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Selections and approximations of convex-valued equivariant mappings

Zdzisław Dzedzej* and Wojciech Kryszewski†

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Abstract

We present some abstract theorems on the existence of selections and graph-approximations of set-valued mappings with convex values in the equivariant setting, i.e., maps commuting with the action of a compact group. Some known results of the Michael, Browder and Cellina type are generalized to this context. The equivariant measurable as well as Carathéodory selection/approximation problems are also studied.

1 Introduction

The purpose of this paper is to collect various results concerning the existence of equivariant selections and graph-approximations of equivariant set-valued maps with convex values. Most of them are motivated by possible applications in the fixed point theory and, e.g., the theory of control and differential inclusions. For instance, if one studies topological invariants for set-valued maps by elementary approximation methods (see e.g. [20], [26]), all the symmetry properties of these invariants are immediate consequences of appropriate results for single-valued maps. The classical Borsuk theorem on a mapping degree may serve as an example.

The paper is organized as follows. In section 2 basic definitions and remarks concerning group actions, equivariant maps and properties of vector Haar integral are formulated and some basic examples of equivariant set-valued maps are presented. Section 3 begins with an equivariant version of Michael's selection theorem and some of it's simple consequences. We address also a less trivial problem of the representation of a set-valued by a sequence of single-valued continuous and equivariant maps. Besides, we present some other results generalizing [10] and [14]. In the next section we establish graph-approximation results starting with a version of the classical theorem of A. Cellina [12] and some of its generalizations. In particular, a constrained approximation theorem from [7] finds its equivariant version. Relative approximation theorems, fundamental for the construction of topological invariants as, e.g., a topological degree theory,

*Faculty of Technical Physics and Applied Mathematics, Gdańsk University of Technology, ul. Narutowicza 11/12, 80-233 Gdańsk, Poland; zdzedzej@mif.pg.gda.pl. Research supported by Polish Ministry of Higher Education Grant nr N N 201 394037.

†Faculty of Mathematics and Computer Sciences, Nicholas Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland; wkrysz@mat.umk.pl. Research supported by Polish Ministry of Higher Education Grant nr N N201 395137

are also proved (comp. [34]). The last section is devoted to measurable and Carathéodory maps. A version of the classical Kuratowski-Ryll-Nardzewski theorem is obtained and then several results on Carathéodory selections, representations and approximations are established. In most of these results we deal with an action of a general compact topological group. We decided to concentrate strictly on abstract selection and approximation theorems and to postpone some possible and fruitful applications to subsequent papers.

Notation: In the sequel we consider only *Hausdorff* topological spaces. If X is a topological space and $A \subset X$, then $\overline{A}$ denotes the *closure* of A. If X is a metric space (with the metric denoted by d by default), $A \subset X$ and $\varepsilon > 0$, then $B(A,\varepsilon) := \{x \in X \mid d(x,A) := \inf_{a\in A} d(x,a) < \varepsilon\}$; in particular $B(a,\varepsilon)$ is the open ball with radius $\varepsilon > 0$ centered at $a \in X$; moreover let $D(A,\varepsilon) := \{x \in X \mid d(x,A) \leqslant \varepsilon\}$. In what follows $\mathbb{E}$ denotes a Banach space with the norm $\|\cdot\|$ over reals or complex numbers and $\mathbb{E}^*$ stands for its *topological dual*, i.e., the Banach space of all continuous bounded forms; if $p \in \mathbb{E}^*$ and $x \in \mathbb{E}$, then $\langle p, x\rangle := p(x)$. Note that if $A \subset \mathbb{E}$ and $\varepsilon > 0$, then $B(A,\varepsilon) = A + \varepsilon B(0,1) = A + B(0,\varepsilon)$. The convex (resp. closed convex) hull of $A \subset \mathbb{E}$ is denoted by $\mathrm{conv} A$ (resp. $\overline{\mathrm{conv}} A$).

2 Preliminaries

Let G be a group. Recall that a *G-set* is a pair (X, ξ_X), where X is a set and $\xi_X : G \times X \to X$ is the action of G on X, i.e., a map such that:

(i) $\xi_X(g_1, \xi(g_2, x)) = \xi_X(g_1 g_2, x)$ for $g_1, g_2 \in G$ and $x \in X$;

(ii) $\xi_X(e, x) = x$ for $x \in X$, where $e \in G$ is the group unit.

In the sequel we write gx instead of $\xi_X(g,x)$, $x \in X$, $g \in G$, unless it leads to ambiguity.

Given G-sets X and Y, a map $f : X \to Y$ is *G-equivariant* if $f(gx) = gf(x)$ for any $x \in X$ and $g \in G$. If the G-action on Y is trivial (or neglected), then we say that f is *G-invariant*.

A subset $A \subset X$ of a G-set X is *G-invariant* if $gA := \{gx \mid \ x \in A\} \subset A$ for all $g \in G$. The set $Gx = G(x) := \{gx \mid g \in G\}$ is the *orbit* through $x \in X$ and X/G denotes the set of all orbits. Observe that if $A \subset X$, then the set $GA := \bigcup_{x\in A} Gx = \bigcup_{g\in G} gA$ is G-invariant.

If G is a topological group, X is a topological space and a G-set, then X is a *G-space*, provided the action ξ_X is (jointly) continuous. We say that a real (resp. complex) Banach space $\mathbb{E}$ is a *real* (resp. *complex*) *Banach representation of* G if $\mathbb{E}$ is G-space and, for each $g \in G$, the map $\xi_{\mathbb{E}}(g,\cdot) : \mathbb{E} \ni x \mapsto gx$ is linear and bounded.

Throughout the whole paper we assume that G is a *compact* group.

Remark 2.1 (1) Observe that if $(\mathbb{E}, \|\cdot\|)$ is a Banach representation of G, then for each $g \in G$, $\xi_{\mathbb{E}}(g,\cdot)$ is a (topological) isomorphism and, in view of the Banach-Steinhaus theorem, there is $M > 0$ such that for all $g \in G$ and $x \in \mathbb{E}$, $M^{-1}\|x\| \leqslant \|gx\| \leqslant M\|x\|$.

(2) If $\mathbb{E}$ is a Banach representation of G, then so is the dual $\mathbb{E}^*$. The action of G on $\mathbb{E}^*$ is defined via conjugation: for $g \in G$ and $p \in \mathbb{E}^*$, $\langle gp, x\rangle := \langle p, g^{-1}x\rangle$, $x \in \mathbb{E}$.

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. As G is a compact group, Borel (and, in particular, continuous) $\mathbb{E}$-valued functions may be integrated over G (see [22] or [21]). More precisely, it is well-known that there is the *Haar measure* on G, i.e., a unique normalized regular Borel measure $\widetilde{\chi}$ which is G-*invariant*: for any $B \in \mathscr{B}(G)$, the σ-algebra of Borel sets in G, $g \in G$, $\widetilde{\chi}(gB) = \widetilde{\chi}(B) = \widetilde{\chi}(Bg)$ and $\widetilde{\chi}(G) = 1$. The measure $\widetilde{\chi}$ admits the *completion* (or *Lebesgue extension*), i.e., if $\mathscr{B}_G$ denotes the σ-algebra in G generated by sets of the form $A = B \cup N$, where $B \in \mathscr{B}(G)$ and $N \subset C \in \mathscr{B}(G)$ with $\widetilde{\chi}(C) = 0$, then there is a unique complete measure χ on $\mathscr{B}_G$ such that $\chi(B) = \widetilde{\chi}(B)$ for any $B \in \mathscr{B}(G)$. It is clear that if $A \in \mathscr{B}_G$, then $gA, Ag \in \mathscr{B}_G$ for any $g \in G$ and it is easy to see that χ is G-invariant.

The space $L^1(G, \mathbb{E})$ of Bochner χ-integrable functions $f : G \to \mathbb{E}$ is well-defined; recall that $f \in L^1(G, \mathbb{E})$ if and only if $f : G \to \mathbb{E}$ is strongly χ-measurable and $\|f\|_{L^1(G,\mathbb{E})} := \int_G \|f(g)\| \, d\chi(g) < \infty$. In particular, if $\mathbb{E}$ is separable, then any bounded χ-measurable map is integrable (for then f is strongly χ-measurable in view of the Pettis theorem – see [37]). Note also that any Borel function $f : G \to \mathbb{E}$ is χ-measurable, but not conversely.

In what follows we write dg instead of $d\chi(g)$. Let us collect the most important properties of the Haar integral.

Proposition 2.2 (1) *For any $h \in G$ and $f \in L^1(G, \mathbb{E})$, maps $G \ni g \mapsto f(hg), f(gh)$ are χ-integrable and*

$$\int_G f(hg) \, dg = \int_G f(g) \, dg = \int_G f(gh) \, dg = \int_G f(g^{-1}) \, dg.$$

(2) *The map $L^1(G, \mathbb{E}) \ni f \mapsto \int_G f(g) \, dg$ is continuous and linear.*
(3) *If $f \in L^1(G, \mathbb{E})$ and $f(G) \subset A \subset \mathbb{E}$, then $\int_G f \, dg \in \overline{\mathrm{conv}}(A)$.*
(4) *If $\mathbb{E}$ is a Banach representation of G, then for any $f \in L^1(G, \mathbb{E})$ and $h \in G$,*

$$\int_G hf(g) \, dg = h \int_G f(g) \, dg. \qquad \square$$

Remark 2.3 (1) It is clear that $C(G, \mathbb{E}) \subset L^1(G, \mathbb{E})$, where $C(G, \mathbb{E})$ denotes the space of all continuous maps $f : G \to \mathbb{E}$ endowed with the usual sup-norm $\|f\|_\infty := \sup_{g \in G} \|f(g)\|$, and

$$\left\| \int_G f \, dg \right\| \leqslant \int_G \|f\| \, dg \leqslant \|f\|_\infty$$

for any $f \in C(G, \mathbb{E})$.

(2) Given a topological space X and $f : X \times G \to \mathbb{E}$ such that for χ-almost all $g \in G$, $f(\cdot, g)$ is continuous at $a \in X$, for each $x \in X$, $f(x, \cdot) \in L^1(G, \mathbb{E})$ and there is $k \in L^1(G, \mathbb{R})$ such that $\|f(x, g)\| \leqslant k(g)$ for χ-almost all $g \in G$ and x from a neighborhood of a, then $F : X \to \mathbb{E}$ given for $x \in X$ by $F(x) := \int_G f(x, g) \, dg$ is continuous at a. In particular if $f : X \times G \to \mathbb{E}$ is continuous, then F is continuous.

(3) If X is a metric space, $f(\cdot, g)$, $g \in G$, is locally Lipschitz (i.e., for any $x_0 \in X$ there are a neighborhood V and $L \in L^1(G, \mathbb{R})$ such that $\|f(x, g) - f(y, g)\| \leqslant L(g) d(x, y)$ for $x, y \in V$ and $g \in G$) and $f(x, \cdot) \in L^1(G, \mathbb{E})$ for $x \in X$, then $F = \int_G f(\cdot, g) \, dg$ is locally Lipschitz.

(4) If $\mathbb{E}$ is a Banach representation of G, then one can define a norm $\| \cdot \|_G$ on $\mathbb{E}$ such that

the action of G on $\mathbb{E}$ is *isometric*, i.e., $\|gx\|_G = \|x\|_G$ for all $g \in G$ and $x \in \mathbb{E}$. Indeed it is sufficient to put

$$\|x\|_G := \int_G \|gx\|\, dg, \quad x \in \mathbb{E}.$$

This new norm is complete since it is equivalent to the original norm $\|\cdot\|$ in $\mathbb{E}$: precisely, in view of Remark 2.1 (1),

$$M^{-1}\|x\| \leqslant \|x\|_G \leqslant M\|x\|, \quad x \in \mathbb{E}.$$

(5) Suppose that (X, d) is a metric G-space and let $d_G(x, y) := \int_G d(gx, gy)\, dg$ for $x, y \in X$. This definition is correct and it is easy to see that d_G is a G-invariant metric on X. Moreover metrics d_G and d are equivalent (i.e., they introduce the same topology in X). To see this fix a sequence $(x_n)_{n=1}^\infty$ in X. Assume that $x_n \to x \in X$. It is clear that there is $m > 0$ such that $\sup_{n\in\mathbb{N},\, g\in G} d(gx_n, gx) \leqslant m$. For all $g \in G$, $d(gx_n, gx) \to 0$; hence by the Lebesgue theorem $d_G(x_n, x) = \int_G d(gx_n, gx)\, dg \to 0$. Conversely if $d_G(x_n, x) \to 0$, then $f_n(g) := d(gx_n, gx)$, $g \in G$, converges to 0 in $L^1(G, \mathbb{R})$. Thus there is a subsequence $(f_{n_k})_{k=1}^\infty$ converging almost everywhere on G to 0; but this implies that $x_{n_k} \to x$ and, finally, it shows that any subsequence of (x_n) has a subsequence converging to x, i.e., $x_n \to x$. Unfortunately metrics d and d_G are not, in general, uniformly equivalent and, hence, (X, d_G) may not be complete even if so is (X, d).

Set-valued maps are the main object of our studies. Recall that given sets X and Y, a *set-valued map* φ from X into Y (written $\varphi : X \multimap Y$) is a map that assigns to each $x \in X$ the *value* $\varphi(x)$ being a *nonempty* subset of Y. If X and Y are topological spaces and, for any closed (resp. open) set $U \subset Y$, the *preimage* $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is closed (resp. open), then we say that φ is *upper* (resp. *lower*) *semicontinuous*; φ is *continuous* if it is upper and lower semicontinuous simultaneously.

If Y is a metric space, then $\varphi : X \multimap Y$ is lower semicontinuous if and only if for any $y \in Y$ the function $X \ni x \mapsto d(y, \varphi(x)) := \inf_{z\in\varphi(x)} d(y, z)$ is upper semicontinuous (as a real function) or, equivalently, given $x_0 \in X$ and $y_0 \in \varphi(x_0)$, $\lim_{x\to x_0} d(y_0, \varphi(x)) = 0$.

A similar characterization of upper semicontinuity is not true, i.e., the lower semicontinuity of $d : X \ni x \mapsto d(y, \varphi(x)) \in \mathbb{R}$ does not imply in general that φ is upper semicontinuous. However if φ has closed values, is *locally compact*, i.e., each point $x \in X$ has a neighborhood U such that $\overline{\varphi(U)}$ is compact, and d is lower semicontinuous, then φ is upper semicontinuous (with compact values). The *graph* $\mathrm{Gr}(\varphi) := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ of an upper semicontinuous map φ with closed values is closed; $\varphi : X \multimap Y$ is upper semicontinuous with *compact* values if and only if the projection $\mathrm{Gr}(\varphi) \to X$ is perfect ([1]). We say that a map φ is *compact* if it is upper semicontinuous and the closure of the *image* $\varphi(X) := \bigcup_{x\in X} \varphi(x)$ is compact. For other details on set-valued maps – see [20], [8] or [23].

Remark 2.4 If X is a G-space, then the *orbit map* $\varphi : X \multimap X$ given by $\varphi(x) := Gx$ is continuous with compact values; if X is compact, then φ is compact. In particular, if $A \subset X$ is G-invariant and U is an open neighborhood of A, then there is an open neighborhood V of A

[1]Recall that a continuous map $f : X \to Y$ is perfect if it is closed and $f^{-1}(y)$ is compact for any $y \in Y$.

such that $GV \subset U$. Indeed, the upper semicontinuity of φ implies that $V := \{x \in X \mid \varphi(x) \subset U\} = X \setminus \varphi^{-1}(X \setminus U)$ fulfils the requirements.

In what follows we shall study G-equivariant set-valued maps.

Definition 2.5 Let X and Y be G-sets. A set-valued map $\varphi : X \multimap Y$ is *G-equivariant* (resp. *G-invariant*) if $\varphi(gx) = g\varphi(x)$ (resp. $\varphi(gx) = \varphi(x)$) for all $g \in G$ and $x \in X$.

Note that φ is G-equivariant if and only if $\varphi(gx) \subset g\varphi(x)$ for all $g \in G$ and $x \in X$ (or $g\varphi(x) \subset \varphi(gx)$ for all $g \in G$ and $x \in X$). Moreover it is easy to see that φ is G-equivariant if and only if its graph $\mathrm{Gr}(F)$ is a G-invariant subset of $X \times Y$ with a natural action $g(x, y) := (gx, gy)$, $x \in X$, $y \in Y$. Observe that if $\varphi : X \multimap Y$, where Y is a topological space, is G-equivariant, then so is its *closure*, i.e., the map $\overline{\varphi} : X \multimap Y$ given by $\overline{\varphi}(x) := \overline{\varphi(x)}$, $x \in X$.

Let us collect some simple examples of G-equivariant set-valued maps.

Example 2.6 (1)(A marginal map, see [4]) Let X, Y be two G-spaces, $H : Y \multimap X$ a G-equivariant map with compact values and $W : X \times Y \to \mathbb{R}$ a continuous G-invariant map, i.e., $W(gx, gy) = W(x, y)$ for all $g \in G, x \in X, y \in Y$. Then a *marginal map* $M : Y \multimap X$ defined by

$$M(y) := \{\overline{x} \in H(y) |\ W(\overline{x}, y) = \inf_{x \in H(y)} W(x, y)\}.$$

is considered in many optimization problems. Clearly M is G-equivariant because of the equality $\inf_{x\in H(y)} W(x, y) = \inf_{gx\in H(gy)} W(gx, gy)$.

(2) Let $\varphi : [0, T] \times U \multimap \mathbb{E}$, where $T > 0$ and $U \subset \mathbb{E}$ is an open G-invariant subset of a Banach G-representation $\mathbb{E}$, be G-equivariant, i.e., $\varphi(t, gx) = g\varphi(t, x)$ for $0 \leqslant t \leqslant T$ and $x \in U$. Consider a differential inclusion (under suitable assumptions assuring the existence of solutions):

$$\begin{cases} x'(t) \in \varphi(t, x(t)) \\ x(0) = x_0. \end{cases}$$

Consider the space $C([0, T], \mathbb{E})$ of continuous maps $[0, T] \to \mathbb{E}$ with the G-action defined by $(g, x) \mapsto gx$, where $(gx)(t) := g(x(t))$ for $x \in C([0, T], \mathbb{E})$, $g \in G$ and $t \in [0, T]$. If $x : [0, T] \to \mathbb{E}$ is a solution to the above problem, i.e., there is an integrable $y : [0, T] \to \mathbb{E}$ such that $x(t) = x_0 + \int_0^t y(s)\, ds$, $t \in [0, T]$, then gx is also a solution to this problem with the initial condition gx_0. Therefore, the solution map $P : U \multimap C([0, T], \mathbb{E})$, that assigns to each initial value $x_0 \in U$ the set of all solutions, is G-equivariant, whenever well-defined.

(3) Let $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$, where $\mathbb{E}$ is a real Banach representation of G, be a convex function. For each $x_0 \in dom(f) := \{x \in \mathbb{E} \mid f(x) < \infty\}$ the subdifferential

$$\partial f(x_0) := \{p \in \mathbb{E}^* \mid f(x) \geqslant f(x_0) + \langle p, x - x_0\rangle \text{ for all } x \in \mathbb{E}\}$$

is defined. Let f be G-invariant. Then the map $\partial f : dom(f) \multimap \mathbb{E}^*$ is G-equivariant. Clearly $dom(f)$ is invariant and if $p \in \partial f(x_0)$, then for all $x \in \mathbb{E}$, $\langle p, x - x_0\rangle \leqslant f(x) - f(x_0)$. Hence

$$\langle gp, x - gx_0\rangle = \langle p, g^{-1}x - x_0\rangle \leqslant f(g^{-1}x) - f(x_0) = f(x) - f(gx_0),$$

which gives the assertion. In a similar manner one shows that the Clarke generalized gradient $\partial f : U \to \mathbb{E}^*$, where U is a G-invariant open in $\mathbb{E}$ and $f : U \to \mathbb{R}$ is a G-invariant locally Lipschitz function, is G-equivariant.

(5) Let $\mathbb{E}$ be a Banach representation of G and $K \subset \mathbb{E}$ be closed G-invariant. For $x \in K$ the Bouligand cone is defined

$$T_K(x) := \limsup_{h \downarrow 0} \frac{K - x}{h}.$$

Then $v \in T_K(x) \iff gv \in T_K(g(x))$. Therefore the map $T_K : K \multimap \mathbb{E}$ is G-equivariant. The same is true for other types of 'tangent' or 'normal' cones, e.g. the Clarke tangent or normal cones:

$$C_K(x) := \liminf_{y \xrightarrow{K} x, h \to 0^+} \frac{K - y}{h}; \quad N_K(x) := \{p \in \mathbb{E}^* \mid \langle p, v \rangle \leqslant 0 \text{ for all } v \in C_K(x)\}, \quad x \in K.$$

(6) Let $\mathbb{E}$ be an isometric Banach representation of G, K be a closed convex G-invariant subset of $\mathbb{E}$ and let $\varepsilon : \mathbb{E} \to \mathbb{R}$ be continuous G-invariant and such that $\varepsilon(x) > d(x, K)$ for $x \notin K$. It is easy to show that $\varphi : \mathbb{E} \multimap \mathbb{E}$, given by $\varphi(x) := D(x, \varepsilon(x)) \cap K$ for $x \in \mathbb{E}$, is lower semicontinuous, G-equivariant and has closed convex values.

3 Equivariant selections and ε-selections

Recall that given sets X and Y, a map $f : X \to Y$ is a *selection* of a set-valued $\varphi : X \multimap Y$ if $f(x) \in \varphi(x)$ for each $x \in X$. It is clear that the axiom of choice implies the existence of selections.

Remark 3.1 (1) The equivariant situation is not that obvious. Let X and Y be G-sets and $\varphi : X \multimap Y$ be G-equivariant. If $f : X \to Y$ is a G-equivariant selection of φ and $x \in X$, then $G_x \subset G_{f(x)}$, where $G_z := \{g \in G \mid gz = z\}$ is the *stabilizer* of $z \in X$, and therefore $f(x) \in \varphi(x) \cap Y^{G_x}$, where $Y^{G_x} := \{y \in Y \mid G_x \subset G_y\}$. On the other hand if $\varphi(x) \cap Y^{G_x} \neq \emptyset$ for all $x \in X$, then there is a G-equivariant selection f of φ. To see this it is sufficient to define f on any orbit $\mathcal{O} \in X/G$. Choose $x_0 \in \mathcal{O}$ and $y_0 \in \varphi(x_0) \cap Y^{G_{x_0}}$; if $g \in G$, we put $f(gx_0) := gy_0$. Then $f : \mathcal{O} \to Y$ is correctly defined and G-equivariant selection of φ restricted to $\mathcal{O}$.

(2) If $\mathbb{E}$ is a Banach representation of G, X is a G-set and $\varphi : X \multimap \mathbb{E}$ is G-equivariant with *convex closed* values, then $\varphi(x) \cap \mathbb{E}^{G_x} \neq \emptyset$ for all $x \in X$. Indeed, take $x \in X$, $y \in \varphi(x)$ and let $z := \int_{G_x} gy \, d\chi_x(g)$, where χ_x is the Haar measure on the (compact) group G_x. For any $g \in G_x$, $gy \in \varphi(gx) = \varphi(x)$; hence $z \in \varphi(x)$ in view of Proposition 2.2 (3). On the other hand, in view of Proposition 2.2 (4), (1), for all $h \in G_x$, $hz = \int_{G_x} hgy \, dg = z$, i.e., $G_x \subset G_z$ and $z \in \varphi(x) \cap \mathbb{E}^{G_x}$. This explains why the problem of the existence of selections of convex-valued maps is well-posed without any additional assumptions.

Here we shall study the existence of *continuous* selections. Suppose that X is a G-space, $\mathbb{E}$ is a Banach representation of G and let $\varphi : X \multimap \mathbb{E}$ be a G-equivariant set-valued map. Given

$f : X \to \mathbb{E}$ define a *symmetrization* of f by

$$F(x) := \int_G g^{-1} f(gx)\, dg = \int_G g f(g^{-1}x)\, dg, \quad x \in X, \tag{1}$$

provided that for each $x \in X$ the integral exists, i.e., the function $G \ni g \mapsto g^{-1}f(gx)$ is integrable. Then, in view of Proposition 2.2 (1), (4), it is easy to see that F is G-equivariant. If f is a selection of a G-equivariant set-valued $\varphi : X \multimap \mathbb{E}$ having closed convex values, then in view of Proposition 2.2 (3) for all $g \in G$ and $x \in X$, $g^{-1}f(gx) \in g^{-1}\varphi(gx) = \varphi(x)$; hence, $F(x) \in \overline{\operatorname{conv}}\{g^{-1}f(gx);\ g \in G\} \subset \varphi(x)$.

If, for example, X is a G-space and f is continuous, then the map $X \times G \ni (x, g) \mapsto g^{-1}f(gx) \in \mathbb{E}$ is continuous and F is well-defined and continuous by Remark 2.3 (2). Therefore we have the following equivariant version of the celebrated Michael theorem [31].

Theorem 3.2 (comp. [1]) *Let X be a paracompact G-space, $\mathbb{E}$ a Banach G-representation and let $\varphi : X \multimap \mathbb{E}$ be a G-equivariant (resp. G-invariant) lower semicontinuous set-valued map with closed convex values. Then φ admits a G-equivariant (resp. G-invariant) continuous selection.*

Proof: In view of the Michael theorem [31], φ has a continuous selection $f : X \to \mathbb{E}$; therefore F, given by (1), is a continuous equivariant selection of φ. □

Let us derive some simple and immediate but useful consequences of Theorem 3.2.

Corollary 3.3 *Any partial continuous G-equivariant selection of φ may be extended to a G-equivariant selection. Precisely, under the assumptions of Theorem 3.2, given a closed G-invariant set $A \subset X$ and a continuous G-map $f : A \to \mathbb{E}$ such that $f(x) \in \varphi(x)$ for $x \in A$, there is a continuous G-equivariant selection F of φ such that $F|_A = f$.*

Proof: It is sufficient to take F as a continuous G-equivariant selection of a lower semicontinuous and G-equivariant set-valued map $\varphi_A : X \multimap \mathbb{E}$ defined for $x \in X$ by

$$\varphi_A(x) := \begin{cases} f(x) & \text{if} \quad x \in A; \\ \varphi(x) & \text{if} \quad x \notin A. \end{cases}$$ □

In a similar manner one can prove the following version of the Tietze-Gleason theorem (the equivariant counterpart of the Tietze-Dugundji extension theorem).

Corollary 3.4 *If X is a paracompact space and $\mathbb{E}$ is a Banach representation of G, then any continuous G-map $f : A \to \mathbb{E}$ admits a continuous G-equivariant extension over X, i.e., there is a G-map $F : X \to \mathbb{E}$ such that $F|_A = f$.*

Proof: It is sufficient to take a continuous G-equivariant selection F of the lower semicontinuous G-equivariant set-valued map $\varphi : X \multimap \mathbb{E}$ with closed convex values defined for $x \in X$ by

$$\varphi(x) := \begin{cases} f(x) & \text{if} \quad x \in A; \\ \mathbb{E} & \text{if} \quad x \notin A. \end{cases}$$ □

The following geometric consequence of Theorem 3.2 might be also very useful for applications (comp. [25], Theorem 1.3).

Corollary 3.5 *Let $\mathbb{E}$ be an isometric Banach representation of G, K a closed convex G-invariant subset in $\mathbb{E}$. Then, for any $\varepsilon > 0$ there is a G-equivariant 'almost orthogonal' retraction $r : \mathbb{E} \to K$, i.e., such that $\|r(x) - x\| \leqslant (1+\varepsilon)d(x, K)$ for $x \in \mathbb{E}$.*

Proof: Let $\varphi(x) := D(x, (1+\varepsilon)d(x, K)) \cap K$ for $x \in K$. In view of Example 2.6 (6), $\varphi : \mathbb{E} \multimap \mathbb{E}$ is lower semicontinuous, G-equivariant with closed convex values. Taking $r : \mathbb{E} \to \mathbb{E}$ as its continuous G-equivariant selection we see that $r(x) \in K$ for $x \in \mathbb{E}$ and $r(x) = x$ if $x \in K$. $\square$

An equivariant version of Bartle-Graves [6] is also easy.

Corollary 3.6 *Let $\mathbb{E}, \mathbb{F}$ be Banach representations of G and $L : \mathbb{E} \to \mathbb{F}$ a linear bounded surjective and G-equivariant operator. Then there exists a continuous G-equivariant map $f : \mathbb{F} \to \mathbb{E}$ such that $L(f(x)) = x$ for $x \in \mathbb{F}$ and $f(0) = 0$.*

Proof: Consider a map $\varphi : \mathbb{F} \multimap \mathbb{E}$ given by $\varphi(x) = L^{-1}(x)$. It has convex closed values and by the Banach theorem for every open $U \subset \mathbb{E}$ $\varphi^{-1}(U) = L(U)$ is open in $\mathbb{F}$. Thus φ is lower semicontinuous and it is obviously G-equivariant. By Theorem 3.2, φ admits a G-equivariant continuous selection $h : \mathbb{F} \to \mathbb{E}$. Then $f(x) = g(x) - g(0)$ is the desired G-equivariant map. $\square$

It is well-known that if X is paracompact, $\mathbb{E}$ is a Banach space and $\varphi : X \multimap \mathbb{E}$ is lower semicontinuous with closed convex values, $x \in X$ and $y \in \varphi(x)$, then there is a continuous selection $f : X \to \mathbb{E}$ of φ such that $f(x) = y$. In the equivariant case it is not necessarily so. However we have the following result.

Proposition 3.7 *Under the assumptions of Theorem 3.2, if $x \in X$ and $y \in \varphi(x) \cap \mathbb{E}^{G_x}$, then there is a continuous G-equivariant selection of φ such that $f(x) = y$.*

Proof: Consider the G-equivariant homeomorphism $\alpha_x : G/G_x \to G(x)$ that assigns to a coset gG_x, $g \in G$, the point $gx \in G(x)$ (see [9, Proposition I.4.1] or [15, Proposition I.3.19 (iii)]). Since $G_x \subset G_y$, there is a continuous G-equivariant map $k : G/G_x \to G/G_y$ given by $k(gG_x) := gG_y$, $g \in G$. Finally take $\beta_y : G/G_y \to G(y)$ (defined as α above) and let $\widetilde{f} : G(x) \to G(y)$ will be given by $\widetilde{f}(z) = \beta_y \circ k \circ \alpha_x^{-1}(z)$ for $z \in G(x)$. Then $\widetilde{f}$ is continuous, G-equivariant and $\widetilde{f}(x) = y$; hence $\widetilde{f}$ is a selection of $\varphi|_A$, where $A = G(x)$ is G-invariant. In view of Corollary 3.3, there is a continuous G-equivariant extension f of $\widetilde{f}$ being the required selection of φ. $\square$

Remark 3.8 (1) Assume that X is a paracompact perfectly normal space (i.e., each open set is F_σ), $\mathbb{E}$ is a separable Banach space and let $\varphi : X \multimap \mathbb{E}$ be lower semicontinuous with closed convex values. As observed by Michael in [31, Lemma 5.2] (see also [23, Proposition 1.4.9]), there is a sequence $f_n : X \to \mathbb{E}$ of continuous maps such that

$$\varphi(x) = \overline{\{f_n(x)\}_{n=1}^{\infty}} \text{ for any } x \in X. \tag{2}$$

In general, there is no equivariant counterpart of this result. Suppose that X is additionally a G-space, $\mathbb{E}$ is a Banach representation of G, φ is G-equivariant and a sequence $\{f_n\}$, consisting of G-equivariant maps, satisfying (2) exists and take $x \in X$. Then, for any $n \geqslant 1$, the stabilizer $G_x := \{g \in G \mid gx = x\}$ of x is contained in the stabilizer $G_{f_n(x)}$. Therefore $f_n(x) \in \mathbb{E}^{G_x}$ and, hence,

$$(3) \qquad \varphi(x) \subset \mathbb{E}^{G_x} := \{y \in \mathbb{E} \mid G_x \subset G_y\},$$

since $\mathbb{E}^{G_x}$ is closed. For example if $\varphi : \mathbb{R} \multimap \mathbb{R}$ is given by $\varphi(x) := [-1, 1]$, $x \in X$, and on $\mathbb{R}$ we consider the usual antipodal action of $G = \mathbb{Z}_2$, then φ is $\mathbb{Z}_2$-equivariant but it does not have the discussed property since $\varphi(x) \not\subset \mathbb{R}^{G_x}$ for $x = 0$.

(2) If X is a G-set, Y is a separable metric G-space, $\varphi : X \multimap Y$ is G-equivariant with closed values and $\varphi(x) \subset Y^{G_x}$, then there is a sequence of G-equivariant maps $f_n : X \to Y$ such that $\varphi(x) = \overline{\{f_n(x)\}_{n=1}^{\infty}}$ on X. Indeed, again it is enough to construct f_n on each orbit $\mathcal{O} \in X/G$. Take $x_0 \in \mathcal{O}$ and choose a dense set $\{y_n\}_{n=1}^{\infty}$ in $\varphi(x_0)$ (observe that $\varphi(x_0)$, as a closed subspace of the separable metric space Y, is separable itself). As in Remark 3.1 (1) we construct a selection $f_n : \mathcal{O} \to Y$ such that $f_n(x_0) = y_n$. It is clear that f_n, $n \in \mathbb{N}$, satisfies the required property.

It seems that condition (3) is not only necessary but also sufficient in the continuous situation; however the authors do not know the proof of this general statement. Nevertheless under some additional conditions the following holds.

Theorem 3.9 *Assume that G is a compact Lie group, X is a separable metric G-space, $\mathbb{E}$ is a separable Banach representation of G and $\varphi : X \multimap \mathbb{E}$ is a G-equivariant lower semicontinuous map with closed convex values such that, for each $x \in X$, $\varphi(x) \subset \mathbb{E}^{G_x}$. Then there is a sequence of continuous G-equivariant maps $f_n : X \to \mathbb{E}$, $n \in \mathbb{N}$, such that for all $x \in X$, $\varphi(x) = \overline{\{f_n(x)\}_{n=1}^{\infty}}$.*

Before we enter the proof let us recall the celebrated Mostow theorem (see e.g. [9, Theorem II.5.4]): *If G is a compact Lie group, X is a completely regular G-space, then given $x \in X$ there is a G-tube around the orbit $G(x)$, i.e., there is a pair (T, r), where T is an open G-invariant neighborhood of $G(x)$ and $r : T \to G(x)$ is a continuous G-equivariant retraction.*

Proof: We shall proceed in three steps.

Claim 1. Let $\Phi(x) := \mathbb{E}^{G_x}$, $x \in X$. Then $\Phi : X \multimap \mathbb{E}$ is G-equivariant lower semicontinuous with closed convex values.

Indeed, for any $x \in X$, $\Phi(x) = \bigcap_{g \in G_x} \ker(g - I)$ (here, given $g \in G$, the map $\mathbb{E} \ni y \mapsto gy \in \mathbb{E}$ is a linear bounded operator and I stands for the identity on $\mathbb{E}$). Hence Φ has closed convex values. To show that Φ is lower semicontinuous take a closed set $C \subset \mathbb{E}$ and let $D := \{x \in X \mid \Phi(x) \subset C\}$. In order to show that D is closed take a sequence $x_n \in D$, $x_n \to x$ and let $y \in \Phi(x)$. Take a tube (T, r) around $G(x)$. Without loss of generality we may assume that $x_n \in T$ for all $n \in \mathbb{N}$. Therefore $r(x_n) \in G(x)$ and $G_{x_n} \subset G_{r(x_n)}$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, $r(x_n) = g_n x$, where $g_n \in G$, and $G_{r(x_n)} = G_{g_n x} = g_n G_x g_n^{-1}$. The compactness of G implies that, passing to a subsequence if necessary, $g_n \to g \in G$ as $n \to \infty$ (recall that G, as a compact Lie group, is metrizable). Hence $g_n x \to gx$; on the other hand $g_n x = r(x_n) \to r(x) = x$, i.e., $g \in G_x \subset G_y$. Let $y_n := g_n y$; then $y_n \to gy = y$. Now $G_{x_n} \subset G_{r(x_n)} = g_n G_x g_n^{-1} \subset g_n G_y g_n^{-1} = G_{g_n y} = G_{y_n}$. Thus $y_n \in \Phi(x_n) \subset C$ and $y \in C$, so $x \in D$.

Claim 2. There is a countable family $\{f_\alpha : X \to \mathbb{E}\}_{\alpha \in A}$ of continuous G-equivariant maps such that, for each $x \in X$, $\overline{\{f_\alpha(x)\}_{\alpha \in A}} = \Phi(x)$, where Φ was defined in Claim 1.

Let $\{y_n\}_{n=1}^\infty$ be a dense countable subset of $\mathbb{E}$. For any $n, m \in \mathbb{N}$, let

$$U_{nm} := \Phi^{-1}(B(y_n, 2^{-m})).$$

Lower semicontinuity of Φ implies that U_{nm} is open. Let $x \in U_{nm}$ and take $y_x \in \Phi(x) \cap B(y_n, 2^{-m})$. In view of Proposition 3.7, there is a G-equivariant selection $k_{x,y_x} : G(x) \to G(y)$ of $\Phi|_{G(x)}$ while, in view of the Mostow theorem there is a tube (T_x, r_x) around the orbit $G(x)$. Let

$$V_{nm}(x) := U_{nm} \cap (k_{x,y_x} \circ r_x)^{-1}(B(y_n, 2^{-m})).$$

It is clear that $V_{nm}(x)$ is an open neighborhood of x. The separability of X (and of U_{nm} as a separable subspace of X) implies that there is a countable family $\{x^i_{nm}\} \subset U_{nm}$ such that

$$U_{nm} = \bigcup_{i=1}^\infty V^i_{nm},$$

where $V^i_{nm} := V_{nm}(x^i_{nm})$. For any $i, n, m \geqslant 1$, let $T^i_{nm} := T_{x^i_{nm}}$. It is easy to see that there is a family $\{T^{ik}_{nm}\}_{k=1}^\infty$ of closed G-invariant sets such that $G(x^i_{nm}) \subset T^{ik}_{nm}$ for all $k \geqslant 1$ and $\bigcup_{k=1}^\infty T^{ik}_{nm} = T^i_{nm}$.

For any $i, n, m \geqslant 1$, let $r^i_{nm} := r_{x^i_{nm}}$, $y^i_{nm} := y_{x^i_{nm}}$, $k^i_{nm} := k_{x^i_{nm}, y_{x^i_{nm}}}$ and let $\widetilde{f}^{ik}_{nm} : T^{ik}_{nm} \to G(y^i_{nm})$ be given by

$$\widetilde{f}^{ik}_{nm} = k^i_{nm} \circ (r^i_{nm}|_{T^{ik}_{nm}}).$$

If $x \in T^{ik}_{nm}$ and $y \in \Phi(r^i_{nm}(x))$, then $G_y \supset G_{r^i_{nm}(x)} \supset G_x$, i.e., $y \in \Phi(x)$. Moreover $k^i_{nm}(r^i_{nm}(x)) \subset \Phi(r^i_{nm}(x)) \subset \Phi(x)$. In other words $\widetilde{f}^{ik}$ is a G-equivariant selection of $\Phi|_{T^{ik}_{nm}}$.

In view of Corollary 3.3, $\widetilde{f}^{ik}_{nm}$ has an extension to an equivariant selection $f^{ik}_{nm} : X \to \mathbb{E}$ of Φ. We claim that the family $\{f^{ik}_{nm}\}_{i,k,n,m=1}^\infty$ satisfies our requirements. Indeed take $x \in X$, $y \in \Phi(x)$ and $\varepsilon > 0$. Let $m \geqslant 1$ be such that $2^{-m+1} < \varepsilon$. There is $n \geqslant 1$ such that $y \in B(y_n, 2^{-m})$. Hence $x \in U_{nm}$ and there is $i \in \mathbb{N}$ such that $x \in V^i_{nm}$. Hence $x \in T^i_{nm}$ and $x \in T^{ik}_{nm}$ for some $k \geqslant 1$. Then $f^{ik}_{nm}(x) = \widetilde{f}^{ik}_{nm}(x) \in \Phi(x)$ and $\|\widetilde{f}^{ik}_{nm}(x) - y_n\| < 2^{-m}$. Therefore $\|f^{ik}_{nm} - y\| < 2 \cdot 2^{-m} = 2^{-m+1} < \varepsilon$. This completes the proof of Claim 2.

Now we pass to the proof of our assertion. Without loss of generality we may assume that $\mathbb{E}$ is an isometric Banach representation of G (see Remark 2.3 (4)). Let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous G-equivariant maps such that $\overline{\{f_n(x)\}_{n=1}^\infty} = \mathbb{E}^{G_x}$ for all $x \in \mathbb{E}$ existing in view of Claim 2. Given $n, m \geqslant 1$, let

$$W_{nm} := \{x \in X \mid \varphi(x) \cap B(f_n(x), 2^{-m}) \neq \emptyset\}.$$

The lower semicontinuity of φ implies that W_{nm} is open; moreover it is easy to see that W_{nm} is G-invariant and there is a family of G-invariant closed sets C^k_{nm} such that $\bigcup_{k=1}^\infty C^k_{nm} = W_{nm}$. For any $n, m \geqslant 1$ such that $W_{nm} \neq \emptyset$ and $k \geqslant 1$ let us define $\varphi^k_{nm} : X \multimap \mathbb{E}$ by

$$\varphi^k_{nm}(x) := \begin{cases} \varphi(x) & \text{if } \ x \in X \setminus C^k_{nm}; \\ \varphi(x) \cap D(f_n(x), 2^{-m}) & \text{if } \ x \in C^k_{nm}. \end{cases}$$

It is standard to check that φ^k_{nm} is G-equivariant lower semicontinuous with closed convex values. Therefore, for all n, m and k, there is a G-equivariant selection $f^k_{nm} : X \to \mathbb{E}$ of φ^k_{nm}.

The family $\{f^k_{nm}\}_{n,m,k\geqslant 1}$ is a desired one. Indeed take $x \in X$, $y \in \varphi(x) \subset \mathbb{E}^{G_x}$, $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $2^{-m+2} < \varepsilon$. There is $n \in \mathbb{N}$ such that $\|f_n(x) - y\| < 2^{-m}$. Hence $x \in W_{nm}$ and $x \in C^k_{nm}$ for some $k \geqslant 1$. Thus $\|f^k_{nm} - f_n(x)\| \leqslant 2^{-m} < 2^{-m+1}$, i.e., $\|f^k_{nm}(x) - y\| < 2^{-m+1} + 2^{-m} < 2^{-m+2} < \varepsilon$. $\square$

Remark 3.10 If φ has closed convex values and has *open fibers*, i.e., for each $y \in \mathbb{E}$, the preimage $\varphi^{-1}(y)$ is *open*, then φ is lower semicontinuous and the assertion of Theorem 3.2 follows (this may be considered as an equivariant version of the Browder theorem [10]). Suppose now that $\mathbb{E}$ is a finite-dimensional representation of G. If a G-equivariant $\varphi : X \to \mathbb{E}$ has convex values (not necessarily closed) and open fibers, then it admits a continuous G-equivariant selection. Indeed, let $\{p_s\}_{s\in S}$ be a partition of unity subordinate to the open cover $\{\varphi^{-1}(y)\}_{y\in Y}$, i.e., for each $s \in S$, $\operatorname{supp} p_s \subset \varphi^{-1}(y_s))$, where $y_s \in \mathbb{E}$. Let $f(x) := \sum_{s\in S} p_s(x) y_s$ and $F(x) := \int_G g^{-1} f(gx)\, dg$ for $x \in X$; then $f : X \to \mathbb{E}$ is a continuous selection of φ, F is equivariant and for each $x \in X$, the integrand is contained in the compact set $Z := \{g^{-1}f(gx) \mid g \in G\} \subset \varphi(x)$. Therefore $F(x) \in \overline{\operatorname{conv}} Z = \operatorname{conv} Z \subset \varphi(x)$ since $\mathbb{E}$ is finite-dimensional.

In what follows we study the existence of G-equivariant ε-selections.

Definition 3.11 Let X be a set and Y a metric space. If $\varepsilon : X \to (0, +\infty)$, then a map $f : X \to Y$ is an *almost selection* of $\varphi : X \multimap Y$, precisely an ε-*selection*, if $d(f(x), \varphi(x)) := \inf_{y\in\varphi(x)} d(f(x), y) < \varepsilon(x)$ for all $x \in X$.

It was proved by Deutsch and Kenderov [14] that $\varphi : X \multimap Y$, where X is a topological space and Y is metric, has an ε-selection for any (constant) $\varepsilon > 0$ if and only if φ is *sub-lower semicontinuous* in the following sense: for any $x \in X$ and $\varepsilon > 0$ there is a neighborhood V of x such that $\bigcap_{y\in V} B(\varphi(y), \varepsilon) \neq \emptyset$. It is immediate to see that lower semicontinuous set-valued maps are sub-lower semicontinuous.

Theorem 3.12 *Let X be a paracompact G-space, $\mathbb{E}$ a Banach representation of G. If $\varphi : X \multimap \mathbb{E}$ is a G-equivariant sub-lower semicontinuous map with convex values, then for any continuous $\varepsilon : X \to (0, +\infty)$ there exists a G-equivariant continuous ε-selection $F : X \to \mathbb{E}$.*

Proof: For any $x \in X$, there is a neighborhood V_x of x and $y_x \in \varphi(x)$ such that $y_x \in B(\varphi(y), \varepsilon(x)/4M)$ for all $y \in V_x$, where M is taken from Remark 2.1. Let $U_x := V_x \cap$

$\varepsilon^{-1}((\varepsilon(x)/2,+\infty))$. Then U_x is an open neighborhood of x. Let $\{p_s\}_{s\in S}$ be a partition of unity subordinate to the open cover $\{U_x\}_{x\in X}$, i.e., for any $s\in S$ there is $x_s\in X$ such that $\operatorname{supp} p_s\subset U_{x_s}$. Let $y_s:=y_{x_s}$ and

$$f(x):=\sum_{s\in S}p_s(x)y_s,\quad x\in X.$$

Then f is continuous and for $x\in X$ if $p_s(x)\neq 0$, then $x\in U_{x_s}$, $y_s\in B(\varphi(x),\varepsilon(x_s)/2)$ and $\varepsilon(x_s)/2<2\varepsilon(x)$. Choose $y_s'\in\varphi(x)$ such that $\|y_s-y_s'\|<\varepsilon(x_s)/4M<\varepsilon(x)/2M$. Hence

$$d(f(x),\varphi(x))\leqslant\left\|f(x)-\sum_{s\in S}p_s(x)y_s'\right\|\leqslant\sum_{s\in S}p_s(x)\|y_s-y_s'\|<\varepsilon(x)/2M.$$

Let, as above,

$$F(x):=\int_G g^{-1}f(gx)\,dg,\quad x\in X.$$

For any $x\in X$ and $g\in G$,

$$d(g^{-1}f(gx),\varphi(x))\leqslant Md(f(gx),\varphi(gx))<\varepsilon(x)/2;$$

therefore $F(x)\in D(\varphi(x),\varepsilon(x)/2)\subset B(\varphi(x),\varepsilon(x))$. $\square$

4 Graph approximations

It is well-known that, in general, upper semicontinuous set-valued maps do not have continuous selections. Nevertheless they often do have graph-approximations.

Definition 4.1 Let X, Y be topological spaces, $\varphi:X\multimap Y$ and let $\mathcal{U}$ be a neighborhood of $\operatorname{Gr}(\varphi)$ in $X\times Y$. A map $f:X\to Y$ is a $\mathcal{U}$-*graph-approximation* (or, simply, $\mathcal{U}$-approximation) of φ if $\operatorname{Gr}(f)\subset\mathcal{U}$.

If X and Y are metric spaces and $\varepsilon:X\to(0,+\infty)$, then $f:X\to Y$ is an ε-*approximation* of φ if, for each $x\in X$, $d((x,f(x)),\operatorname{Gr}(\varphi))<\varepsilon(x)$ [2] or, equivalently, $f(x)\in B(\varphi(B(x,\varepsilon(x))),\varepsilon(x))$ for all $x\in X$.

Remark 4.2 (1) Let X,Y be metric spaces. It is easy to see that if $\varepsilon:X\to(0,+\infty)$ is continuous, then there is a neighborhood $\mathcal{U}(\varepsilon)$ of $\operatorname{Gr}(\varphi)$ such that any $\mathcal{U}(\varepsilon)$-approximation of φ is an ε-approximation. To this end it is sufficient to take

$$\mathcal{U}(\varepsilon):=\bigcup_{(x,y)\in Gr(\varphi)}[\varepsilon^{-1}((\varepsilon(x)/2,+\infty))\cap B(x,\varepsilon(x)/2)]\times B(y,\varepsilon(x)/2)=$$

$$\bigcup_{x\in X}[\varepsilon^{-1}((\varepsilon(x)/2,+\infty))\cap B(x,\varepsilon(x)/2)]\times B(\varphi(x),\varepsilon(x)/2).$$

[2] On the product $X\times Y$ the max-metric is considered, i.e. $d((x,y),(x',y')):=\max\{d_X(x,x'),d_Y(y,y')\}$ for any $(x,y),(x',y')\in X\times Y$.

Conversely if φ is upper semicontinuous with compact values, then for any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ there is a continuous $\varepsilon : X \to (0, +\infty)$ such that any ε-approximation of φ is a $\mathcal{U}$-approximation.

(2) Let X be a paracompact space. It may be easily shown that if φ is lower semicontinuous with compact values, then for a continuous $\varepsilon : X \to (0, +\infty)$ there is a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ such that every $\mathcal{U}$-approximation of φ is an ε-selection. Conversely, if φ is upper semicontinuous with compact values, then given a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ there is a continuous $\varepsilon : X \to (0, +\infty)$ such that an ε-selection of φ is a $\mathcal{U}$-approximation.

(3) On many occasions, especially those involving compactness, it is sufficient to consider ε-approximations with constant $\varepsilon > 0$. However, apart from the situation when X or Y are not metrizable (and the notion of ε-approximation makes no sense), general graph-approximations play an important role. To see this observe, for instance, that if $h : Y \to Z$, where Z is a topological space, is continuous and φ admits arbitrarily close graph-approximations, then so does the composition $h \circ \varphi$. Indeed, given a neighborhood $\mathcal{U}$ of $\mathrm{Gr}(h \circ \varphi)$ let $\mathcal{V} := H^{-1}(\mathcal{U})$, where $H(x, y) := (x, h(y))$ for $(x, y) \in X \times Y$. Then $\mathcal{V}$ is a neighborhood of $\mathrm{Gr}(\varphi)$ and $h \circ f$ is a $\mathcal{U}$-approximation of $h \circ \varphi$ provided $f : X \to Y$ is a $\mathcal{V}$-approximation of φ.

When studying the existence of continuous $\mathcal{U}$-approximations one has to observe that it is necessary to restrict the choice of $\mathcal{U}$ even when a map φ has closed convex values. To see this consider $\varphi : [0, +\infty) \multimap \mathbb{R}$ given by $\varphi(x) = \frac{1}{x}$ for $x > 0$ and $\varphi(0) = [0, +\infty)$. Then φ is upper semicontinuous with closed convex values. However φ does not admit continuous $\mathcal{U}$-approximations, where

$$\mathcal{U} := \{(x, y) \in [0, +\infty) \times \mathbb{R} \mid 1/2 < xy < 2 \text{ or } 3xy < 1\}. \tag{4}$$

Definition 4.3 Let X be a topological space and $\mathbb{E}$ a Banach space. A neighborhood $\mathcal{U}$ of the graph of $\varphi : X \multimap \mathbb{E}$ is *thick* if for any $x \in X$ there are a neighborhood U_x of x in X and a *convex* neighborhood V_x of $\varphi(x)$ such that $U_x \times \overline{V}_x \subset \mathcal{U}$.

Remark 4.4 (a) If X is a metric space, φ has convex values and $\varepsilon : X \to (0, +\infty)$ is continuous, then $\mathcal{U}(\varepsilon)$ defined in Remark 4.2 (1) is thick.

(b) If φ has compact convex values, then *any* neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ is thick.

(c) If φ has convex values (not necessarily compact), then there are neighborhoods without the thickness property: see the above example (4).

(d) If in Remark 4.2 (2), φ is upper semicontinuous (not necessarily compact-valued) and $\mathcal{U}$ is thick, then there is a continuous $\varepsilon > 0$ such that an ε-selection is a $\mathcal{U}$-approximation of φ.

The problem of the existence of equivariant graph-approximations is not that easy unless some additional, rather restrictive, assumptions are undertaken since, unlike selections, the symmetrization of a graph-approximation is G-equivariant but, in general, it is no longer a desired graph-approximation.

Below we provide a rather detailed discussion of this apparently important question and study the equivariant approximability with no additional conditions concerning, e.g., the action of G on X.

First let us collect some useful facts concerning open coverings of G-spaces – see [32].

Definition 4.5 Let X be a G-space. A covering $\mathscr{W} = \{W_\lambda\}_{\lambda\in\Lambda}$ of X is *G-invariant* if, for each $\lambda \in \Lambda$, W_λ is G-invariant. We say that $\mathscr{W}$ is a *G-covering* if Λ is a G-set and $gW_\lambda = W_{g\lambda}$ (and thus $gW_\lambda \in \mathscr{W}$) for all $\lambda \in \Lambda$ and $g \in G$.

If $\mathscr{W} = \{W_\lambda\}_{\lambda\in\Lambda}$ is a G-covering, then the *saturation* $\widetilde{\mathscr{W}} = \{\widetilde{W}_\alpha := \bigcup_{\lambda\in\alpha} W_\lambda\}_{\alpha\in\Lambda/G}$, is a G-invariant covering. Note that $\widetilde{W}_\alpha = GW_\lambda$, where λ is an arbitrary element of $\alpha \in \Lambda/G$. Conversely a G-invariant covering $\mathscr{W}$ is a G-covering (it is sufficient to consider the trivial action of G on the index set Λ) and $\widetilde{\mathscr{W}} = \mathscr{W}$.

Proposition 4.6 *Let X be a G-space and $\mathscr{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open locally finite covering of X. Then any $x \in X$ has an open G-invariant neighborhood V such that the set $\{\lambda \in \Lambda \mid U_\lambda \cap V \neq \emptyset\}$ is finite; hence the covering $\{GU_\lambda\}_{\lambda\in\Lambda}$ is open G-invariant and locally finite.*

Assume that X is a paracompact G-space. Then:

(1) *any open covering $\mathscr{U}$ of X admits an open G-covering $\mathscr{W} = \{W_\lambda\}_{\lambda\in\Lambda}$ star-refining $\mathcal{U}$, i.e., for each $\lambda \in \Lambda$, there is $U \in \mathscr{U}$ such that* $\mathrm{st}(W_\lambda, \mathscr{W}) := \bigcup\{W \in \mathscr{W} \mid W \cap W_\lambda \neq \emptyset\} \subset U$;

(2) *any open G-covering $\mathscr{U}$ of X admits an open G-covering $\mathscr{W}$ refining $\mathscr{U}$ with the locally finite saturation $\widetilde{\mathscr{W}}$;*

(3) *for every open G-invariant covering $\mathscr{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ there is a partition of unity $\{p_\lambda\}_{\lambda\in\Lambda}$ such that p_λ is G-invariant and* $\operatorname{supp} p_\lambda := \overline{\{x \in X \mid p_\lambda(x) \neq 0\}} \subset U_\lambda$ *for every $\lambda \in \Lambda$.* $\square$

Theorem 4.7 *Suppose that X is a paracompact G-space, $\mathbb{E}$ is a Banach representation of G and let $\varphi : X \multimap \mathbb{E}$ be an upper semicontinuous G-equivariant map. If $\mathcal{U}$ is a thick neighborhood of* $\mathrm{Gr}(\varphi)$, *then there exists a continuous G-equivariant $\mathcal{U}$-approximation $F : X \to \mathbb{E}$.*

The above example (4) shows that the thickness assumption can not be avoided.

Proof: For every $x \in X$ choose U_x and V_x as in the definition of thickness. Upper semicontinuity of φ implies that, diminishing U_x if necessary, we can assume without loss of generality that $\varphi(U_x) \subset V_x$. According to Proposition 4.6 (1) there is an open G-covering $\mathscr{W} = \{W_\lambda\}_{\lambda\in\Lambda}$ star-refining $\mathscr{U} := \{U_x\}_{x\in X}$. Let $\{p_s\}_{s\in S}$ be a partition of unity subordinated to $\mathscr{W}$, i.e., for each $s \in S$ there is $\lambda_s \in \Lambda$ such that $\operatorname{supp} p_s \subset W_{\lambda_s}$. For each $s \in S$ choose $y_s \in \varphi(W_{\lambda_s})$. For $x \in X$, let $S(x) := \{s \in S \mid p_s(x) \neq 0\}$ and

$$f(x) := \sum_{s\in S} p_s(x) y_s = \sum_{s\in S(x)} p_s(x) y_s.$$

Clearly $f : X \to \mathbb{E}$ is a well-defined continuous map.

Now let $x \in X$. We shall show that there is $y \in X$ such that $x \in U_y$ and if $s \in S(gx)$ for some $g \in G$, then $g^{-1}y_s \in V_y$. To this end fix $t \in S(x)$. Then $p_t(x) \neq 0$ and thus

$$x \in \operatorname{supp} p_t \subset W_{\lambda_t} \subset \mathrm{st}(W_{\lambda_t}, \mathscr{W}) \subset U_y$$

for some $y \in X$. Take $g \in G$ and $s \in S(gx)$. Therefore $gx \in \operatorname{supp} p_s \subset W_{\lambda_s}$ and $x \in g^{-1}W_{\lambda_s} = W_{g^{-1}\lambda_s}$ (see Definition 4.5). Hence $W_{g^{-1}\lambda_s} \cap W_{\lambda_t} \neq \emptyset$ and

$$W_{g^{-1}\lambda_s} \subset \operatorname{st}(W_{\lambda_t}, \mathscr{W}) \subset U_y.$$

Now $y_s \in \varphi(W_{\lambda_s})$ and thus

$$g^{-1}y_s \in g^{-1}\varphi(W_{\lambda_s}) = \varphi(W_{g^{-1}\lambda_s}) \subset \varphi(U_y) \subset V_y.$$

For each $g \in G$,

$$g^{-1}f(gx) = g^{-1} \sum_{s \in S(gx)} p_s(gx)y_s = \sum_{s \in S(gx)} p_s(gx)g^{-1}y_s \in V_y$$

since V_y is convex. This implies that $F(x)$, where the equivariant F is given by (1), belongs to $\overline{V_y}$ and thus $(x, F(x)) \in U_y \times \overline{V}_y \subset \mathcal{U}$. □

In view of Remark 4.4 (b), (a) we have the following results being generalizations of the Cellina approximation theorem [12].

Corollary 4.8 (1) *If X is a paracompact G-space, $\mathbb{E}$ is a Banach representation of G and $\varphi : X \multimap \mathbb{E}$ is a G-equivariant upper semicontinuous map with convex compact values, then φ has a continuous G-equivariant $\mathcal{U}$-approximation for any neighborhood $\mathcal{U}$ of* $\operatorname{Gr}(\varphi)$.

(2) *If X is a metric G-space, $\mathbb{E}$ is a Banach representation of G and $\varphi : X \to \mathbb{E}$ is a G-equivariant map with convex values, then φ admits a continuous G-equivariant ε-approximation for any continuous $\varepsilon : X \to (0, +\infty)$.* □

Remark 4.9 (1) In course of the proofs of Theorems 3.12, 4.7 (and thus in Corollary 4.8) the constructed G-equivariant maps are symmetrizations of maps taking values in $\operatorname{conv}\varphi(X)$. Therefore if φ is compact, then so are their single-valued almost selections and graph-approximations.

(2) In the context of Corollary 4.8 (2) if we assume that G acts on X by isometries (or equivalently that the metric d on X is G-invariant – comp. Remark 2.3 (5)), then there exist G-equivariant locally Lipschitz ε-approximations because one can use locally Lipschitz partitions of unity in the proof and Remark 2.3 (3).

Now we address an equivariant version of a constrained approximation problem studied in [7]: given G-spaces X, Y, G-equivariant maps $\varphi, \psi : X \multimap Y$ and a neighborhood $\mathcal{U}$ of $\operatorname{Gr}(\varphi)$, does there exist an G-equivariant $\mathcal{U}$-approximation $f : X \to Y$ such that $f(x) \in \psi(x)$?

We start with some lemmata.

Lemma 4.10 *Let X be a G-space. If $\varepsilon : X \to (0, +\infty)$ is continuous, then $\eta : X \to (0, +\infty)$, given by $\eta(x) := \inf_{g \in G} \eta(gx)$, is well-defined, continuous and G-invariant.*

Proof: It is clear that, for each $x \in X$, there is $g_x \in G$ such that $\eta(x) = \varepsilon(g_x x) > 0$ since G is compact. It is immediate to see that η is G-invariant. Take $0 < \alpha < \beta$ and $x \in \eta^{-1}((\alpha, \beta))$. Then $\alpha < \eta(x) = \varepsilon(g_x x) < \beta$. The continuity of ε implies that there is a neighborhood V_x of x (in X) such that $\varepsilon(g_x y) < \beta$ for all $y \in V_x$. Hence, for all $y \in V_x$, $\eta(y) \leqslant \varepsilon(g_x y) < \beta$. On the other hand, for some $\alpha < \alpha' < \eta(x)$ and any $g \in G$, $\alpha' < \varepsilon(gx)$. Hence, in view of continuity of ε, for any $g \in G$ there are neighborhoods H_g of g and W_g of x such that $\alpha' < \eta(hy)$ if $h \in H_g$ and $y \in W_g$. The compactness of G implies that there are $g_1, ..., g_n \in G$ such that $\bigcup_{i=1}^n H_{g_i} = G$. Let $W_x := \bigcap_{i=1}^n W_{g_i}$. Let $y \in W_x$. For any $g \in G$, then there is $1 \leqslant j \leqslant n$ such that $g \in H_{g_j}$. Since $y \in W_{g_j}$, we get that $\varepsilon(gy) > \alpha'$. Therefore $\eta(y) = \inf_{g \in G} \varepsilon(gy) \geqslant \alpha' > \alpha$. Finally if $y \in V_x \cap W_x$, then $\alpha < \eta(y) < \beta$. □

Lemma 4.11 *Suppose that $\varphi : X \multimap \mathbb{E}$ is upper semicontinuous, where X is paracompact and $\mathbb{E}$ is a Banach space, and let $\mathcal{U}$ be a neighborhood of $\mathrm{Gr}(\varphi)$ having the following property:*
$(*)$ for each $y \in X$, there is a neighborhood U_y of y and $r_y > 0$ such that $U_y \times B(\varphi(y), r_y) \subset \mathcal{U}$. Then there is a neighborhood $\mathcal{W}$ of $\mathrm{Gr}(\varphi)$ and a continuous function $\varepsilon : X \to (0, +\infty)$ such that given a $\mathcal{W}$-approximation $f : X \to \mathbb{E}$ of φ and a map $f' : X \to \mathbb{E}$ such that $\|f'(x) - f(x)\| < \varepsilon(x)$ for all $x \in X$, f' is a $\mathcal{U}$-approximation of φ.

Proof: Let $s_y := \frac{1}{4} r_y$, $y \in X$. The upper semicontinuity of φ implies that, for each $y \in X$, there is a neighborhood $V_y \subset U_y$ of y such that $\varphi(V_y) \subset B(\varphi(y), 2s_y)$. Let $\{\lambda_j\}_{j \in J}$ be a partition of unity subordinated to the open cover $\{V_y\}_{y \in X}$, i.e., for any $j \in J$, there is $y_j \in X$ such that $\operatorname{supp} \lambda_j \subset V_{y_j}$. Let $\varepsilon(x) = \sum_{j \in J} \lambda_j(x) s_{y_j}$, $x \in X$. Then $\varepsilon : X \to (0, +\infty)$ and is continuous.

Let $x \in X$. Then there is $j(x) \in J$ such that $\lambda_{j(x)}(x) > 0$ (thus $x \in V_{y_{j(x)}}$) and $\varepsilon(x) \leqslant s_{y_{j(x)}}$. Let $W_x := \varepsilon^{-1}((0, 2s_{y_{j(x)}})) \cap V_{y_{j(x)}}$ and

$$\mathcal{W} := \bigcup_{x \in X} W_x \times B(\varphi(x), \varepsilon(x)).$$

Obviously if W_x is an open neighborhood of x and $\mathcal{W}$ is a thick neighborhood of $\mathrm{Gr}(\varphi)$.

Suppose that $f : X \to \mathbb{E}$ is a $\mathcal{W}$-approximation of φ and $f' : X \to \mathbb{E}$ is such that $\|f(x) - f'(x)\| < \varepsilon(x)$ for any $x \in X$. Let us fix $x' \in X$. There is $x \in X$ such that $(x', f(x')) \in W_x \times B(\varphi(x), \varepsilon(x))$, i.e., $x' \in W_x$ and $f(x') \in B(\varphi(x), \varepsilon(x))$. Therefore $x' \in V_{y_{j(x)}}$ and $\varepsilon(x') < 2s_{y_{j(x)}}$. At the same time $x \in V_{y_{j(x)}}$ and $\varepsilon(x) \leqslant s_{y_{j(x)}}$, so $\varphi(x) \in B(\varphi(y_{j(x)}), s_{y_{j(x)}})$. Moreover

$$\begin{aligned} f'(x') \in B(\varphi(x), \varepsilon(x) + \varepsilon(x')) \subset B(\varphi(y_{j(x)}), s_{y_{j(x)}} + s_{y_{j(x)}} + 2s_{y_{j(x)}}) = \\ B(\varphi(y_{j(x)}), 4s_{y_{j(x)}}) = B(\varphi(y_{j(x)}, r_{y_{j(x)}}). \end{aligned}$$

Hence $(x', f'(x')) \in V_{y_{j(x)}} \times B(\varphi(y_{j(x)}), r_{y_{j(x)}}) \subset U_{y_{j(x)}} \times B(\varphi(y_{j(x)}, r_{y_{j(x)}}) \subset \mathcal{U}$. □

Remark 4.12 (1) Observe that if φ has compact values, then every neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ has the property $(*)$ from Lemma 4.11. Similarly if X is a metric space and $\varepsilon : X \to (0, +\infty)$ is continuous, the neighborhood $\mathcal{U}(\varepsilon)$ from Remark 4.2 (1) has this property, too.

(2) It is clear that if φ has convex values, then the neighborhood $\mathcal{W}$ constructed in the proof of the above Lemma is thick.

Theorem 4.13 *Let X be a paracompact G-space and $\mathbb{E}$ a Banach representation of G. Let $\psi : X \multimap \mathbb{E}$ be G-equivariant lower semicontinuous map with closed convex values, and $\varphi : X \multimap \mathbb{E}$ a G-equivariant upper semicontinuous map with convex values such that $\varphi(x) \cap \psi(x) \neq \emptyset$ for all $x \in X$. Suppose that $\mathcal{U}$ is a neighborhood of $\mathrm{Gr}(\varphi)$ with property $(*)$, i.e., for any $x \in X$, there are a neighborhood T_x and $r_x > 0$ such that $T_x \times B(\varphi(x), r_x) \subset \mathcal{U}$. Then there is a continuous G-equivariant map $f : X \to \mathbb{E}$ being both a selection of ψ and a $\mathcal{U}$-approximation of φ.*

Proof: We assume without loss of generality that $\mathbb{E}$ is an isometric Banach representation of G. According to Lemma 4.11 and Remark 4.12 (2) there is a thick neighborhood $\mathcal{W}$ of $\mathrm{Gr}(\varphi)$ and a continuous function $\varepsilon : X \to (0, +\infty)$ such that if $H : X \to \mathbb{E}$ is a $\mathcal{W}$-approximation of φ and $F : X \to \mathbb{E}$ is continuous, then F is a $\mathcal{U}$-approximation of φ, provided $\|H(x) - F(x)\| \leqslant \varepsilon(x)$ for all $x \in X$. In view of Lemma 4.10 we may assume without loss of generality that ε is G-invariant. Since $\mathcal{W}$ is thick, for any $x \in X$, there are a neighborhood U_x of x and a convex neighborhood V_x of $\varphi(x)$ such that $U_x \times \overline{V_x} \subset \mathcal{W}$. We may assume that $\varphi(U_x) \subset V_x$.

Let $\mathscr{W}$ be a G-covering of X star refining $\{U_x\}_{x \in X}$. For each $x \in X$ choose $z_x \in \varphi(x) \cap \psi(x)$ and $W \in \mathscr{W}$ such that $x \in W$ and let

$$B_W(x) := \{y \in W \mid \psi(y) \cap B(z_x, 4^{-1}\varepsilon(x)) \neq \emptyset\} \cap \varepsilon^{-1}(2^{-1}\varepsilon(x), +\infty).$$

It is clear that $x \in B_W(x)$ and the covering $\mathscr{B} := \{B_W(x)\}_{W \in \mathscr{W}, x \in X}$ refines $\mathscr{W}$. Let $\{p_s\}_{s \in S}$ be a partition of unity subordinated to $\mathscr{B}$, i.e., for each $s \in S$, there is $W_s \in \mathscr{W}$ and $x_s \in W_s$ such that $\operatorname{supp} p_s \subset B_s := B_{W_s}(x_s)$. Let

$$h(x) := \sum_{s \in S} p_s(x) z_s, \quad x \in X,$$

where $z_s := z_{x_s}$. Then h is an $\varepsilon/2$-selection of ψ. Indeed, if $x \in X$ and $p_s(x) \neq 0$, then $x \in B_s$, i.e., $x \in W_s$, $\psi(x) \cap B(z_s, \varepsilon(x_s)/4) \neq \emptyset$ and $\varepsilon(x_s)/2 < \varepsilon(x)$. Hence there is $z_s' \in \psi(x)$ such that $\|z_s' - z_s\| < \varepsilon(x_s)/4 < \varepsilon(x)/2$ and, thus, for $y := \sum_{s \in S} p_s(x) z_s'$ we have

$$\|h(x) - y\| \leqslant \sum_{s \in S} p_s(x) \|z_s - z_s'\| < \varepsilon(x)/2.$$

Therefore, for all $x \in X$ and $g \in G$,

$$d(g^{-1}h(gx), \psi(x)) = d(h(gx), \psi(gx)) < \varepsilon(gx)/2 = \varepsilon(x)/2,$$

i.e., $g^{-1}h(gx) \in B(\psi(x), \varepsilon(x)/2) \subset D(\psi(x), 3\varepsilon(x)/4)$.

Now let

$$H(x) := \int_G g^{-1}h(gx)\, dg, \quad x \in X.$$

It follows that, for all $x \in X$,

$$d(H(x), \psi(x)) < \varepsilon(x).$$

Moreover, as in the proof of Theorem 4.7, one checks that H is a $\mathcal{W}$-approximation of φ.

Now consider a map $X \ni x \mapsto D(H(x), \varepsilon(x)) \cap \psi(x)$. It is standard to show that it is lower semicontinuous, G-equivariant and has closed convex values. Hence, in view of Theorem 3.2, it has a continuous G-equivariant selection $F : X \to \mathbb{E}$. Then $\|F(x) - H(x)\| \leqslant \varepsilon(x)$ for $x \in X$ and F is a desired selection of ψ and a $\mathcal{U}$-approximation of φ. □

Quite often in applications the map ψ from Theorem 4.13 is specified. The following result is an example (see [5] or [27], Theorem 2.8.4).

Theorem 4.14 *Let $\mathbb{E}$ be an isometric Banach representation of G and let $K \subset \mathbb{E}$ be closed convex and G-invariant. Further assume that a G-equivariant upper semicontinuous map $\varphi : K \multimap \mathbb{E}$ with closed convex values is weekly tangent to K, i.e., for each $x \in K$, $\varphi(x) \cap T_K(x) \neq \emptyset$. Then, for each $\varepsilon > 0$, there exists a G-equivariant locally Lipschitz map $f : K \to \mathbb{E}$ being an ε-approximation of φ such that for all $x \in K$, $f(x) \in T_K(x)$.*

Proof: Since the map $K \ni x \multimap T_K(x)$ is lower semicontinuous (see e.g. [27], Remark 1.3.9 or [23]) and G-equivariant (Example 2.6(5)), to obtain a continuous G-equivariant $f : K \to \mathbb{E}$ it is sufficient to apply Theorem 4.13. However, in our case one can modify the proof by applying locally Lipschitz partitions of unity, since G acts on K by isometries (see Remark 4.9 (2)). We omit the details and leave them to the reader. □

5 Extending approximations and homotopy

The problem of the extension properties of a given partial $\mathcal{U}$-approximation seems to be a natural question – comp. [34]. We have the following general result in the equivariant setting.

Theorem 5.1 *Given a paracompact G-space X, a closed G-invariant subset $A \subset X$ and a Banach representation $\mathbb{E}$ of G, an upper semicontinuous G-equivariant map $\varphi : X \multimap \mathbb{E}$ and a thick neighborhood $\mathcal{U} \subset X \times \mathbb{E}$ of $Gr(\varphi)$, there exists a thick neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Gr(\varphi)$ such that any continuous G-equivariant $\mathcal{V}$-approximation $f : A \to \mathbb{E}$ of $\varphi|_A$ admits a continuous G-equivariant extension $F : X \to \mathbb{E}$ being a $\mathcal{U}$-approximation of φ.*

Proof: Let $\mathcal{U}$ be a thick neighborhood of φ; then for every $x \in X$ there are an open neighborhood $U_x \subset \mathbb{E}$ of x in X and a convex open set $V_x \subset \mathbb{E}$ such that $\varphi(x) \subset V_x$ and $U_x \times V_x \subset \mathcal{U}$. Since φ is upper semicontinuous, we can assume (diminishing U_x if necessary) that $\varphi(U_x) \subset V_x$. Let $\mathscr{T} = \{T_s\}_{s \in S}$ be a locally finite star-refinement of $\{U_x\}_{x \in X}$. Thus for every $s \in S$ there is $x_s \in X$ such that $\mathrm{st}(T_s, \mathscr{T}) \in U_{x_s}$.

For any $s \in S$, let $K(s) := \{t \in S \mid T_t \cap T_s \neq \emptyset\}$ (observe that if $t \in K(s)$, then $T_s \in \mathrm{st}(T_t, \mathscr{T}) \subset U_{x_t}$ and $\varphi(T_s) \subset V_{x_t}$) and define a set

$$V_s := \bigcap_{t \in K(s)} V_{x_t} \subset \mathbb{E}.$$

Note that $V_s \neq \emptyset$ since $\varphi(T_s) \subset V_s$ and V_s is convex and open, since $K(s)$ is finite. Moreover, for all $s \in S$, $T_s \times V_s \subset \mathcal{U}$. Let

$$\mathcal{V} := \bigcup_{s \in S} T_s \times V_s.$$

One easily sees that $\mathcal{V}$ is a thick neighborhood of $\mathrm{Gr}(\varphi)$ since any x belongs to some T_s and, thus, $\varphi(x) \subset V_s$ because $\varphi(x) \subset V_{x_t}$ for all $t \in K(s)$.

Now let $f : A \to \mathbb{E}$ be a G-equivariant $\mathcal{V}$-approximation of φ (i.e., $\mathrm{Gr}(f) \subset \mathcal{V}$). By Corollary 3.3 there exists an equivariant continuous extension $k : X \to \mathbb{E}$ of f. Since $\mathcal{V}$ is an open subset of $X \times \mathbb{E}$, there exists an open G-invariant neighborhood W of A such that $(x, g(x)) \in \mathcal{V}$ for all $x \in W$. Additionally choose an open G-invariant neighborhood V of A such that $A \subset V \subset \overline{V} \subset W$. Let $\{\alpha, \beta\}$ be a G-invariant partition of unity subordinate to the G-invariant covering $\{W, X \setminus \overline{V}\}$ (see Proposition 4.6 (3)), i.e.,

$$\operatorname{supp} \alpha \subset W, \;\; \operatorname{supp} \beta \subset X \setminus \overline{V} \;\text{ and }\; \alpha(x) + \beta(x) = 1.$$

By Theorem 4.7 there exists another G-equivariant $\mathcal{V}$-approximation $h : X \to \mathbb{E}$ of φ. Define

$$F(x) := \alpha(x) \cdot k(x) + \beta(x) \cdot h(x) \quad \text{for} \quad x \in X.$$

Then obviously $F : X \to \mathbb{E}$ is continuous, G-equivariant and $F|_A = f$.

If $\alpha(x) \neq 0$, then $x \in W$ and $(x, k(x)) \in T_s \times V_s$, $(x, h(x)) \in T_t \times V_t$ for some $s, t \in S$. Therefore $x \in T_t \cap T_s \subset \mathrm{st}(T_t, \mathscr{T}) \subset U_{x_t}$ and $t \in K(s)$. Hence $V_s \subset V_{x_t}$ and, since $t \in K(t)$, $V_t \subset V_{x_t}$. This implies that

$$(x, F(x)) \in U_{x_t} \times V_{x_t} \subset \mathcal{U},$$

i.e., F is a desired G-equivariant $\mathcal{U}$-approximation of φ extending f. $\square$

Combining Theorem 5.1 with Remark 4.4 (d) we get the following

Corollary 5.2 *If X, $\mathbb{E}$, φ and $\mathcal{U}$ are as in Theorem 5.1, then there is a continuous function $\varepsilon : X \to (0, +\infty)$ such that any G-equivariant continuous ε-selection of φ extends to a continuous G-equivariant $\mathcal{U}$-approximation of φ.* $\square$

Remark 5.3 Assume that X is a metric G-space, $A \subset X$ is closed and G-invariant, $\mathbb{E}$ is a Banach representation of G and $f : A \to \mathbb{E}$ is a partial almost selection of an equivariant $\varphi : X \multimap \mathbb{E}$. Does there exist a G-equivariant almost selection of φ extending f? The answer is positive under appropriate assumptions on φ. We state (without proof) the following result: *If φ is sub-lower semicontinuous and Hausdorff lower semicontinuous at each $a \in A$* ([3]), *then for any $\varepsilon > 0$ and $0 < \delta < \varepsilon$ any G-equivariant continuous partial δ-selection $f : A \to \mathbb{E}$ of φ admits a G-equivariant continuous extension $F : X \to \mathbb{E}$ being an ε-selection of φ* (comp. [27]).

The following corollary is of importance in order to define various homotopy invariants by the graph-approximation method. Roughly speaking it says that sufficiently close G-equivariant approximations of a given map are equivariantly homotopic via homotopy being arbitrarily close to this map.

[3] I.e., for any $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{y \in \varphi(a)} d(y, \varphi(x)) < \varepsilon$ provided $d(x, a) < \delta$.

Corollary 5.4 *Let X be a paracompact G-space and $\mathbb{E}$ a Banach representation of G. Let $\varphi : X \multimap \mathbb{E}$ be a G-equivariant upper semicontinuous map. For every thick neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$ there exists a thick neighborhood $\mathcal{V}$ of $\mathrm{Gr}(\varphi)$, such that every two G-equivariant continuous $\mathcal{V}$-approximations of φ are joined by a homotopy $h : X \times [0,1] \to \mathbb{E}$ such that, for each $t \in [0,1]$, the map*

$$h(\cdot, t) : X \to \mathbb{E}$$

is a G-equivariant $\mathcal{U}$-approximation of φ.

Proof: Let $\pi : X \times [0,1] \to X$ be the projection onto X and let $\varphi' := \varphi \circ \pi : X \times [0,1] \multimap \mathbb{E}$. The space $X \times [0,1]$ is a G-space with an action $g(x,t) := (gx, t)$. The set $A := X \times \{0,1\}$ is a closed G-invariant subset of $X \times [0,1]$. Let $\mathcal{U}$ be a thick neighborhood of $\mathrm{Gr}(\varphi)$ in $X \times \mathbb{E}$. Then

$$\mathcal{U}' := \{(x,t,y) |\ (x,y) \in \mathcal{U}, t \in [0,1]\}$$

is a thick neighborhood of $\mathrm{Gr}(\varphi')$ since, for any $(x,t) \in X \times [0,1]$, $U_{(x,t)} \times \overline{V}_x \subset \mathcal{U}'$, where $U_{(x,t)} := U_x \times [0,1]$, and $\varphi'(U_{(x,t)}) \subset V_x$ (U_x and V_x are taken form Definition 4.3). In view of Theorem 5.1 there is a thick neighborhood of $\mathrm{Gr}(\varphi')$ such any partial G-equivariant continuous approximation $h' : A \to \mathbb{E}$ admits a G-equivariant continuous extension onto $X \times [0,1]$ being a $\mathcal{U}'$-approximation of φ'.

Let $\mathcal{V}_i := \{(x,y) \in X \times \mathbb{E} \mid (x,i,y) \in \mathcal{V}'\}$, $i = 0,1$, and $\mathcal{V} := \mathcal{V}_0 \cap \mathcal{V}_1$. Let $f_i : X \to \mathbb{E}$, $i = 0,1$, be a G-equivariant continuous $\mathcal{V}$-approximation of φ and let $h' : A \to \mathbb{E}$ be given by

$$h'(x,t) := \begin{cases} f_0(x) & \text{if} \quad x \in X, t = 0 \\ f_1(x) & \text{if} \quad x \in X, t = 1. \end{cases}$$

Then h' is a G-equivariant continuous $\mathcal{V}'$-approximation of φ'. Therefore it admits an extension $h : X \times [0,1] \to \mathbb{E}$ being a G-equivariant continuous $\mathcal{U}'$-approximation of φ'. Thus $Gr(h) \subset \mathcal{U}'$, i.e., for each $x \in X$ and $t \in [0,1]$, $(x,t,h(x,t)) \in \mathcal{U}'$. This means that $(x, h(x,t)) \in \mathcal{U}$. Therefore, for each $t \in [0,1]$, $h(\cdot,t) : X \to \mathbb{E}$ is a $\mathcal{U}$-approximation of φ. $\square$

6 Equivariant measurable and Carathéodory-type selections and approximations

In many applications one considers certain classes of maps between G-spaces or, more generally, G-sets and their transformations. Given G-sets Ω and X, by $\mathrm{Map}(\Omega, X)$ we denote the class of *all* maps $f : \Omega \to X$. It appears that $\mathrm{Map}(\Omega, X)$ is a G-set itself: if $g \in G$ and $x : \Omega \to X$, then $g \cdot x : \Omega \to X$ is defined by $(g \cdot x)(\omega) = gx(g^{-1}\omega)$ for $\omega \in \Omega$; it is easy to see that $G \times \mathrm{Map}(\Omega, X) \ni (g,x) \mapsto g \cdot x \in \mathrm{Map}(\Omega, X)$ is indeed a G-action. Many specific subclasses of $\mathrm{Map}(\Omega, X)$ are invariant with respect to this action; for instance if Ω and X are G-spaces and $\mathcal{C}(\Omega, X)$ denotes the space of all continuous maps $\Omega \to X$ with compact open topology, then, for each $g \in G$ and $x \in \mathcal{C}(\Omega, X)$, $g \cdot x \in \mathcal{C}(\Omega, X)$ and the action is continuous.

Note that the subclass $\mathrm{Map}_G(\Omega, X)$ consisting of G-equivariant maps $\Omega \to X$ is equal to the set of fixed points of the described action, i.e., $\{x \in \mathrm{Map}(\Omega, X) \mid g \cdot x = x \text{ for all } g \in G\}$.

As concerns (equivariant) transformations between mapping spaces the superposition operators play an important role. Assume that Y is a G-set and let $\varphi : \Omega \times X \multimap Y$ be a set-valued map. By the *superposition* (or *Nemytskii*) operator generated by φ we mean a map $N_\varphi : \mathrm{Map}(\Omega, X) \multimap \mathrm{Map}(\Omega, Y)$ given by

$$N_\varphi(x) := \{y : \Omega \to Y \mid y(\omega) \in \varphi(\omega, x(\omega)) \text{ for all } \omega \in \Omega\}, \quad x \in \mathrm{Map}(\Omega, X).$$

Observe that N_φ is G-equivariant if and only if φ is G-equivariant ([4]), i.e., for all $\omega \in \Omega$ and $x \in X$, $\varphi(g\omega, gx) = g\varphi(\omega, x)$. In this context some questions arise: given an invariant subclass $\mathcal{M}(\Omega, X)$ in $\mathrm{Map}(\Omega, X)$ and the corresponding (invariant) subclass $\mathcal{M}(\Omega, Y) \subset \mathrm{Map}(\Omega, Y)$, which properties of φ guarantee that N_φ is well-defined as a map from $\mathcal{M}(\Omega, X)$ to $\mathcal{M}(\Omega, Y)$ (i.e., is there $y \in \mathcal{M}(\Omega, Y)$ such that $y(\omega) \in \varphi(\omega, x(\omega))$ for any $\omega \in \Omega$, where $x \in \mathcal{M}(\Omega, X)$)? When N_φ, if correctly defined, has appropriate regularity properties (semicontinuity, measurability etc.)? Under what conditions can one state the existence of its (sufficiently regular) single-valued selections and/or approximations? The full answer to these questions is beyond the scope of this paper. Here we shall study only the properties of φ in the context of the existence of its (equivariant) selections and approximations.

To this end we shall first deal with equivariant selections and approximations of set-valued maps defined on measure spaces.

Let $(\Omega, \mathscr{A})$ be a measurable space and $\mathbb{E}$ a Banach space. We say that a set-valued map $\varphi : \Omega \multimap \mathbb{E}$ is $\mathscr{A}$*-measurable* (or, for short, *measurable*) if it has closed values and for any open $U \subset \mathbb{E}$ the preimage $\varphi^{-1}(U) \in \mathscr{A}$ ([5]). It is well-known (see the Castaing theorem [4, Theorem 8.3.1]) that if $\mathbb{E}$ is separable, then the following conditions are equivalent:

- φ is measurable;
- for each $z \in \mathbb{E}$, the map $\Omega \ni \omega \mapsto d(z, \varphi(\omega)) := \inf_{y \in \varphi(\omega)} \|z - y\| \in \mathbb{R}$ is measurable;
- there is a sequence of measurable functions $f_n : \Omega \to \mathbb{E}$, $n \geqslant 1$, such that $\varphi(\omega) = \overline{\{f_n(\omega)\}_{n=1}^\infty}$, $\omega \in \Omega$.

If $\mathbb{E}$ is separable and $\varphi : \Omega \multimap \mathbb{E}$ is measurable, then its graph $\mathrm{Gr}(\varphi) \in \mathscr{A} \otimes \mathscr{B}(\mathbb{E})$, where $\mathscr{B}(\mathbb{E})$ stands for the σ-algebra of Borel subsets in $\mathbb{E}$ and $\mathscr{A} \otimes \mathscr{B}(\mathbb{E})$ is the σ-algebra in $\Omega \times \mathbb{E}$ generated by products $A \times B$, $A \in \mathscr{A}$, $B \in \mathscr{B}(\mathbb{E})$; conversely if $\mathrm{Gr}(\varphi) \in \mathscr{A} \otimes \mathscr{B}(\mathbb{E})$ and $(\Omega, \mathscr{A})$ is *complete* (i.e., $\mathscr{A}$ admits a complete σ-finite measure), then φ is measurable ([6]).

We shall be concerned with measurable spaces with additional structure of a G-set. In what follows, unless stated otherwise, we assume for simplicity that:

Assumption 6.1 Ω is a G-space, $\mathscr{A}$ is a σ-algebra in Ω containing $\mathscr{B}(\Omega)$, the σ-algebra of all Borel subsets in Ω, and $\mu : \mathscr{A} \to [0, +\infty]$ is a complete, σ-finite regular measure (i.e., for any $A \in \mathscr{A}$ and $\varepsilon > 0$, there is a closed set F such that $F \subset A$ and $\mu(A \setminus F) < \varepsilon$).

[4] On $\Omega \times X$ the group G acts component-wise, i.e. given $g \in G$ and $(\omega, x) \in \Omega \times X$, $g(\omega, x) := (g\omega, gx)$.

[5] The reader should be warned that the notion of measurability we use here is sometimes called *weak measurability*; elsewhere a map φ is said to be *measurable* (or *strongly measurable*) if preimages of closed sets are measurable. The latter notion is slightly stronger, however if $\mathscr{A}$ admits a complete σ-finite measure, then both notions coincide since the preimage of any Borel subset in $\mathbb{E}$ is measurable in this case.

[6] The proofs of many results below rely implicitly on the following *projection theorem*: *If $(\Omega, \mathscr{A})$ is complete, X is a separable and complete metric space and $A \in \mathscr{A} \otimes \mathscr{B}(X)$, then the projection $pr_\Omega(A) := \{\omega \in \Omega \mid (\omega, x) \in A$ for some $x \in X\} \in \mathscr{A}$* (see e.g. [11]).

If Ω is a locally compact σ-compact (e.g. locally compact separable metric) space and μ is a positive Radon measure (corresponding to some positive real linear functional on the space of all continuous real functions with compact support defined on Ω) – see e.g. [21, §11], see also [17, Chapter 7], then μ is a complete σ-finite regular measure defined on a σ-algebra $\mathscr{A}$ containing $\mathscr{B}(\Omega)$ (comp. [21, (11.34)]); in fact $(\mathscr{A},\mu)$ is the Lebesgue completion of $(\mathscr{B}(\Omega),\mu|_{\mathscr{B}(\Omega)})$ (7).

Having this we get an equivariant counterpart of the Kuratowski-Ryll-Nardzewski theorem [30] for convex-valued maps.

Theorem 6.2 *Assume that $(\Omega,\mathscr{A},\mu)$ is as in Assumption 6.1 and $\varphi:\Omega\multimap\mathbb{E}$, where $\mathbb{E}$ is a separable Banach representation of G, is $\mathscr{A}$-measurable with closed convex values and G-equivariant. Then:*

(1) there is a G-equivariant measurable selection $f:\Omega\to\mathbb{E}$ of φ, i.e., $f(\omega)\in\varphi(\omega)$ for all $\omega\in\Omega$;

(2) there is a sequence $\{f_n:\Omega\to\mathbb{E}\}$ of G-equivariant measurable maps such that $\varphi(\omega)=\overline{\{f_n(\omega)\}_{n=1}^\infty}$ for any $\omega\in\Omega$, provided G is a compact Lie group, Ω is separable metric and $\varphi(\omega)\subset\mathbb{E}^{G_\omega}$ for all $\omega\in\Omega$.

Proof: The measurability of φ implies that there is a sequence $h_n:\Omega\to\mathbb{E}$, $n\in\mathbb{N}$, of measurable maps such that $\varphi(\omega)=\overline{\{h_n(\omega)\}_{n=1}^\infty}$, $\omega\in\Omega$. Take $\varepsilon>0$ and fix an arbitrary $y\in\mathbb{E}$. For each $n\geqslant 1$, the function $\Omega\ni\omega\mapsto\widetilde{h}_n(\omega):=d(y,h_n(\omega))$ is measurable. In view of the Lusin theorem (see e.g. [17, Th.7.10]), for any $n\geqslant 1$ there is a closed set $\widetilde{\Omega}_n$ with $\mu(\Omega\setminus\widetilde{\Omega}_n)<2^{-n}\varepsilon$ such that the restriction $\widetilde{h}_n|_{\widetilde{\Omega}_n}$ is continuous. Then $\mu(\Omega\setminus\widetilde{\Omega}_\varepsilon)<\varepsilon$, where $\widetilde{\Omega}_\varepsilon=\bigcap_{n=1}^\infty\widetilde{\Omega}_n$. It is clear that $\widetilde{\Omega}_\varepsilon$ is closed and $\widetilde{h}_n|_{\widetilde{\Omega}_\varepsilon}$ is continuous for all $n\geqslant 1$. Since

$$d(y,\varphi(\omega))=\inf_{n\in\mathbb{N}}d(y,h_n(\omega))=\inf_{n\in\mathbb{N}}\widetilde{h}_n(\omega),\ \ \omega\in\Omega,$$

we see that the function $\widetilde{\Omega}_\varepsilon\ni\omega\mapsto d(y,\varphi(\omega))$ is upper semicontinuous (as the lower envelope of continuous functions), i.e., the restriction $\varphi|_{\widetilde{\Omega}_\varepsilon}$ is lower semicontinuous (8).

Now let $\Omega_\varepsilon:=G\widetilde{\Omega}_\varepsilon=\{g\omega\mid g\in G,\ \omega\in\widetilde{\Omega}_\varepsilon\}$. Then Ω_ε is G-invariant closed, $\mu(\Omega\setminus\Omega_\varepsilon)<\varepsilon$ and $\varphi|_{\Omega_\varepsilon}$ is lower semicontinuous. Indeed take $\omega_0\in\Omega_\varepsilon$, $y_0\in\varphi(\omega_0)$ and a (generalized) sequence $(\omega_\lambda)_{\lambda\in\Lambda}$ in Ω_ε such that $\omega_\lambda\to\omega_0$. For each $\lambda\in\Lambda$, there are $\widetilde{\omega}_\lambda\in\widetilde{\Omega}_\varepsilon$ and $g_\lambda\in G$ such that $\omega_\lambda=g_\lambda\widetilde{\omega}_\lambda$. Without loss of generality we may assume that $g_\lambda\to g_0\in G$. Hence $\widetilde{\omega}_\lambda=g_\lambda^{-1}\omega_\lambda\to\widetilde{\omega}_0:=g_0^{-1}\omega_0\in\widetilde{\Omega}_\varepsilon$ and, hence, $\omega_\lambda=g_\lambda\widetilde{\omega}_\lambda\to g_0\widetilde{\omega}_0=\omega_0$. Therefore $y_0\in\varphi(g_0\widetilde{\omega}_0)=g_0\varphi(\widetilde{\omega}_0)$. Since $\varphi|_{\widetilde{\Omega}_\varepsilon}$ is lower semicontinuous, if $\widetilde{y}_0:=g_0^{-1}y_0\in\varphi(\widetilde{\omega}_0)$ and $\widetilde{\omega}_\lambda\to\widetilde{\omega}_0$, for each $\lambda\in\Lambda$ there is $\widetilde{y}_\lambda\in\varphi(\widetilde{\omega}_\lambda)$ such that $\widetilde{y}_\lambda\to\widetilde{y}_0$. Therefore $y_\lambda:=g_\lambda\widetilde{y}_\lambda\in g_\lambda\varphi(\widetilde{\omega}_\lambda)=\varphi(g_\lambda\widetilde{\omega}_\lambda)=\varphi(\omega_\lambda)$ and $y_\lambda=g_\lambda\widetilde{y}_\lambda\to g_0\widetilde{y}_0=y_0$.

For any $n\geqslant 1$ choose a G-invariant closed set $\Omega_n\subset\Omega$ such that $\mu(\Omega\setminus\Omega_n)<1/n$ and $\varphi|_{\Omega_n}$ is lower semicontinuous. Let $\Omega_0:=\Omega\setminus\bigcup_{n=1}^\infty\Omega_n$. Clearly $\mu(\Omega_0)=0$ and Ω_0 is G-invariant. Let $f_0:\Omega_0\to\mathbb{E}$ be an arbitrary G-equivariant selection of $\varphi|_{\Omega_0}$ (resp. let $f_{0m}:\Omega_0\to\mathbb{E}$, $m\in\mathbb{N}$,

7 Observe that the Haar measure χ may be described in the same way (see [21, §15] or [17, Sec. 11.1]).

8 The provided argument is rather standard; however in order to keep the paper self-contained we decided to include it, see also [24].

be a sequence of arbitrary maps such that $\varphi(\omega) = \overline{\{f_{0m}(\omega)\}_{m=1}^{\infty}}$ for $\omega \in \Omega_0$) – see Remarks 3.1 (2) and 3.8 (2). In view of Theorem 3.2 (resp. 3.9 in the second case) for any $n \geqslant 1$ there is a G-equivariant continuous map $f : \Omega_n \to \mathbb{E}$ (resp. a sequence $f_{nm} : \Omega_n \to \mathbb{E}$, $m \in \mathbb{N}$, of continuous G-equivariant maps) such that $f_n(\omega) \in \varphi(\omega)$ (resp. $\varphi(\omega) = \overline{\{f_{nm}(\omega)\}_{m=1}^{\infty}}$) for every $\omega \in \Omega_n$.

Define $f : \Omega \to \mathbb{E}$ by

$$f(\omega) := \begin{cases} f_n(\omega) & \text{if} \quad \omega \in \Omega_n \setminus \bigcup_{k=1}^{n-1} \Omega_k; \\ f_0(\omega) & \text{if} \quad \omega \in \Omega_0; \end{cases}$$

respectively we define $f_m : \Omega \to \mathbb{E}$, $m \geqslant 1$, by

$$f_m(\omega) := \begin{cases} f_{nm}(\omega) & \text{if} \quad \omega \in \Omega_n \setminus \bigcup_{k=1}^{n-1} \Omega_k; \\ f_{0m}(\omega) & \text{if} \quad \omega \in \Omega_0. \end{cases}$$

Then the map f (resp. the sequence f_m, $m \geqslant 1$) is measurable (resp. consists of measurable maps) and fulfils the requirements of the assertion. □

Now, in order to get a partial answer to questions formulated at the beginning of this section, we shall address the problem of the existence of equivariant Carathéodory-type selections and graph-approximations of equivariant Carathéodory set-valued maps (comp. [18], [19], [29] and [36]). Recall that, given a topological space X and an *arbitrary* complete measure space $(\Omega, \mathscr{A}, \mu)$, a set-valued map $\varphi : \Omega \times X \multimap \mathbb{E}$, where $\mathbb{E}$ is a Banach space, is a *lower-Carathéodory* map (resp. *strict lower-Carathéodory* map) if it has closed values, for all $x \in X$, the map $\varphi(\cdot, x)$ is $\mathscr{A}$-measurable and, for μ-almost all $\omega \in \Omega$ (resp. for all $\omega \in \Omega$), the map $\varphi(\omega, \cdot) : X \multimap \mathbb{E}$ is lower semicontinuous. In a similar manner we define (strict) upper-Carathédory map. In particular a map $f : \Omega \times X \to \mathbb{E}$ is (strict) Carathéodory if for all $x \in X$, $f(\cdot, x)$ is measurable and (for all) for μ-almost all $\omega \in \Omega$, $f(\omega, \cdot)$ is continuous.

We say that $\varphi : \Omega \times X \multimap \mathbb{E}$ is *almost product measurable* (precisely *almost $\mathscr{A} \otimes \mathscr{B}(X)$-measurable*) if there is a μ-nullset $N \subset \Omega$ such that the restriction $\varphi|_{(\Omega \setminus N) \times X}$ is $\mathscr{A}_N \otimes \mathscr{B}(X)$-measurable, where $\mathscr{A}_N$ stands for the restriction of $\mathscr{A}$ to $\Omega \setminus N$ (i.e. $\mathscr{A}_N := \{A \cap (\Omega \setminus N) \mid A \in \mathscr{A}\}$), i.e., for each open $U \in Y$, the set $\{(\omega, x) \in (\Omega \setminus N) \times X \mid \varphi(\omega, x) \cap U \neq \emptyset\} \in \mathscr{A}$ ([9]).

In order to understand the nature of assumptions we are about to undertake, observe that:

Proposition 6.3 *(1) If φ is almost product measurable, then for any $x \in X$, $\varphi(\cdot, x)$ is $\mathscr{A}$-measurable;*

(2) if X is separable and $f : \Omega \times X \to \mathbb{E}$ is a Carathéodory map, then f is almost product measurable;

(3) if there is a sequence $(f_n)_{n=1}^{\infty}$ of Carathéodory maps $f_n : \Omega \times X \to \mathbb{E}$ such that $\varphi(\omega, x) = \overline{\{f_n(\omega, x)\}_{n=1}^{\infty}}$ for all $x \in X$ and μ-almost all $\omega \in \Omega$, then φ is lower Carathéodory; if, additionally, X is metrizable and separable, then φ is almost product measurable.

(4) If φ is almost product measurable and $x : \Omega \to X$ is measurable, then there is a measurable $y : \Omega \to \mathbb{E}$ such that $y(\omega) \in \varphi(\omega, x(\omega))$ for all $\omega \in \Omega$.

[9]Observe that $\mathscr{A}_N \times \mathscr{B}(X)$ is equal to the restriction of the σ-algebra $\mathscr{A} \otimes \mathscr{B}(X)$ to $(\Omega \setminus N) \times X$.

Proof: (1) Let $x \in X$ and let $i : \Omega \to \Omega \times X$ be given by $i(\omega) := (\omega, x)$. Then $i^{-1}(C) \in \mathscr{A}$ for any $C \in \mathscr{A} \otimes \mathscr{B}(X)$. Clearly, for an open $U \subset \mathbb{E}$, $\varphi(\cdot, x)^{-1}(U) = i^{-1}\varphi^{-1}(U) = \{\omega \in \Omega \setminus N \mid \varphi(\omega, x) \cap U \neq \emptyset\} \cup \{\omega \in N \mid \varphi(\omega, x) \cap U \neq \emptyset\} \in \mathscr{A}$ since $\{\omega \in N \mid \varphi(\omega, x) \cap U \neq \emptyset\} \in \mathscr{A}$ as a μ-nullset in a complete measure space.

(2) Since f is a Carathéodory map, there is a μ-nullset N such that $f|_{(\Omega\setminus N)\times X}$ is strict Carathéodory. In view of [4, Th. 8.2.6], $f|_{(\Omega\setminus N)\times X}$ is $\mathscr{A}_N \otimes \mathscr{B}(X)$-measurable.

(3) For each $n \in \mathbb{N}$, there is a μ-nullset $N_n \subset \Omega$ such that $f_n|_{(\Omega\setminus N_n)\times X}$ is strict Carathéodory; moreover there is a μ-nullset $N_0 \subset \Omega$ such that $\varphi(\omega, x) = \overline{\{f_n(\omega, x)\}_{n=1}^{\infty}}$ if $\omega \in \Omega \setminus N_0$. Let $N = \bigcup_{n=0}^{\infty} N_n$. Then $N \in \mathscr{A}$, $\mu(N) = 0$ and, for each $n \in \mathbb{N}$ and $f_n|_{(\Omega\setminus N)\times X}$ is strict Carathéodory and $\varphi(\omega, x) = \overline{\{f_n(\omega, x)\}_{n=1}^{\infty}}$ for $\omega \in \Omega \setminus N$ and $x \in X$. Let $\omega \in \Omega \setminus N$ and $y \in \mathbb{E}$. It is clear that, for each $x \in X$, $d(y, \varphi(\omega, x)) = \inf_{n\in\mathbb{N}} d(y, f_n(\omega, x))$. Therefore the function $d(y, \varphi(\omega, \cdot))$ is upper semicontinuous, i.e., $\varphi(\omega, \cdot)$ is lower semicontinuous. On the other hand, for any $x \in X$, the function $d(y, \varphi(\cdot, x))$ is measurable, i.e., $\varphi(\cdot, x)$ is measurable. Now suppose that X is metrizable and separable. For each $n \in \mathbb{N}$, $f_n|_{(\Omega\setminus N)\times X}$ is $\mathscr{A}_N \otimes \mathscr{B}(X)$-measurable in view of (2). Hence so is φ.

(4) Let $\widetilde{x} : \Omega \to \Omega \times X$ be defined by $\widetilde{x}(\omega) := (\omega, x(\omega))$, $\omega \in \Omega$, and let $N \subset \Omega$ be a μ-nullset such that $\varphi|_{(\Omega\setminus N)\times X}$ is $\mathscr{A}_N \otimes \mathscr{B}(X)$-measurable. We claim that if $C \in \mathscr{A}_N \otimes \mathscr{B}(X)$, then $\widetilde{x}^{-1}(C) \in \mathscr{A}_N$. Indeed if $C = A \times B$, where $A \in \mathscr{A}_N$ and $B \in \mathscr{B}(X)$, then $\widetilde{x}^{-1}(C) = A \cap x^{-1}(B) \in \mathscr{A}_N$. Let $U \subset \mathbb{E}$ be open. Then

$$\{\omega \in \Omega \setminus N \mid \varphi(\omega, x(\omega)) \cap U \neq \emptyset\} = \widetilde{x}^{-1}((\varphi|_{(\Omega\setminus N)\times X})^{-1}(U)) \in \mathscr{A}_N.$$

Hence, in view of the completeness of μ we see that $\varphi(\cdot, x(\cdot))^{-1}(U) \in \mathscr{A}$. The Kuratowski-Ryll-Nardzewski Theorem [30] implies the existence of a desired measurable y. □

Theorem 6.4 *Let $(\Omega, \mathscr{A}, \mu)$ be as in Assumption 6.1 and suppose that Ω is paracompact perfectly normal (e.g. metrizable), X is a complete separable metric G-space, $\mathbb{E}$ is a separable Banach representation of G and a set-valued map $\varphi : \Omega \times X \multimap \mathbb{E}$ with closed convex values is lower-Carathéodory, almost product measurable and G-equivariant. Then:*

(1) *there is a G-equivariant Carathéodory map $f : \Omega \times X \to \mathbb{E}$ such that $f(\omega, x) \in \varphi(\omega, x)$ for all $x \in X$ and $\omega \in \Omega$.*

(2) *there is a sequence $f_n : \Omega \times X \to \mathbb{E}$ of G-equivariant Carathéodory selections of φ such that $\varphi(\omega, x) = \overline{\{f_n(\omega, x)\}_{n=1}^{\infty}}$ for all $x \in X$ and all $\omega \in \Omega$, provided G is a compact Lie group, Ω is separable metric and $\varphi(\omega, x) \subset \mathbb{E}^{G_{(\omega,x)}}$ for all $\omega \in \Omega$ and $x \in X$* ([10]).

Proof: Here, instead of the Lusin property, we shall use a version of the Scorza Dragoni theorem (see [28] or [2]) which states that: *If $\varphi : \Omega \times X \multimap \mathbb{E}$ is a product measurable strict lower-Carathéodory map, then for each $\varepsilon > 0$, there is a closed $\Omega_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and $\varphi|_{\Omega_\varepsilon \times X}$ is lower semicontinuous* (of course in this statement the G-structure of Ω plays no role).

Now the proof goes similarly to the above one. We may assume that there is a μ-nullset N such that $\varphi|_{(\Omega\setminus N)\times X}$ is strict lower-Carathéodory and $\mathscr{A}_N \times \mathscr{B}(X)$-measurable.

For each $n \in \mathbb{N}$ there is a closed $\Omega_n \subset \Omega$ such that $\mu(\Omega \setminus \Omega_n) < 1/n$ and $\varphi|_{\Omega_n \times X}$ is

[10]Clearly $G_{(\omega,x)} = \{g \in G \mid g(\omega, x) = (\omega, x)\} = G_\omega \cap G_x$.

lower semicontinuous. Indeed, the regularity of μ shows that there is an open $V \subset \Omega$ such that $N \subset V$ and $\mu(V \setminus N) < 1/2n$. It is clear that $\varphi|_{(\Omega\setminus V)\times X}$ is strict Carathéodory and $\mathscr{A}_V \otimes \mathscr{B}(X)$-measurable. Therefore, in view of the above-mentioned Scorza Dragoni property, there is a closed subset $\Omega_n \subset \Omega \setminus V$ (thus Ω_n is closed in Ω) with $\mu(\Omega \setminus (V \cup \Omega_n)) < 1/2n$ (thus $\mu(\Omega \setminus \Omega_n) < 1/n$) such that $\varphi|_{\Omega_n \times X}$ is lower semicontinuous.

Let $\Omega_0 := \Omega \setminus \bigcup_{n=1}^{\infty} \Omega_n$; then $\mu(\Omega_0) = 0$ and $N \subset \Omega_0$. Arguing as in the proof of Theorem 6.2, we may assume with no loss of generality that Ω_n and, thus, Ω_0 are G-invariant. Since the product $\Omega_n \times X$ is paracompact (resp. separable metrizable), in view of Theorem 3.2 (resp. 3.9), for each $n \geqslant 1$ there is a continuous G-equivariant map $f_n : \Omega_n \times X \to \mathbb{E}$ such that $f_n(\omega, x) \in \varphi(\omega, x)$ for all $\omega \in \Omega_n$ and $x \in X$ (resp. there is a sequence $f_{nm} : \Omega_n \times X \to \mathbb{E}$, $m \geqslant 1$, of continuous G-equivariant maps such that $\overline{\{f_{nm}(\omega, x)\}_{m=1}^{\infty}} = \varphi(\omega, x)$ on $\Omega_n \times X$). Additionally we find an arbitrary G-equivariant selection $f_0 : \Omega_0 \times X \to \mathbb{E}$ of $\varphi|_{\Omega_0 \times X}$ (resp. a sequence (f_{0m}), $f_{0m} : \Omega_0 \times X \to \mathbb{E}$, of G-equivariant selections of $\varphi|_{\Omega_0 \times X}$ such that $\varphi(\omega, x) = \overline{\{f_{0m}(\omega, x)\}_{m=1}^{\infty}}$ for $\omega \in \Omega_0$ and $x \in X$) by the use of Remark 3.1 (2) (resp. 3.8 (2)).

Define $f : \Omega \times X \to \mathbb{E}$ by

$$f(\omega, x) := \begin{cases} f_n(\omega, x) & \text{if} \quad \omega \in \Omega_n \setminus \bigcup_{k=1}^{n-1} \Omega_k, \ x \in X; \\ f_0(\omega, x) & \text{if} \quad \omega \in \Omega_0, \ x \in X; \end{cases} \tag{5}$$

respectively we define $f_m : \Omega \times X \to \mathbb{E}$, $m \geqslant 1$, by

$$f_m(\omega) := \begin{cases} f_{nm}(\omega, x) & \text{if} \quad \omega \in \Omega_n \setminus \bigcup_{k=1}^{n-1} \Omega_k, \ x \in X; \\ f_{0m}(\omega, x) & \text{if} \quad \omega \in \Omega_0, \ x \in X. \end{cases}$$

It is again clear that this completes the proof. □

Now we turn our attention to equivariant Carathéodory approximations (comp. [29]).

Theorem 6.5 *Let Assumption 6.1 be satisfied, Ω be a paracompact perfectly normal G-space, X a separable complete metric G-space and $\varphi : \Omega \times X \multimap \mathbb{E}$, where $\mathbb{E}$ is a separable Banach representation of G, be a G-equivariant almost product measurable upper-Carathéodory set-valued map with compact convex values. For any neighborhood $\mathcal{U}$ of $\mathrm{Gr}(\varphi)$, there is a G-equivariant Carathéodory map $f : \Omega \times X \to \mathbb{E}$ such that $\mathrm{Gr}(f) \subset \mathcal{U}$.*

Proof: We use an important result from [38] stating that: *A strict upper-Carathéodory map $\varphi : \Omega \times X \multimap \mathbb{E}$ with compact values has the Scorza Dragoni property: for any $\varepsilon > 0$ there is a closed $\Omega_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and $\varphi|_{\Omega_\varepsilon \times X}$ is upper semicontinuous if and only if φ is product measurable* (here again the G-structure of Ω has no significance).

The proof now is almost the same as before: for any $n \in \mathbb{N}$, we choose a closed G-invariant subset $\Omega_n \subset \Omega$ with $\mu(\Omega \setminus \Omega_n) < 1/n$ such that $\varphi|_{\Omega_n \times X}$ is upper semicontinuous. In view of Theorem 4.7 and Corollary 4.8, for each $n \in \mathbb{N}$, there is a continuous G-invariant $f_n : \Omega_n \times X \to \mathbb{E}$ such that $(u, f_n(u)) \in \mathcal{U}$ for $u \in \Omega_n \times X$. Additionally let $f_0 : \Omega_0 \times X \to \mathbb{E}$ be an arbitrary G-equivariant selection of $\varphi|_{\Omega_0 \times X}$, where $\Omega_0 = \Omega \setminus \bigcup_{n=1}^{\infty} \Omega_n$. Defining $f : \Omega \times X \to \mathbb{E}$ by (5) we complete the proof. □

Combining Theorems 6.4 and 6.5 we get the following 'Carathéodory' counterpart of Theorem 4.13 (comp. [33]).

Corollary 6.6 *Let Ω be as in Assumption 6.1 and, additionally, perfectly normal paracompact, let X be a separable metric space and $\mathbb{E}$ a separable Banach representation of G. If $\varphi : \Omega\times X \multimap \mathbb{E}$ is upper-Carathéodory G-equivariant and almost product measurable with compact convex values, $\psi : \Omega \times X \multimap \mathbb{E}$ is lower-Carathéodory G-equivariant and almost product measurable with closed convex values, $\mathcal{U}$ is a neighborhood of $\mathrm{Gr}(\varphi)$ and $\varphi(\omega, x)\cap\psi(\omega, x) \neq \emptyset$ for all $\omega \in \Omega$ and $x \in X$, then there exists a G-equivariant Carathéodory map $f : \Omega \times X \multimap \mathbb{E}$ such that $\mathrm{Gr}(f) \subset \mathcal{U}$ and f is a selection of ψ.* □

For the sake of completeness we include a result allowing to get the existence of strict Carathéodory selections.

Theorem 6.7 *Suppose that X is a σ-compact metric G-space, $\mathbb{E}$ is a separable Banach representation of G and $(\Omega, \mathscr{A})$ satisfies Assumption 6.1. Let $\varphi : \Omega \times X \multimap \mathbb{E}$ be a G-equivariant strict lower-Carathéodory and product measurable map with closed convex values. Then:*

(1) *there exists a G-equivariant strict Carathéodory map $F : \Omega\times X \to \mathbb{E}$ such that, for each $\omega \in \Omega$ and $x \in X$, $F(\omega, x) \in \varphi(\omega, x)$;*

(2) *there is a sequence $F_n : \Omega \times X \to \mathbb{E}$, $n \in \mathbb{N}$, of G-equivariant strict Carathéodory maps such that $\varphi(\omega, x) = \overline{\{F_n(\omega, x)\}_{n=1}^{\infty}}$ for all $\omega \in \Omega$ and $x \in X$, provided G is a compact Lie group, Ω is a separable metric space space and $\varphi(\omega, x) \subset \mathbb{E}^{G_{(\omega,x)}}$ for any $\omega \in \Omega$ and $x \in X$.*

Proof: In the proof we follow the ideas from [18] and [29]. Assume for a while that X is compact. Then $\mathbb{F} := \mathcal{C}(X, \mathbb{E})$, the space of all continuous maps, is a separable Banach space (with the usual sup-norm $\|\cdot\|_\infty$) and a representation of G with the G-action described above: $(g \cdot f)(x) := gf(g^{-1}x)$ for $f \in \mathbb{F}$, $g \in G$ and $x \in X$.

(1) Consider a set-valued map $\Phi : \Omega \multimap \mathbb{F}$ given by

$$\Phi(\omega) := \{f \in \mathbb{F} \mid f(x) \in \varphi(\omega, x) \text{ for all } x \in X\}, \ \ \omega \in \Omega.$$

In view of the Michael theorem, Φ has nonempty values; moreover $\Phi(\omega)$ is closed and convex for any $\omega \in \Omega$.

We claim that Φ is measurable. Since $(\Omega, \mathscr{A})$ is complete, it is sufficient to show that if $D := \{f \in \mathbb{F} \mid \|f - f_0\| \leqslant \varepsilon\}$ is the closed ε-ball centered at f_0 in $\mathbb{F}$, where $f_0 \in \mathbb{F}$ and $\varepsilon > 0$, then $\Phi^{-1}(D) \in \mathscr{A}$. If $\omega \in \Phi^{-1}(D)$, then there is $f \in \Phi(\omega) \cap D$ such that, for all $x \in X$, $f(x) \in \varphi(\omega, x) \cap D(f_0(x), \varepsilon)$, i.e.,

$$\Phi^{-1}(D) \subset A := \bigcap_{x\in X} \{\omega \in \Omega \mid \varphi(\omega, x) \cap D(f_0(x), \varepsilon) \neq \emptyset\}.$$

On the other hand, if $\omega \in A$, then the map $X \ni x \mapsto \varphi(\omega, x) \cap D(f_0(x), \varepsilon)$ (with nonempty values) is lower semicontinuous with closed convex values and, in view of the Michael theorem, there is $f \in \mathbb{F}$ such that $f(x) \in D(f_0(x), \varepsilon) \cap \varphi(\omega, x)$, i.e., $A \subset \Phi^{-1}(D)$. It is clear that the map $\Omega \times X \ni (\omega, x) \to \varphi(\omega, x) - f_0(x)$ is product measurable, hence

$$\begin{aligned} C &:= \{(\omega, x) \in \Omega \times X \mid \varphi(\omega, x) \cap D(f_0(x), \varepsilon) = \emptyset\} \\ &= \{(\omega, x) \mid [\varphi(\omega, x) - f_0(x)] \cap D(0, \varepsilon) = \emptyset\} \in \mathscr{A} \otimes \mathscr{B}(X). \end{aligned}$$

Now observe that (see footnote on page 21)

$$\Omega \setminus A = \bigcup_{x \in X} \{\omega \in \Omega \mid \varphi(\omega, x) \cap D(f_0(x), \varepsilon) = \emptyset\} = pr_\Omega(C) \in \mathscr{A},$$

i.e., $\Phi^{-1}(D) = A \in \mathscr{A}$.

Next we claim that Φ is G-equivariant. If $g \in G$ and $f \in \Phi(g\omega)$, then $f(gx) \in \varphi(g\omega, gx) = g\varphi(\omega, x)$ for all $x \in X$; therefore $g^{-1}f(gx) \in \varphi(\omega, g^{-1}x)$, i.e., $(g^{-1} \cdot f)(x) \in \varphi(\omega, x)$ for all $x \in X$. This means that $(g^{-1} \cdot f) \in \Phi(\omega)$ and $f \in g \cdot \Phi(\omega)$.

Theorem 6.2 implies that there is a measurable G-equivariant $s : \Omega \to \mathbb{F}$ such that $s(\omega) \in \Phi(\omega)$ for any $\omega \in \Omega$. Let $F : \Omega \times X \to \mathbb{E}$ be defined by $F(\omega, x) := s(\omega)(x)$ for $\omega \in \Omega$ and $x \in X$. For $\omega \in \Omega$, $F(\omega, \cdot) = s(\omega)$ is continuous; for $x \in X$, $F(\cdot, x) = s(\cdot)(x) = e_x \circ s$, where $e_x : \mathbb{F} \ni f \mapsto f(x) \in \mathbb{E}$ is the evaluation, is measurable since e_x is continuous. Thus F is a strict Carathéodory map and $F(\omega, x) \in \varphi(\omega, x)$ for all $\omega \in \Omega$ and $x \in X$. Finally we check that F is G-equivariant. Indeed, if $\omega \in \Omega$, $x \in X$ and $g \in G$, then $F(g\omega, gx) = s(g\omega)(gx) = (g \cdot s(\omega))(gx) = gs(\omega)(x) = gF(\omega, x)$.

(2) Let $\Psi : \Omega \multimap \mathbb{F}$ be given by

$$\Psi(\omega) := \{f \in \mathbb{F} \mid \ f \ \text{ is } G_\omega\text{-equivariant and } \ f(x) \in \varphi(\omega, x) \ \text{ for all } \ x \in X\}, \ \ \omega \in \Omega,$$

where G_ω is the stabilizer of $\omega \in \Omega$, i.e., $f \in \Psi(\omega)$ provided $f \in \mathbb{F}^{G_\omega} = \{f \in \mathbb{F} \mid G_f \supset G_\omega\}$ (note that $f \in \mathbb{F}^{G_\omega}$ if and only if $gf(x) = f(gx)$ for $x \in X$ and $g \in G_\omega$) and f is a selection of $\varphi(\omega, \cdot)$. In other words we see that $\Psi(\omega) = \Phi(\omega) \cap \mathbb{F}^{G_\omega}$. It is easy to see that values of Ψ are nonempty (in view of Theorem 3.2) closed and convex. Observe that the map $\Omega \ni \omega \mapsto \mathbb{F}^{G_\omega} \subset \mathbb{F}$ is G-equivariant lower semicontinuous with closed convex values (see the first step in the proof of Theorem 3.9) and, hence, measurable. This implies that so is Ψ. In view of Theorem 6.2 (2) there is a sequence $s_n : \Omega \to \mathbb{F}$, $n \geqslant 1$, of measurable G-equivariant maps such that $\overline{\{s_n(\omega)\}_{n=1}^\infty} = \Psi(\omega)$ for $\omega \in \Omega$, $x \in X$. Let $F_n : \Omega \times X \to \mathbb{E}$, $n \geqslant 1$, be given by $F_n(\omega, x) := s_n(\omega)(x)$ for $\omega \in \Omega$ and $x \in X$. As above we show that each F_n, $n \geqslant 1$, is a G-equivariant strict Carathéodory map. Moreover the density of $\{F_n(\omega, x)\}_{n=1}^\infty$ in $\varphi(\omega, x)$ for $\omega \in \Omega$ and $x \in X$ follows from the fact that for any $y \in \varphi(\omega, x)$ there is $f \in \Psi(\omega)$ such that $f(x) = y$ – see Proposition 3.7 with φ replaced by $\varphi(\omega, \cdot)$ and G by G_ω observing that $\mathbb{E}$ is a Banach representation of G_ω and, in this setting, $\mathbb{E}^{G_x} = \mathbb{E}^{G_{(\omega,x)}}$.

Now let X be σ-compact, i.e., there is an increasing sequence X_n of compact subsets in X such that $X = \bigcup_{n=1}^\infty X_n$. In case (1) we have a G-equivariant strict Carathéodory map $F_1 : \Omega \times X_1 \to \mathbb{E}$ being a selection of $\varphi_1 := \varphi|_{\Omega \times X_1}$. We define $\varphi_2 : \Omega \times X_2 \multimap \mathbb{E}$ by

$$\varphi_2(\omega, x) := \begin{cases} F_1(\omega, x) & \text{if} \quad (\omega, x) \in \Omega \times X_1; \\ \varphi(\omega, x) & \text{if} \quad \omega \in \Omega, x \in X_2 \setminus X_1. \end{cases}$$

It is easy to see that φ_2 is a G-equivariant strict lower-Carathéodory map with closed convex values. In view of the first part of the proof, there is a G-equivariant strict Carathéodory map $F_2 : \Omega \times X_2 \to \mathbb{E}$ being a selection of φ_2. Continuing this procedure inductively we get a sequence $F_n : \Omega \times X_n \to \mathbb{E}$, $n \geqslant 1$, of G-equivariant strict Carathéodory selections of $\varphi|_{\Omega \times X_n}$ such that $F_n|_{\Omega \times X_{n-1}} = F_{n-1}$ for all $n \geqslant 2$. Finally we may define $F : \Omega \times X \to \mathbb{E}$ putting $F(\omega, x) = F_n(\omega, x)$ provided $\omega \in \Omega$ and $x \in X_n$. In the case (2) one may employ the same procedure in order to get the assertion. $\square$

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