TORSION OF RESTRAINED THIN-WALLED BARS OF OPEN CONSTANT BISYMMETRIC CROSS-SECTION

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Abstract: Elastic and geometric stiffness matrices were derived using Castigliano's first theorem, for the case of torsion of restrained thin-walled bars of open constant bisymmetric cross-section. Functions which describe the angles of torsion were adopted from the solutions of the differential equation for restrained torsion. The exact solutions were simplified by expanding them in a power series. Numerical examples were taken from Kujawa M 2009 *Static and Sensitivity Analysis of Grids...* **97**, GUT Publishing House, and Szymczak C 1978 *Engineering Transaction* **26** 323. Convergence of the solutions was analyzed using the matrices derived for torsion angles, warping, bimoments and critical forces.

Keywords: thin-walled bars, elastic stiffness matrix, geometric stiffness matrix, restrained torsion, Castigliano's method, torsional buckling

1. Introduction

Elastic and geometric stiffness matrices for a thin-walled bar of a constant open cross-section have already been derived in [1] using variational principles, and in [2] using the principle of virtual displacement. In both of these works the torsion angle was described by third-order polynomials. These polynomials did not result from the solution of the differential equation of restrained torsion. In this article, the function of the torsion angle was adopted from the solution of the differential equation of restrained torsion of thin-walled bars of open constant bisymmetric cross-section. Matrices were derived using Castigliano's first theorem [3, 4]. The analysis was performed using commercial computing packages – Mathematica [5] and Matlab [6].

2. Elastic stiffness matrix

Let us consider an element ik cut from a thin-walled bar of open cross-section. The element has a length l, constant cross-section A and is made of a homogeneous material with a Young's modulus E. For the element ik let us define a function of the torsion angle based on the known differential equation of restrained torsion [7]:

$$\frac{d^4\theta}{dx^4} - \kappa^2 \frac{d^2\theta}{dx^2} = 0 \tag{1}$$

where $\kappa = \sqrt{GJ_d/EJ_\omega}$ is a characteristic number which depends on the ratio of the pure torsional rigidity GJ_d, in the sense of the Saint-Venant theory, to the sectoral rigidity EJ_ω. The general solution of the Equation (1) is:

$$\theta(x) = (e^{\kappa x}/\kappa^2)\overline{C}_1 + (e^{-\kappa x}/\kappa^2)\overline{C}_2 + x\ C_3 + C_4 \tag{2}$$

By transforming function (2) from the exponential into a hyperbolic form and by adopting new variables C_1 and C_2 instead of $(\overline{C}_1 + \overline{C}_2)/\kappa^2$, $(\overline{C}_1 - \overline{C}_2)/\kappa^2$, we obtain the following expression for the torsion angle:

$$\theta(x) = \cosh(\kappa x)C_1 + \sinh(\kappa x)C_2 + x\ C_3 + C_4 \tag{3}$$

By substituting the boundary conditions of the form:

$$\theta = \theta_i,\, \theta' = \theta'_i \quad \text{for} \quad x = 0 \tag{4}$$

$$\theta = \theta_k,\, \theta' = \theta'_k \quad \text{for} \quad x = l \tag{5}$$

at both ends of the bar ik into Equation (3), we determine the values of the constants C_1, C_2, C_3, C_4. Let us write the expression (3) in matrix form:

$$\theta(x) = \boldsymbol{\Phi}^T \boldsymbol{C}\ \boldsymbol{q} \tag{6}$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{q}$ are vectors of the form:

$$\boldsymbol{\Phi}^T = \{\cosh(\kappa x), \sinh(\kappa x), x, 1\} \tag{7}$$

$$\boldsymbol{q}^T = \{\theta_i, \theta'_i, \theta_k, \theta'_k\} \tag{8}$$

and $\boldsymbol{C}$ is a matrix of elements depending on the characteristic number κ and on the length of the bar l. The elastic energy is given by:

$$U = \frac{1}{2}\left\{EJ_\omega \int_0^l \left[(\theta'')^2 + \kappa^2(\theta')^2\right] dx\right\} \tag{9}$$

By substituting the previously derived expression for the torsion angle (6) into relation (9), we obtain:

$$\frac{1}{2}\left\{\boldsymbol{q}^T EJ_\omega \int_0^l \left[(\boldsymbol{C}^T\boldsymbol{\Phi}'')(\boldsymbol{\Phi}'')^T\boldsymbol{C} + \kappa^2(\boldsymbol{C}^T\boldsymbol{\Phi}')(\boldsymbol{\Phi}')^T\boldsymbol{C}\right] dx\ \boldsymbol{q}\right\} = \frac{1}{2}\boldsymbol{q}^T\boldsymbol{k}_L\boldsymbol{q} \tag{10}$$

where $\boldsymbol{k}_L$ is an unknown elastic stiffness matrix.

From Castigliano's theorem we know that the derivative of a linear function of the elastic energy of the nodal displacement vector $\boldsymbol{q}$ corresponds to the vector of nodal forces $\boldsymbol{f}^T = \{M_{Si}, B_i, M_{Sk}, B_k\}$:

$$\frac{\partial}{\partial \boldsymbol{q}} = \left(\frac{1}{2}\boldsymbol{q}^T \boldsymbol{k}_L \boldsymbol{q}\right) = \boldsymbol{k}_L \boldsymbol{q} = \boldsymbol{f} \tag{11}$$

where M_{Si} and M_{Sk} denote the torsional moments at nodes ik, and B_i, B_k are the bimoments at these nodes.

By determining the individual components of the stiffness matrix arising from relation (10), we find the elastic stiffness matrix in the torsion form:

$$\boldsymbol{k}_L = \begin{bmatrix} [k_{L,11}] & [k_{L,12}] \\ \text{sym.} & [k_{L,22}] \end{bmatrix} \tag{12}$$

related to the internal forces by the relation:

$$\begin{Bmatrix} M_{Si} \\ B_i \\ M_{Sk} \\ B_k \end{Bmatrix} = \begin{bmatrix} [k_{L,11}] & [k_{L,12}] \\ \text{sym.} & [k_{L,22}] \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_i' \\ \theta_k \\ \theta_k' \end{Bmatrix} \tag{13}$$

where:

$$[k_{L,11}] = EJ_\omega \begin{bmatrix} \frac{\kappa^3}{\kappa l - 2\tanh\left(\frac{\kappa l}{2}\right)} & \frac{\kappa^2}{\kappa l \coth\left(\frac{\kappa l}{2}\right) - 2} \\ \frac{\kappa^2}{\kappa l \coth\left(\frac{\kappa l}{2}\right) - 2} & \frac{\kappa\left(\kappa l \cosh(\kappa l) - \sinh(\kappa l)\right)}{-2\cosh(\kappa l) + \kappa l \sinh(\kappa l) + 2} \end{bmatrix} \tag{14}$$

$$[k_{L,12}] = EJ_\omega \begin{bmatrix} \frac{\kappa^3}{2\tanh\left(\frac{\kappa l}{2}\right) - \kappa l} & \frac{\kappa^2}{\kappa l \coth\left(\frac{\kappa l}{2}\right) - 2} \\ \frac{\kappa^2}{2 - \kappa l \coth\left(\frac{\kappa l}{2}\right)} & \frac{\kappa(\sinh(\kappa l) - \kappa l)}{-2\cosh(\kappa l) + \kappa l \sinh(\kappa l) + 2} \end{bmatrix} \tag{15}$$

$$[k_{L,21}] = EJ_\omega \begin{bmatrix} \frac{\kappa^3}{2\tanh\left(\frac{\kappa l}{2}\right) - \kappa l} & \frac{\kappa^2}{2 - \kappa l \coth\left(\frac{\kappa l}{2}\right)} \\ \frac{\kappa^2}{\kappa l \coth\left(\frac{\kappa l}{2}\right) - 2} & \frac{\kappa(\sinh(\kappa l) - \kappa l)}{-2\cosh(\kappa l) + \kappa l \sinh(\kappa l) + 2} \end{bmatrix} \tag{16}$$

$$[k_{L,22}] = EJ_\omega \begin{bmatrix} \frac{\kappa^3}{\kappa l - 2\tanh\left(\frac{\kappa l}{2}\right)} & \frac{\kappa^2}{2 - \kappa l \coth\left(\frac{\kappa l}{2}\right)} \\ \frac{\kappa^2}{2 - \kappa l \coth\left(\frac{\kappa l}{2}\right)} & \frac{\kappa\left(\kappa l \cosh(\kappa l) - \sinh(\kappa l)\right)}{-2\cosh(\kappa l) + \kappa l \sinh(\kappa l) + 2} \end{bmatrix} \tag{17}$$

By expanding the hyperbolic functions in (12) in a power series in κ, we obtain a simplified form of the stiffness matrix. In the first approximation we truncated the expansion of the simplified stiffness matrix $\boldsymbol{k}_{2Lu}$ at κ^2:

$$\boldsymbol{k}_{2Lu} = \begin{bmatrix} [k_{2Lu,11}] & [k_{2Lu,12}] \\ \text{sym.} & [k_{2Lu,22}] \end{bmatrix} \tag{18}$$

where:

$$[k_{2Lu,11}] = EJ_\omega \begin{bmatrix} \frac{12}{l^3} + \frac{6\kappa^2}{5l} & \frac{6}{l^2} + \frac{\kappa^2}{10} \\ \frac{6}{l^2} + \frac{\kappa^2}{10} & \frac{4}{l} + \frac{2l\kappa^2}{15} \end{bmatrix} \tag{19}$$

$$[k_{2Lu,12}] = EJ_\omega \begin{bmatrix} -\frac{12}{l^3} - \frac{6\kappa^2}{5l} & \frac{6}{l^2} + \frac{\kappa^2}{10} \\ -\frac{6}{l^2} - \frac{\kappa^2}{10} & \frac{2}{l} - \frac{l\kappa^2}{30} \end{bmatrix} \tag{20}$$

$$[k_{2Lu,21}] = EJ_\omega \begin{bmatrix} -\frac{12}{l^3} - \frac{6\kappa^2}{5l} & -\frac{6}{l^2} - \frac{\kappa^2}{10} \\ \frac{6}{l^2} + \frac{\kappa^2}{10} & \frac{2}{l} - \frac{l\kappa^2}{30} \end{bmatrix} \tag{21}$$

$$[k_{2Lu,22}] = EJ_\omega \begin{bmatrix} \frac{12}{l^3} + \frac{6\kappa^2}{5l} & -\frac{6}{l^2} - \frac{\kappa^2}{10} \\ -\frac{6}{l^2} - \frac{\kappa^2}{10} & \frac{4}{l} + \frac{2l\kappa^2}{15} \end{bmatrix} \tag{22}$$

The resulting simplified stiffness matrix (18) corresponds to the matrices simplified and derived by Barsoum and Gallagher [1] and Meek *et al.* [2], where third-order polynomials were assumed as the functions describing the torsion angle. The elastic stiffness matrices $\boldsymbol{k}_L$ (12) and $\boldsymbol{k}_{2Lu}$ (18) derived in this article were also consistent with the matrices derived by Szymczak [4]. We also propose a power series approximation to fourth order. The elastic stiffness matrix (12) simplified in this way has the form:

$$\boldsymbol{k}_{4Lu} = \begin{bmatrix} [k_{4Lu,11}] & [k_{4Lu,12}] \\ \text{sym.} & [k_{4Lu,22}] \end{bmatrix} \tag{23}$$

where:

$$[k_{4Lu,11}] = EJ_\omega \begin{bmatrix} \frac{12}{l^3} + \frac{6\kappa^2}{5l} - \frac{l\kappa^4}{700} & \frac{6}{l^2} + \frac{\kappa^2}{10} - \frac{l^2\kappa^4}{1400} \\ \frac{6}{l^2} + \frac{\kappa^2}{10} - \frac{l^2\kappa^4}{1400} & \frac{4}{l} + \frac{2l\kappa^2}{15} - \frac{11l^3\kappa^4}{6300} \end{bmatrix} \tag{24}$$

$$[k_{4Lu,12}] = EJ_\omega \begin{bmatrix} -\frac{12}{l^3} - \frac{6\kappa^2}{5l} + \frac{l\kappa^4}{700} & \frac{6}{l^2} + \frac{\kappa^2}{10} - \frac{l^2\kappa^4}{1400} \\ -\frac{6}{l^2} - \frac{\kappa^2}{10} + \frac{l^2\kappa^4}{1400} & \frac{2}{l} - \frac{l\kappa^2}{30} + \frac{13l^3\kappa^4}{12600} \end{bmatrix} \tag{25}$$

$$[k_{4Lu,21}] = EJ_\omega \begin{bmatrix} -\frac{12}{l^3} - \frac{6\kappa^2}{5l} + \frac{l\kappa^4}{700} & -\frac{6}{l^2} - \frac{\kappa^2}{10} + \frac{l^2\kappa^4}{1400} \\ \frac{6}{l^2} + \frac{\kappa^2}{10} - \frac{l^2\kappa^4}{1400} & \frac{2}{l} - \frac{l\kappa^2}{30} + \frac{13l^3\kappa^4}{12600} \end{bmatrix} \tag{26}$$

$$[k_{4Lu,22}] = EJ_\omega \begin{bmatrix} \frac{12}{l^3} + \frac{6\kappa^2}{5l} - \frac{l\kappa^4}{700} & -\frac{6}{l^2} - \frac{\kappa^2}{10} + \frac{l^2\kappa^4}{1400} \\ -\frac{6}{l^2} - \frac{\kappa^2}{10} + \frac{l^2\kappa^4}{1400} & \frac{4}{l} + \frac{2l\kappa^2}{15} - \frac{11l^3\kappa^4}{6300} \end{bmatrix} \tag{27}$$

3. Geometric stiffness matrix

In a similar manner, we can determine the geometric stiffness matrix $\boldsymbol{k}_G$. The strain ε_x towards the axis of the bar for a finite torsion angle $\theta(x)$ can be calculated from the general theory of elasticity:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{28}$$

where u, v, w are the displacement vector components in the direction of the x, y and z axes, respectively. Based on the linear theory of restrained torsion [7], the first term of the expression (28) can be written as:

$$\frac{\partial u}{\partial x} = u'_o - \omega\theta'' \tag{29}$$

where u_o is the axial strain on the bar, and ω is the sectoral area. Substituting the relation (29) into expression (28) and taking into account the relations (*cf.* Figure 1):

$$v = z\theta, \quad w = y\theta, \quad z^2 + y^2 = r^2 \tag{30}$$

we obtain:

$$\varepsilon_x = u'_o - \omega\theta'' + \frac{1}{2}r^2(\theta')^2 \tag{31}$$

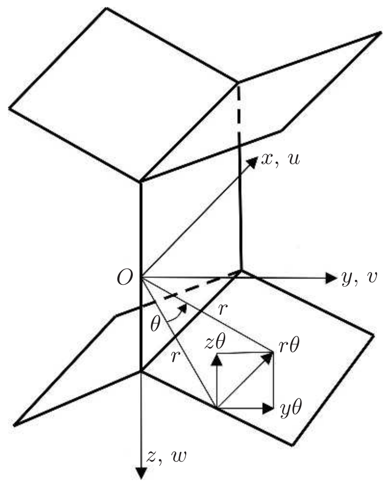

Figure 1. Bar sector ik

Thus the elastic energy of internal forces would be:

$$U = \frac{1}{2}\left\{ EJ_o \int_0^l \left[u_o'(\theta')^2\right] dx \right\} \tag{32}$$

where J_o is the polar moment of inertia, relative to the center of gravity of the cross-section, and u_o' is:

$$u_o' = \frac{P}{EA} \tag{33}$$

where P is the axial force. By substituting the previously derived expression for the angle of torsion in the matrix form (6) into relation (32), we obtain:

$$\frac{1}{2}\boldsymbol{q}^T P r^2 \int_0^l \left[(\boldsymbol{C}^T \boldsymbol{\Phi}')(\boldsymbol{\Phi}')^T \boldsymbol{C}\right] dx \;\; \boldsymbol{q} = \frac{1}{2}\boldsymbol{q}^T \boldsymbol{k}_G \boldsymbol{q} \tag{34}$$

where $r^2 = J_o/A$ denotes the squared radius of inertia of the cross-section, relative to the origin of the coordinate system, and $\boldsymbol{k}_G$ is the geometric stiffness matrix. By determining the individual components of the matrix resulting from formula (34), we obtain the geometric stiffness matrix:

$$\boldsymbol{k}_G = \mathrm{Pr}^2 \begin{bmatrix} k_{G,11} & k_{G,12} & k_{G,13} & k_{G,14} \\ & k_{G,22} & k_{G,23} & k_{G,24} \\ & & k_{G,33} & k_{G,34} \\ \text{sym.} & & & k_{G,44} \end{bmatrix} \tag{35}$$

related to the internal forces by the relation:

$$\begin{Bmatrix} M_{Si} \\ B_i \\ M_{Sk} \\ B_k \end{Bmatrix} = \mathrm{Pr}^2 \begin{bmatrix} k_{G,11} & k_{G,12} & k_{G,13} & k_{G,14} \\ & k_{G,22} & k_{G,23} & k_{G,24} \\ & & k_{G,33} & k_{G,34} \\ \text{sym.} & & & k_{G,44} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_i' \\ \theta_k \\ \theta_k' \end{Bmatrix} \tag{36}$$

where:

$$[k_{G,11}] = \frac{\kappa\left(\kappa l(\cosh(\kappa l)+2) - 3\sinh(\kappa l)\right)}{2\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{37}$$

$$[k_{G,12}] = \frac{\kappa^2 l^2 + \kappa l \sinh(\kappa l) - 4\cosh(\kappa l) + 4}{4\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{38}$$

$$[k_{G,13}] = \frac{\kappa\left(3\sinh(\kappa l) - \kappa l(\cosh(\kappa l) + 2)\right)}{2\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{39}$$

$$[k_{G,14}] = \frac{\kappa^2 l^2 + \kappa l \sinh(\kappa l) - 4\cosh(\kappa l) + 4}{4\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{40}$$

$$[k_{G,22}] = \frac{2\left(\kappa^2 l^2 + 2\right)\sinh(\kappa l) - \kappa l\left(8\cosh(\kappa l) + \kappa l(\kappa l - 3\sinh(\kappa l))\operatorname{csch}^2\left(\frac{\kappa l}{2}\right) + 4\right)}{8\kappa\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{41}$$

$$[k_{G,23}] = -\frac{\kappa^2 l^2 + \kappa l \sinh(\kappa l) - 4\cosh(\kappa l) + 4}{4\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{42}$$

$$\begin{aligned}[k_{G,24}] = &-\frac{\left(\kappa l \coth\left(\frac{\kappa l}{2}\right) - 2\right)^2\left(\left(3\kappa^2 l^2 - 2\right)\sinh(\kappa l) - \kappa l\left(\kappa^2 l^2 + 6\right)\cosh(\kappa l)\right)}{8\kappa\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^4} \\ &-\frac{\left(\kappa l \coth\left(\frac{\kappa l}{2}\right) - 2\right)^2\left(6\kappa l + \sinh(2\kappa l)\right)}{8\kappa\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^4}\end{aligned} \tag{43}$$

$$[k_{G,33}] = \frac{\kappa\left(\kappa l(\cosh(\kappa l) + 2) - 3\sinh(\kappa l)\right)}{2\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{44}$$

$$[k_{G,34}] = -\frac{\kappa^2 l^2 + \kappa l \sinh(\kappa l) - 4\cosh(\kappa l) + 4}{4\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{45}$$

$$[k_{G,44}] = \frac{2(\kappa^2 l^2 + 2)\sinh(\kappa l) - \kappa l\left(8\cosh(\kappa l) + \kappa l(\kappa l - 3\sinh(\kappa l))\operatorname{csch}^2\left(\frac{\kappa l}{2}\right) + 4\right)}{8\kappa\left(\kappa l \cosh\left(\frac{\kappa l}{2}\right) - 2\sinh\left(\frac{\kappa l}{2}\right)\right)^2} \tag{46}$$

By expanding the hyperbolic functions in Equation (35) in a power series (truncating the expansion after constant terms), we obtain:

$$\boldsymbol{k}_{Gu} = \mathrm{Pr}^2 \begin{bmatrix} \frac{6}{5l} & \frac{1}{10} & -\frac{6}{5l} & \frac{1}{10} \\ \frac{1}{10} & \frac{2l}{15} & -\frac{1}{10} & -\frac{l}{30} \\ -\frac{6}{5l} & -\frac{1}{10} & \frac{6}{5l} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{l}{30} & -\frac{1}{10} & \frac{2l}{15} \end{bmatrix} \tag{47}$$

The simplified geometric matrix (47) corresponds to the matrices derived by Barsoum and Gallager [1], Meek *et al.* [2] and Szymczak [4].

4. Analysis of convergence – numerical examples

The derived matrices can be used in the analysis of thin-walled rod structures with open bisymmetric cross-sections under torque load. In this article, the matrices were used in order to calculate the torsion angles and the forces of critical torsional buckling for a thin-walled I-section bar. The examples were taken from [4, 8]. The aim of the analysis is to study the convergence of solutions for the matrices derived in this article. The matrices in the simplified form are widely used in engineering calculations owing to [1]. The analysis of the convergence or divergence resulting from the adopted simplifications is necessary in order to assess whether their application is justified. The methods for constructing the global stiffness matrices $\boldsymbol{K}_L$, $\boldsymbol{K}_G$ for the entire system, the implementation of the boundary conditions and the manner in which numerical analysis is performed are well known [3], therefore we will not discuss them here. In this study the commercial computational package Matlab [6] was used.

4.1. Example 1

Let us consider an I-section cantilever ($E=70\,\mathrm{GPa}$, $v=0.33$), which was loaded at the free end with a torque of $M_S=100\,\mathrm{Nm}$ (Figure 2).

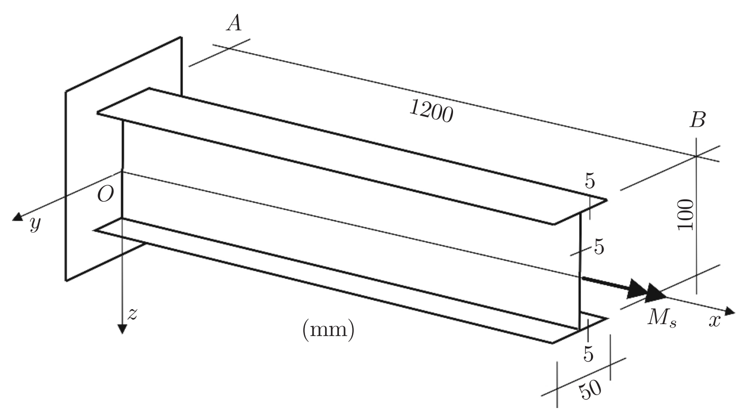

Figure 2. Schematic diagram of the cantilever beam under the action of torsion

This example illustrates the analysis of the convergence for the simplified elements $\boldsymbol{k}_{2Lu}$, $\boldsymbol{k}_{4Lu}$ as compared with the results of the analysis of the exact solution using element $\boldsymbol{k}_L$ (12) [7]. We analyzed the convergence of the torsion angles, the warping measured at the location of the load at point $\boldsymbol{B}$, as well as the values of the bimoments at point $\boldsymbol{A}$ depending on the number of elements (the density of discretization). The results can be found in Figures 3–8.

In the case of the simplified elements $\boldsymbol{k}_{2Lu}$ (18) and $\boldsymbol{k}_{4Lu}$ (23), full convergence of the numerical solution to the exact solution using the element $\boldsymbol{k}_L$ (12) was obtained after the bar was discretized with 4 or more elements (Figures 3, 5 and 7). The discrepancies in the solutions for the analyzed example due to coarser discretization were insignificant. In the case of the matrix element $\boldsymbol{k}_L$, with a very fine discretization (more than 400 elements) (*cf.* Figures 4, 6 and 8), the global stiffness matrix was almost singular (degenerate). The solutions obtained with a very dense discretization would be less satisfying because of the near-singularity of the stiffness matrix of the system. In the case of simplified matrices $\boldsymbol{k}_{2Lu}$ (18) and $\boldsymbol{k}_{4Lu}$ (23) the problems with the singularity of the global stiffness matrix do not lead to diverging solutions.

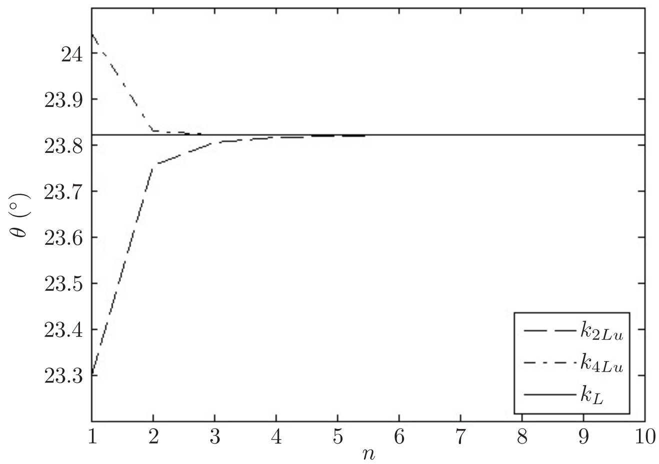

Figure 3. Distribution of the torsion angle θ (in degrees) at the point $\boldsymbol{B}$, depending on the discretization density, n

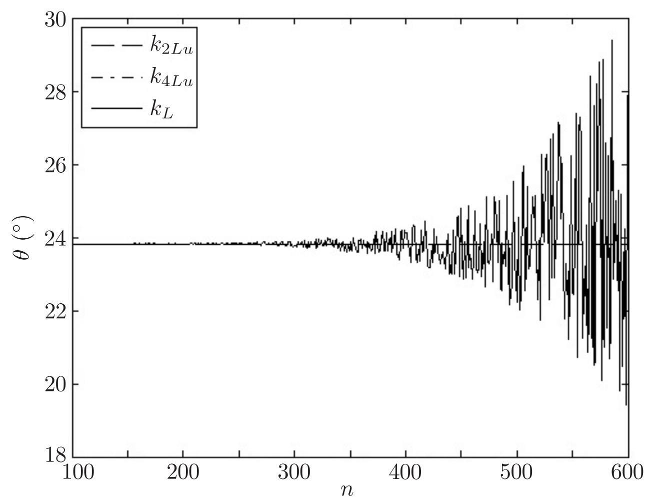

Figure 4. Distribution of the torsion angle θ (in degrees) at the point $\boldsymbol{B}$, depending on the discretization density, n

4.2. Example 2

The second numerical example is identical to the example analyzed in [4]. It must be emphasized that the examples analyzed in [4] are not realistic due to the plasticization of the bar that occurs before the appearance of the first torsional buckling.

In the case of the beam shown in Figure 9 ($E = 210\,\mathrm{GPa}$, $v = 0.3$), the results of the calculations of the critical load using the exact matrices $\boldsymbol{k}_L$ (12), $\boldsymbol{k}_G$ (35) were compared with the results obtained using the simplified matrices $\boldsymbol{k}_{2Lu}$ (18) and $\boldsymbol{k}_{Gu}$ (47). For comparison, the exact critical force for the beam (Figure 9) was 3 343 kN. The results were summarized in Table 1.

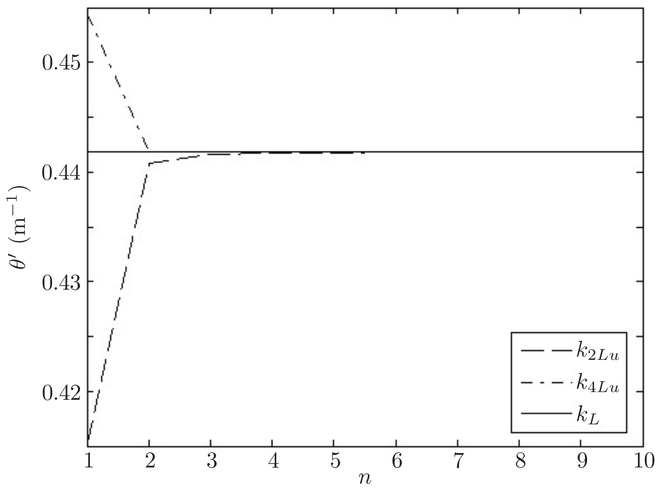

Figure 5. Distribution of the warping θ' (in m^{-1}) at the point $\boldsymbol{B}$, depending on the discretization density, n

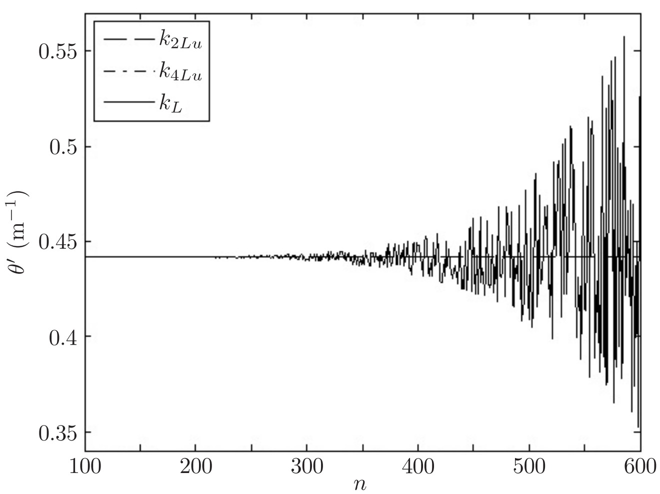

Figure 6. Distribution of the warping θ' (in m^{-1}) at the point $\boldsymbol{B}$, depending on the discretization density, n

The value of the critical force calculated using the derived matrices coincides closely with the expected value. It should be noted that better convergence of the solutions was obtained for the simplified matrices. Using the simplified matrices can prevent the instability of the solutions resulting from a very fine discretization.

5. Summary

The study of the convergence of the solutions obtained with the matrices derived in this article allows to assess the extent of their application to engineering problems. Commonly used matrices presented in [1] perform very well during static analysis and during stability analysis of thin-walled bars of open

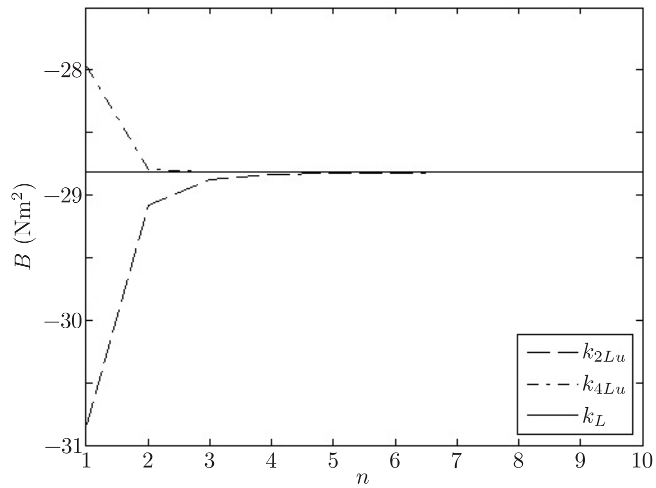

Figure 7. Distribution of the bimoment B (in Nm^2) at the point $\boldsymbol{A}$, depending on the discretization density, n

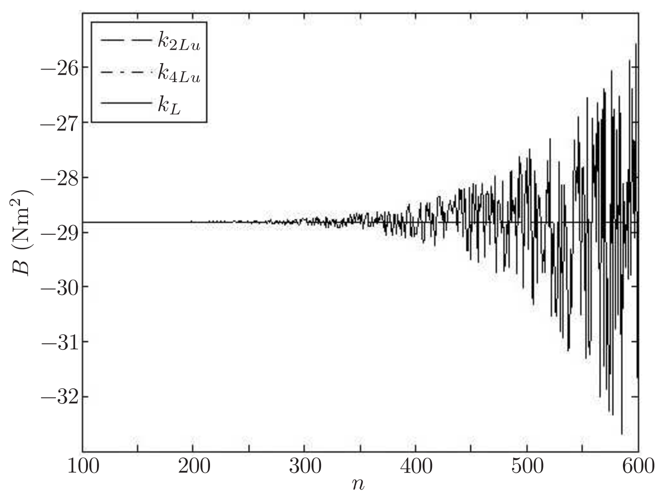

Figure 8. Distribution the bimoment B (in Nm^2) at the point $\boldsymbol{A}$, depending on the discretization density, n

bisymmetric cross-sections. The instability of the solutions arising when the exact stiffness matrices $\boldsymbol{k}_L$ (12), $\boldsymbol{k}_G$ (35) (hyperbolic functions) are used, associated with the high-density of elements covering the rod (the length of the elements) undoubtedly constitutes a serious numerical problem. Thus, we recommend that numerical calculations be carried out using the simplified matrices.

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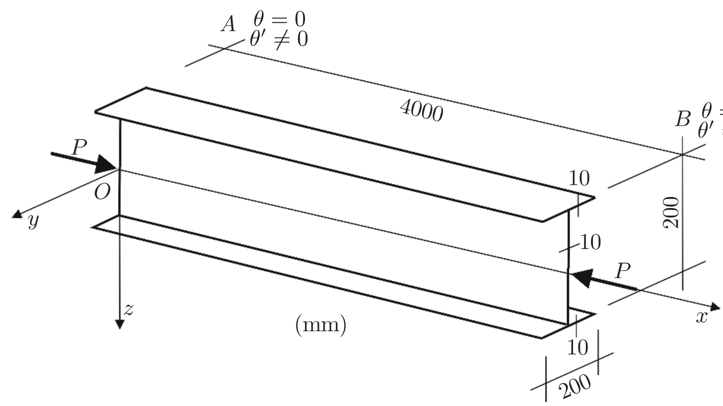

Figure 9. Schematic diagram of the bar under the action of compressive force

Table 1. Critical force (kN)

n	2	4	6	10	20	50	100	200
$\boldsymbol{k}_L$, $\boldsymbol{k}_G$	3 387	3 346	3 343	3 343	3 343	3 339	3 006	70
$\boldsymbol{k}_{2Lu}$, $\boldsymbol{k}_{Gu}$	3 356	3 343	3 343	3 343	3 343	3 343	3 343	3 343

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