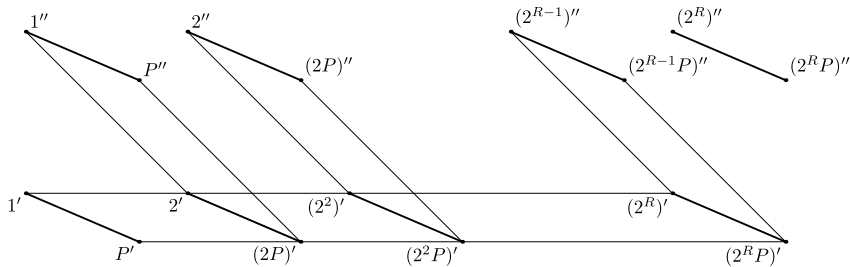


Fig. 2.  $\mathcal{GOR}(f; r)$  for self-maps of  $\mathbb{R}P^m$ .

Now, let  $f$  be a map satisfying Standing Assumptions and  $\mathcal{GOR}(f; r)$  be the Reidemeister graph for  $f, r = P \cdot 2^R, P$  is odd. For a fixed  $0 \leq s \leq R$  we consider a part of this graph, defined as:

$$\mathcal{GOR}(2^s P)'' = \{(2^s k)'' : k \mid P\}.$$

**Lemma 5.5.** *In order to realize all coefficients of  $\mathcal{GOR}(2^s P)''$ , maybe except for the coefficient at the vertex  $(2^s)''$ , one needs  $h_P \cdot DD^m(1)$  sequences attached at  $(2^s)''$ . They give the contribution  $h_P \cdot 2^s$  to  $NJD_r[f]$ .*

**Proof.** The equality

$$\mathcal{GOR}(2^s P)'' = \{(2^s k)'' : k \mid P\}, \tag{5.2}$$

shows that  $\mathcal{GOR}(2^s P)''$  is isomorphic to the graph of all divisors of  $P$ . Thus, by Remark 5.2, to realize (in the sense of Definition 4.3) all coefficients of  $\mathcal{GOR}(2^s P)''$  modulo  $a_{(2^s)''}$  it is enough to attach  $(D_P^m[f] \bmod \text{reg}_1) \cdot DD^m(1)$  sequences at  $(2^s)''$ . Furthermore, due to item (3) of our Standing Assumptions the condition (\*) of Remark 5.3 is satisfied for the isomorphic graph of all divisors of  $P$ . As a consequence,  $(D_P^m[f] \bmod \text{reg}_1)$  does not depend on  $f$  and is equal  $h_P$ . This ends the proof.  $\square$

The example of  $\mathcal{GOR}(2^s P)''$  for  $P = p^3 q$ , where  $p$  and  $q$  are primes, is given in Fig. 1. We are now in a position to formulate the main result of the paper.

**Theorem 5.6.** *Let  $f$  be a map satisfying our Standing Assumptions i.e.*

- (1)  $f : M \rightarrow M$  is a self-map of a smooth closed connected manifold of dimension  $\geq 4$  and  $r$  is a given natural number of the form  $r = 2^P \cdot R$ , where  $P$  is odd,
- (2)  $\pi_1 M = \mathbb{Z}_2, f_{\#} = \text{id}$ ,
- (3) all coefficients  $a_{l^*}$  in the general periodic expansion of index function (4.2) (i.e. the coefficients in the Reidemeister graph standing at  $\text{Reg}_{l^*}$ ) are nonzero.

Then:

$$2^{R+1} \cdot h_P \leq NJD_r[f] \leq 2^{R+1} \cdot (h_P + 1).$$

**Proof.** In Fig. 2 a symbolic representation of  $\mathcal{GOR}(f; r)$  is given. This graph can be interpreted literally in the case  $P$  is a prime number. In the general case the aslope lines (in bold), joining  $(2^s)^*$  and  $(2^s P)^*$ , represent a graph isomorphic to a graph of all divisors of an odd number  $P$ . This means that on the bold edges there are other vertices, which are not specified. For example, the line joining  $(2^s)^*$  and  $(2^s P)^*$  for  $P = p^3 q$ , where  $p, q$  are primes, is in fact the graph given in Fig. 1.

Now, we fix  $0 \leq s \leq R$ . Then, by Lemma 5.5, to realize all the coefficients at vertices  $\{(l \cdot 2^s)'' : l \mid P, l \neq 1\}$  one needs  $h_P$  sequences attached at  $(2^s)''$ , which gives the contribution to  $NJD_r[f]$  equal to  $2^s \cdot h_P$ .

Similarly, to realize the coefficients at  $\{l': l \mid P, l \neq 1\}$  one needs  $h_P$  sequences attached at  $1'$ . In sum this gives the following estimates of  $NJD_r[f]$  from below:

$$\begin{aligned} NJD_r[f] &\geq \sum_{s=0}^R h_P \cdot 2^s + h_P \\ &= h_P \left[ 1 + \sum_{s=0}^R 2^s \right] = h_P \cdot 2^{R+1}. \end{aligned} \quad (5.3)$$

Now we give an upper estimate. Let us notice that realizing the graph  $\mathcal{GOR}(2^s P)''$  we can also realize the graph  $\mathcal{GOR}(2^{s+1} P)' = \{(2^{s+1} \cdot l)'\} : l \mid P\}$  for  $s = 0, \dots, R-1$ , modulo the coefficient at  $(2^{s+1})'$ . In fact, let  $c(n)$  be a sequence attached at  $(2^s)''$  realizing the coefficient at  $(2^s l)''$ , where  $l \mid P$ . Then, as  $P$  is odd,  $c(n)$  is of one of the types  $(B^0)$ ,  $(C^0)$  or  $(D^0)$  of Theorem 2.9. We can change it for the sequence  $c'(n)$  of the type  $(E^0)$  or  $(F^0)$  of Theorem 2.9 realizing also  $(2^{s+1} l)'$  for  $0 \leq s \leq R-1$ . This is possible, since the dimension of the manifold  $M$  is odd by the assumption.

As the result, the only coefficients which may still remain unrealized are  $\{(2^s)^* : 0 \leq s \leq R\}$ .

The vertices  $1'', 2'', \dots, (2^R)''$  are irreducible, so to realize the coefficients at these vertices it is necessary (and sufficient) to attach a single sequence at each of them, which gives the contribution to  $NJD_r[f]$  equal to:  $1 + 2 + \dots + 2^R = 2^{R+1} - 1$ . Furthermore, if we use for that purpose the sequences of the type  $(A)$ , then they realize also the coefficients at the vertices  $2', \dots, (2^R)'$ . The remaining coefficient at  $1'$  can be realized by one sequence of the type  $(A)$  attached at this vertex. Finally, summing the contributions of the three parts of  $\mathcal{GOR}(f; r)$  considered above, we obtain:

$$NJD_r[f] \leq h_P \cdot 2^{R+1} + (2^{R+1} - 1) + 1, \quad (5.4)$$

which ends the proof.  $\square$

**Remark 5.7.** For a self-map  $f : \mathbb{R}P^m \rightarrow \mathbb{R}P^m$ , where  $m$  is even  $\mathcal{GOR}(f; r)$  is the same as in the case of  $m$  odd. However, the forms of local fixed point indices of iterations in even dimensions are much different (cf. [10]). This makes it impossible to apply the same trick which allowed us, in the proof of the inequality (5.3), to realize both  $\mathcal{GOR}(2^s P)''$  and  $\mathcal{GOR}(2^{s+1} P)'$  by one sequence, and consequently it is much more difficult to find the reasonable estimate for  $NJD_r[f]$  from above in the case of even  $m$ .

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