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The shadowing chain lemma for singular Hamiltonian systems involving strong forces

Research Article

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Abstract: We consider a planar autonomous Hamiltonian system $\ddot{q} + \nabla V(q) = 0$, where the potential $V\colon \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ has a single well of infinite depth at some point ξ and a strict global maximum 0 at two distinct points a and b. Under a strong force condition around the singularity ξ we will prove a lemma on the existence and multiplicity of heteroclinic and homoclinic orbits – the shadowing chain lemma – via minimization of action integrals and using simple geometrical arguments.

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1. Introduction

In this paper we consider the second order Hamiltonian system (Newtonian system) of the form

$$\ddot{q} + \nabla V(q) = 0, \tag{1}$$

where $\ddot{} = d^2/dt^2$, $q \in \mathbb{R}^2$, and ∇V denotes the gradient of a potential V. We denote by $|\cdot|$ the norm in $\mathbb{R}^2$ induced by the standard inner product $(\cdot,\cdot)$. Throughout this work we assume that the potential V satisfies the following conditions:

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(V_1) there exists $\xi \in \mathbb{R}^2$ such that $V \in C^1(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$,

(V_2) $\lim_{x \to \xi} V(x) = -\infty$,

(V_3) there is a neighbourhood $\mathcal{N}$ of the point ξ and there is a function $U \in C^1(\mathcal{N} \setminus \{\xi\}, \mathbb{R})$ such that $|U(x)| \to \infty$ as $x \to \xi$ and $|\nabla U(x)|^2 \leq -V(x)$ for all $x \in \mathcal{N} \setminus \{\xi\}$,

(V_4) $V(x) \leq 0$ and there are two distinct points $a, b \in \mathbb{R}^2 \setminus \{\xi\}$ such that $V(x) = 0$ if and only if $x \in \{a, b\}$,

(V_5) there is a negative constant V_0 such that $\limsup_{|x| \to \infty} V(x) \leq V_0$.

From now on, $\mathcal{M}$ stands for the set of stationary points of the system (1), i.e.

$$\mathcal{M} = \{a, b\}.$$

Under the above assumptions, applying a variational approach we study the existence and multiplicity of heteroclinic and (nonstationary) homoclinic orbits of (1) which, as $t \to \pm\infty$, are asymptotic to a pair of different stationary points or a stationary point, respectively, and omit the singularity ξ.

During the last twenty years, there has been a great progress in the use of variational methods to find homoclinic and heteroclinic solutions for Hamiltonian systems. Such solutions are global in time, therefore it is natural and reasonable to use global methods to obtain them. Moreover, there are classical principles such as the Maupertuis principle of least action and Hamilton's principle that give a variational characterization of solutions of Hamiltonian systems. The existence of connecting orbits is an important problem in the study of the behaviour of dynamical systems. For example, their existence may give horseshoe chaos, see [11] and the references given there. The presence of infinitely many geometrically distinct homoclinic and/or heteroclinic orbits is an indication of nonintegrability and chaotic behaviour for the system (1), see [2].

Condition (V_3) is the strong force condition introduced by Gordon, see [6]. It governs the rate at which $V(x) \to -\infty$ as $x \to \xi$ and holds, for example, if $\alpha \geq 2$ for $V(x) = -|x - \xi|^{-\alpha}$ near ξ. (V_3) implies that $W^{1,2}_{\mathrm{loc}}$-collisions are not possible for the system (1), i.e. no solution of (1) in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$ can enter the singularity ξ in finite time. The case of singular Hamiltonian systems seems to be important since potentials arising in physics have infinitely deep wells. However, as pointed out by Gordon, it is disappointing that the strong force condition excludes the gravitational potential, i.e. $V(x) = -|x - \xi|^{-1}$.

There are works on homoclinic and heteroclinic orbits for singular Hamiltonian systems involving a strong force: we refer the reader to [1, 3–5, 10, 12]. We underline that they concern problems which are similar, both in the spirit and in the approach to the one studied in this work, but they regard a case of Hamiltonian systems with one stationary point and/or multiple singularities.

The main result of this paper is the starting point for showing the existence of infinitely many heteroclinic and homoclinic orbits to the system (1) with some condition introduced by Bolotin, see [2].

Let
$$E = \left\{ q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) : \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt < \infty \right\}.$$
Then E is a Hilbert space under the norm

$$\|q\|_E^2 = \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt + |q(0)|^2.$$

Let
$$\Lambda = \{q \in E : q(t) \neq \xi \text{ for all } t \in \mathbb{R}\},$$
the set of curves in E that avoid ξ. For $q \in \Lambda$, set

$$I(q) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\dot{q}(t)|^2 - V(q(t)) \right) dt. \tag{2}$$

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To shorten notation, $q(\pm\infty) = \lim_{t\to\pm\infty} q(t)$. If $q \in \Lambda$ and $I(q) < \infty$ then $q(\pm\infty) \in \mathcal{M}$ (see Corollary 2.3).

We define the family $\mathcal{F}$ as follows. A set $Z \subset \Lambda$ is a member of $\mathcal{F}$ if it has the following properties:

- $I(q) < \infty$ for all $q \in Z$,
- if $p, q \in Z$ then $p(\pm\infty) = q(\pm\infty)$,
- for each $q \in Z$ and for each $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ there exists $\delta > 0$ such that if $s \in (-\delta, \delta)$ then $q + s\psi \in Z$.

Let us remark that if q is a minimizer of I on a set $Z \in \mathcal{F}$ then

$$\frac{d}{ds} I(q+s\psi)\big|_{s=0} = 0 = \int_{-\infty}^{\infty} \big((\dot q(t), \dot\psi(t)) - (\nabla V(q(t)), \psi(t))\big)\, dt,$$

and consequently, q is a weak solution of (1). Analysis similar to that in the proof of [9, Proposition 3.18] shows that q is a classical solution of (1).

Let us now introduce the polar coordinate system in $\mathbb{R}^2$ with the pole ξ and the polar axis $\{x \in \mathbb{R}^2 : x = \xi + t \cdot \vec{\xi a},\ t \geq 0\}$. It is well-known that each point on the plane is determined by a distance r from the pole called a radius and an angle φ from the axis called a polar angle. In this work, polar angles are measured counterclockwise from the axis. In this polar coordinate system one has $q(t) = (r(t)\cos\varphi(t), r(t)\sin\varphi(t))$ for all $q \in \Lambda$. There is no uniqueness of a function $\varphi(t)$. If $q(t)$ is continuous then we can assume that $r(t)$ and $\varphi(t)$ are continuous, too.

Definition 1.1.

For each $q \in \Lambda$ such that $q(\pm\infty) = a$, $q(\pm\infty) = b$ or $q(-\infty) = a$ and $q(\infty) = b$ we define the *rotation number* $\operatorname{rot}(q)$ (the winding number) as follows:

$$\operatorname{rot}(q) = \left[\frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}\right],$$

where $[s]$ denotes the integral part of $s \in \mathbb{R}$. If $q(-\infty) = b$ and $q(\infty) = a$, set

$$\operatorname{rot}(q) = \left[\frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}\right] + 1.$$

This definition is independent of the choice of $\varphi(t)$. Set

$$R = \frac{1}{3}\min\{|b-a|, |b-\xi|, |a-\xi|\}.$$

From now on, $B_r(x)$ stands for a ball in $\mathbb{R}^2$ of radius $r > 0$, centered at a point $x \in \mathbb{R}^2$.

Remark 1.2.

If $q \in \Lambda$ and there are $t_1, t_2 \in \mathbb{R}$ and $0 < \varepsilon \leq R$ such that if $q((-\infty, t_1]) \subset B_\varepsilon(a)$ and $q([t_2, \infty)) \subset B_\varepsilon(b)$, then

$$\operatorname{rot}\big(q\restriction_{[t_1,t_2]}\big) = \operatorname{rot}(p),$$

where $p\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$ is given by

$$p(t) = \begin{cases} a & \text{if } t < t_1 - 1, \\ a(t_1 - t) + q(t_1)(t - t_1 + 1) & \text{if } t \in [t_1 - 1, t_1), \\ q(t) & \text{if } t \in [t_1, t_2], \\ q(t_2)(1 + t_2 - t) + b(t - t_2) & \text{if } t \in (t_2, t_2 + 1], \\ b & \text{if } t > t_2 + 1. \end{cases}$$

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Similarly, one can define the rotation number $\mathrm{rot}(q\restriction_{[t_1,t_2]})$ for each $q \in \Lambda$ such that $q((-\infty, t_1]) \cup q([t_2, \infty)) \subset B_\varepsilon(a)$ or $q((-\infty, t_1]) \cup q([t_2, \infty)) \subset B_\varepsilon(b)$.

Remark 1.3.
Assume that $q \in \Lambda$, $q(-\infty) = a$, and $q(\infty) = b$. For each $T \in \mathbb{R}$ such that $q(T) \in B_\varepsilon(a)$ or $q(T) \in B_\varepsilon(b)$ we will denote by $\mathrm{rot}(q\restriction_{(-\infty,T]})$ and $\mathrm{rot}(q\restriction_{[T,\infty)})$ the rotation numbers of appropriate paths in Λ that arise from $q\restriction_{(-\infty,T]}$ and $q\restriction_{[T,\infty)}$ by connecting $q(T)$ to a or b, respectively, by a line segment.

We can introduce similar notations for $q \in \Lambda$ such that $q(\pm\infty) = a$, $q(\pm\infty) = b$ and $q(-\infty) = b$, $q(\infty) = a$.

In this paper we continue the research started by the second author in [8], where the following result was proved:

Theorem 1.4.
Under the assumptions (V_1)–(V_5), the Hamiltonian system (1) possesses at least two solutions which wind around ξ and join $\mathcal{M}$ to $\mathcal{M}$. One of them is a heteroclinic orbit joining a to b. The second is either heteroclinic with a rotation number different from the first or homoclinic.

Our purpose now is to prove the shadowing chain lemma which states:

Lemma 1.5.
Let $Z \in \mathcal{F}$ be an arbitrary set all of whose elements have the same rotation number $M \in \mathbb{Z}$. Set

$$z = \inf\{I(q) : q \in Z\}.$$

Under the conditions (V_1)–(V_5), there are a finite number of homoclinic and heteroclinic solutions $Q_1, Q_2, \ldots, Q_l \in \Lambda$ of the Hamiltonian system (1) such that

$$z = I(Q_1) + I(Q_2) + \cdots + I(Q_l) \qquad \textit{and} \qquad M = \mathrm{rot}(Q_1) + \mathrm{rot}(Q_2) + \cdots + \mathrm{rot}(Q_l).$$

As we have already mentioned, this lemma is the starting point for showing the existence of infinitely many homoclinic and heteroclinic orbits to the system (1) under a certain condition introduced by Bolotin. Precisely, we want to adapt from Bolotin the condition on the existence of a minimal noncontractible periodic orbit around ξ. A result similar to Lemma 1.5 was obtained by Caldiroli and Jeanjean in [4]. They considered the Hamiltonian system (1) with the potential V possessing a global maximum at 0. Their lemma is about a chain of homoclinics to 0. We would like to underline that our method of the proof differs from theirs. An example of a shadowing chain of solutions of (1) can be found in the proof of [8, Lemma 4.6]. That chain was composed of two elements and at least one of them was a heteroclinic orbit.

The paper is organized as follows: In Section 2 we study the Lagrangian functional associated with the Hamiltonian system (1). In Section 3 the shadowing chain lemma is proved. Appendix provides a detailed proof of two technical lemmas stated in Section 3.

2. The Lagrangian functional

In this section we will be concerned with the study of the Lagrangian functional given by (2). We will use its properties in the proof of the shadowing chain lemma.

Set

$$\alpha_\varepsilon = \inf\{-V(x) : x \notin B_\varepsilon(\mathcal{M})\},$$

where $0 < \varepsilon \le R$ and $B_\varepsilon(\mathcal{M}) = B_\varepsilon(a) \cup B_\varepsilon(b)$. By ($V_2$), ($V_4$) and ($V_5$) it follows that $\alpha_\varepsilon > 0$.

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Lemma 2.1.
Suppose that $q \in \Lambda$ and $q(t) \notin B_\varepsilon(\mathcal{M})$ for each $t \in \bigcup_{i=1}^{k}[r_i, s_i]$, where $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ for $i \neq j$. Then

$$I(q) \geq \sqrt{2\alpha_\varepsilon} \sum_{i=1}^{k} |q(s_i) - q(r_i)|.$$

The proof of Lemma 2.1 is the same as that of [9, Lemma 3.6] or [7, Lemma 2.1].

Corollary 2.2.
If $q \in \Lambda$ and $I(q) < \infty$ then $q \in L^\infty(\mathbb{R}, \mathbb{R}^2)$.

Corollary 2.3.
If $q \in \Lambda$ and $I(q) < \infty$ then $q(\pm\infty) \in \mathcal{M}$.

We can easily prove these two corollaries by the use of Lemma 2.1. For more details we refer the reader to [9, Remark 3.10 and Proposition 3.11] and [7, Corollary 2.2 and Lemma 2.4].

Proposition 2.4.
Let $Z \in \mathcal{F}$. If $\{q_m\}_{m\in\mathbb{N}} \subset Z$ and $\{I(q_m)\}_{m\in\mathbb{N}} \subset \mathbb{R}$ is bounded, then $\{q_m\}_{m\in\mathbb{N}}$ possesses a subsequence that converges weakly in E, and hence strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$.

Proof. It is sufficient to show that $\{q_m\}_{m\in\mathbb{N}}$ is a bounded sequence in E. By assumption, there is $M > 0$ such that for all $m \in \mathbb{N}$, $0 < I(q_m) \leq M$. From this and (2) we get

$$\|\dot{q}_m\|^2_{L^2} \leq 2M.$$

Moreover, from Corollary 2.2 it follows that $q_m \in L^\infty(\mathbb{R}, \mathbb{R}^2)$ for all $m \in \mathbb{N}$.

Let x_0 and y_0 denote the starting and ending point, respectively, of a function $q \in Z$. Fix $0 < \varepsilon \leq R$. Then for each $m \in \mathbb{N}$ there are $\tau_m, t_m \in \mathbb{R}$ such that $q_m(\tau_m) \in \partial B_\varepsilon(x_0)$, $q_m(t) \in B_\varepsilon(x_0)$ for all $t < \tau_m$, $q_m(t_m) \in \partial B_\varepsilon(y_0)$ and $q_m(t) \in B_\varepsilon(y_0)$ for all $t > t_m$. Finally, for $q_m\restriction_{[\tau_m, t_m]}$ there is $s_m \in [\tau_m, t_m]$ such that

$$|q_m(s_m)| = \max_{t\in[\tau_m, t_m]} |q_m(t)|.$$

Applying Lemma 2.1 we conclude that the sequence $\{q_m(s_m)\}_{m\in\mathbb{N}}$ is bounded. Hence $\{q_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}, \mathbb{R}^2)$. In consequence, $\{q_m\}_{m\in\mathbb{N}}$ is bounded in E. By the reflexivity of E there is $Q \in E$ such that, passing to a subsequence, $q_m \rightharpoonup Q$ in E, which implies that $q_m \to Q$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. □

Lemma 2.5.
If $q \in \Lambda$ and $q(t) \in \mathcal{N}$ for all $t \in [\sigma, \mu]$ then

$$|U(q(\mu))| \leq |U(q(\sigma))| + \left(\int_\sigma^\mu -V(q(t))\,dt\right)^{1/2} \cdot \left(\int_\sigma^\mu |\dot{q}(t)|^2 dt\right)^{1/2}.$$

The proof of this lemma can be found in [10, (2.21), p. 271]. It is based on the strong force condition. Applying the above inequality and (2), for $q \in \Lambda$ such that $q(t) \in \mathcal{N}$ for all $t \in [\sigma, \mu]$ we get

$$|U(q(\mu))| \leq |U(q(\sigma))| + \sqrt{2}I(q).$$

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Proposition 2.6.

Let $Z \subset \Lambda$ be a set such that the functional I restricted to Z is bounded. Then there exists $r > 0$ such that $|q(t) - \xi| \geq r$ for all $q \in Z$ and $t \in \mathbb{R}$.

Proof. By Corollary 2.3, $q(\pm\infty) \in \mathcal{M}$ for each $q \in Z$.

On the contrary, suppose that there exist sequences $\{q_m\}_{m\in\mathbb{N}} \subset Z$ and $\{\mu_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ such that $q_m(\mu_m) \to \xi$ as $m \to \infty$. Fix $0 < \delta \leq R$ such that $\overline{B_\delta(\xi)} \subset \mathcal{N}$. There is $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $|q_m(\mu_m) - \xi| < \delta$. For each $m \geq m_0$ there exists $\sigma_m < \mu_m$ such that $q_m(\sigma_m) \in \partial B_\delta(\xi)$ and $q_m(t) \in B_\delta(\xi)$ for all $t \in (\sigma_m, \mu_m)$. Then

$$|U(q_m(\mu_m))| \leq |U(q_m(\sigma_m))| + \sqrt{2}I(q_m).$$

As $\{U(q_m(\sigma_m))\}_{m\in\mathbb{N}}$ and $\{I(q_m)\}_{m\in\mathbb{N}}$ are bounded, we get $\{U(q_m(\mu_m))\}_{m\in\mathbb{N}}$ is bounded, too. On the other hand, by (V_3), we receive $|U(q_m(\mu_m))| \to \infty$ as $m \to \infty$, a contradiction. □

Proposition 2.7.

If $Q \in \Lambda$ is a homoclinic or heteroclinic solution of (1) then

$$\dot{Q}(t) \to 0 \qquad \text{as} \quad t \to \pm\infty.$$

This can be found in [8, Proposition 2.8, p. 477].

Proposition 2.8.

Let $Z \subset \Lambda$ be a set such that the functional I restricted to Z is bounded. Then there exists $M > 0$ such that $|\mathrm{rot}(q)| \leq M$ for all $q \in Z$.

Proof. Suppose, contrary to our claim, that for every $m \in \mathbb{N}$ there exists $q_m \in Z$ such that $|\mathrm{rot}(q_m)| > m$. From Proposition 2.6 it follows that there is $r > 0$ such that $|q_m(t) - \xi| > r$ for all $m \in \mathbb{N}$ and $t \in \mathbb{R}$.

Fix $0 < \varepsilon \leq R$. It is clear that one can associate with q_m a family of mutually disjoint intervals $\{[r_i^m, s_i^m]\}_{i=1}^{k_m}$ such that $q_m([r_i^m, s_i^m]) \cap B_\varepsilon(\mathcal{M}) = \emptyset$ and $|q_m(s_i^m) - q_m(r_i^m)| \geq r$. Moreover, k_m increases with m. By Lemma 2.1,

$$I(q_m) \geq \sqrt{2\alpha_\varepsilon}\, k_m r,$$

a contradiction. □

From now on, to simplify notation we write

$$I_{T_1}^{T_2}(q) = \int_{T_1}^{T_2} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt, \quad I_{-\infty}^{T}(q) = \int_{-\infty}^{T} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt, \quad I_{T}^{\infty}(q) = \int_{T}^{\infty} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt,$$

where $T_1, T_2, T \in \mathbb{R}$.

Remark 2.9.

For all $T_1, T_2 \in \mathbb{R}$ such that $T_1 < T_2$, the functional given by

$$\Lambda \ni q \mapsto I_{T_1}^{T_2}(q) \in \mathbb{R}$$

is weakly lower semi-continuous.

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Remark 2.10.
If $q \in \Lambda$ then for each $\theta \in \mathbb{R}$, $\theta q = q(\cdot - \theta) \in \Lambda$ and $I(q) = I(\theta q)$. Moreover, $I(q) = I(q(-t))$.

Lemma 2.11.
For each $\eta > 0$ there is $0 < r \leq R$ such that for all $x, y \in B_r(a)$ (resp. $x, y \in B_r(b)$) and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2} |y - x|^2 - V(l_{x,y}(t)) \right) dt < \eta,$$

where $l_{x,y}(t) = (T + 1 - t)x + (t - T)y$ for each $t \in [T, T + 1]$.

The proof of this lemma is straightforward.

Lemma 2.12.
Let $\{q_m\}_{m \in \mathbb{N}}$ be a minimizing sequence of the Lagrangian functional I restricted to a set $Z \in \mathfrak{F}$. If $q_m \rightharpoonup Q$ in E then Q is a homoclinic or heteroclinic solution of (1).

Proof. Set $z = \inf \{I(q) : q \in Z\}$. By assumption, $I(q_m) \to z$ as $m \to \infty$. From Remark 2.9 we conclude that $I(Q) \leq z$. Corollary 2.3 now implies $Q(\pm\infty) \in \mathcal{M}$. Finally, Proposition 2.6 gives $Q(t) \neq \xi$ for all $t \in \mathbb{R}$.

Define

$$\widetilde{Z} = \left\{ q \in \Lambda : q(\pm\infty) = Q(\pm\infty), \operatorname{rot}(q) = \operatorname{rot}(Q) \right\}.$$

It is clear that $\widetilde{Z} \in \mathfrak{F}$. Let $\widetilde{z}$ be given by $\widetilde{z} = \inf \{I(q) \colon q \in \widetilde{Z}\}$. We shall have the lemma established if we prove that $I(Q) = \widetilde{z}$.

Conversely, suppose that $I(Q) > \widetilde{z}$. Fix $\eta > 0$ such that $I(Q) = \widetilde{z} + 8\eta$. By the definition of infimum there exists $q \in \widetilde{Z}$ such that $I(q) < \widetilde{z} + \eta$. By Lemma 2.11 there is $0 < r \leq R$ such that for all $x, y \in B_r(a)$ (resp. $x, y \in B_r(b)$) and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2} |y - x|^2 - V(l_{x,y}(t)) \right) dt < \eta.$$

Choose now $T > 0$ such that the sets $q((-\infty, -T])$, $q([T, \infty))$, $Q((-\infty, -T])$, $Q([T, \infty))$ are contained in $B_r(\mathcal{M})$ and $I(Q) - I_{-T}^{T}(Q) < \eta$. Since q_m goes to Q uniformly on compact subsets of $\mathbb{R}$, there is $m_0 \in \mathbb{N}$ such that $q_m([-T-1, -T])$ and $q_m([T, T+1])$ are subsets of $B_r(\mathcal{M})$ for all $m \geq m_0$.

We will consider the behaviour of the sequence $\{p_m\}_{m \in \mathbb{N}}$ defined by

$$p_m(t) = \begin{cases} q_m(t) & \text{if } t < -T - 1, \\ (-T - t) q_m(-T - 1) + (T + 1 + t) q(-T) & \text{if } t \in [-T-1, -T], \\ q(t) & \text{if } t \in [-T, T], \\ (T + 1 - t) q(T) + (t - T) q_m(T + 1) & \text{if } t \in [T, T+1], \\ q_m(t) & \text{if } t > T + 1. \end{cases}$$

We have $p_m(-\infty) = q_m(-\infty)$, $p_m(\infty) = q_m(\infty)$, and moreover, for $m \geq m_0$,

$$\begin{aligned} \operatorname{rot}(p_m) &= \operatorname{rot}\left(q_m\restriction_{(-\infty,-T-1]}\right) + \operatorname{rot}(q) + \operatorname{rot}\left(q_m\restriction_{[T+1,\infty)}\right) = \operatorname{rot}\left(q_m\restriction_{(-\infty,-T-1]}\right) + \operatorname{rot}(Q) + \operatorname{rot}\left(q_m\restriction_{[T+1,\infty)}\right) \\ &= \operatorname{rot}\left(q_m\restriction_{(-\infty,-T-1]}\right) + \operatorname{rot}\left(q_m\restriction_{[-T-1,T+1]}\right) + \operatorname{rot}\left(q_m\restriction_{[T+1,\infty)}\right) = \operatorname{rot}(q_m). \end{aligned}$$

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Hence $\{p_m\}_{m=m_0}^{\infty} \subset Z$. From Remark 2.9 it follows that there is $m_1 \in \mathbb{N}$ such that for $m \geq m_1$, $I_{-T}^{T}(q_m) > I_{-T}^{T}(Q) - \eta$. By the above, for $m \in \mathbb{N}$ sufficiently large,

$$I(q_m) - I(p_m) \geq I_{-T-1}^{T+1}(q_m) - I_{-T}^{T}(q) - 2\eta \geq I_{-T}^{T}(q_m) - I_{-T}^{T}(q) - 2\eta > I_{-T}^{T}(Q) - I_{-T}^{T}(q) - 3\eta > I(Q) - I(q) - 4\eta > 3\eta.$$

Thus $I(q_m) > I(p_m) + 3\eta$. Letting $m \to \infty$ we get

$$z \geq \liminf_{m\to\infty} I(p_m) + 3\eta \geq z + 3\eta,$$

a contradiction. □

3. The shadowing chain lemma

Let $Z \in \mathcal{F}$ be an arbitrary but fixed set all of whose elements have the same rotation $M \in \mathbb{Z}$. We will denote by x_0 and y_0 the starting and ending point, respectively, of a function $q \in Z$. From Corollary 2.3 it follows that $x_0, y_0 \in \mathcal{M}$. Set

$$z = \inf\{I(q) : q \in Z\}.$$

Choose a minimizing sequence $\{q_m\}_{m\in\mathbb{N}}$ of the functional I restricted to Z, i.e. $\{q_m\}_{m\in\mathbb{N}} \subset Z$ and

$$\lim_{m\to\infty} I(q_m) = z.$$

For each $i \in \mathbb{N}$ let a set C_i be defined by

$$C_i = \overline{\bigcup_{m=i}^{\infty} q_m(\mathbb{R})}\,.$$

It is easily seen that the sets C_i are compact and connected. Moreover, $x_0, y_0 \in C_i$ and $C_i \supset C_{i+1}$ for each $i \in \mathbb{N}$. Set

$$\mathcal{C} = \bigcap_{i=1}^{\infty} C_i.$$

By the above, the set $\mathcal{C}$ is a nonempty continuum.

Lemma 3.1.
For each $x \in \mathcal{C}$ there exists a homoclinic or heteroclinic solution Q_x of (1) such that $Q_x(0) = x$.

Proof. Fix $x \in \mathcal{C} \setminus \{x_0, y_0\}$. Thus $q_{m_n}(t_n) \to x$, $n \to \infty$. Define $\widehat{q}_{m_n}(t) = q_{m_n}(t + t_n)$, where $n \in \mathbb{N}$. From Proposition 2.4 we conclude that there is $Q_x \in E$ such that, passing to a subsequence if necessary, $\widehat{q}_{m_n} \rightharpoonup Q_x$ in E and $\widehat{q}_{m_n} \to Q_x$ in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. Lemma 2.12 yields Q_x is a homoclinic or heteroclinic solution of (1). Moreover, we have $x = \lim_{n\to\infty} \widehat{q}_{m_n}(0) = Q_x(0)$. □

In fact, $\mathcal{C}$ is a sum of chains each of which has a structure described by the following shadowing chain lemma.

Lemma 3.2.
There are a finite number of homoclinic and heteroclinic solutions $Q_1, Q_2, \ldots, Q_l$ of (1) such that

$$z = I(Q_1) + I(Q_2) + \cdots + I(Q_l) \qquad \textit{and} \qquad M = \mathrm{rot}(q_m) = \mathrm{rot}(Q_1) + \mathrm{rot}(Q_2) + \cdots + \mathrm{rot}(Q_l).$$

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Proof. Fix $0 < \varepsilon \le R$. Remark 2.10 implies there is no loss of generality in assuming that $q_m(0) \in \partial B_\varepsilon(x_0)$ and $|q_m(t) - x_0| < \varepsilon$ for all $m \in \mathbb{N}$ and $t < 0$. By Proposition 2.4, there is $Q_1 \in E$ such that, passing to a subsequence if necessary, $q_m \rightharpoonup Q_1$ in E and $q_m \to Q_1$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. From Lemma 2.12 we obtain Q_1 is a homoclinic or heteroclinic solution of (1). Obviously, $Q_1(0) \in \partial B_\varepsilon(x_0)$ and $|Q_1(t) - x_0| \le \varepsilon$ for all $t < 0$. Hence $Q_1(-\infty) = x_0$. On account of Remark 2.9, we have $I(Q_1) \le z$.

Set $x_1 = Q_1(\infty)$. Fix $\eta > 0$. By Lemma 2.11 there is $0 < r \le R$ such that for all $x, y \in B_r(a)$ (resp. $x, y \in B_r(b)$) and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2}|y - x|^2 - V(l_{x,y}(t)) \right) dt < \eta.$$

Choose $T > 0$ such that $Q_1([T, \infty)) \subset B_r(x_1)$ and $I(Q_1) - I^T_{-T}(Q_1) < \eta$. Since $q_m \to Q_1$ uniformly on $[T, T+1]$, there exists $m(T) \in \mathbb{N}$ such that if $m \ge m(T)$ then $q_m([T, T+1]) \subset B_r(x_1)$. Let q^T_m be given by

$$q^T_m(t) = \begin{cases} x_1 & \text{if} \quad t < T, \\ (T+1-t)x_1 + (t-T)\, q_m(T+1) & \text{if} \quad t \in [T, T+1], \\ q_m(t) & \text{if} \quad t > T+1. \end{cases}$$

We see at once that $q^T_m(-\infty) = x_1$, $q^T_m(\infty) = y_0$ and for $m \ge m(T)$,

$$\mathrm{rot}(q^T_m) = \mathrm{rot}(q_m) - \mathrm{rot}(Q_1) = M - \mathrm{rot}(Q_1).$$

Define

$$Z_1 = \left\{ q \in \Lambda : q(-\infty) = x_1, \, q(\infty) = y_0, \, \mathrm{rot}(q) = M - \mathrm{rot}(Q_1) \right\}.$$

Set

$$z_1 = \inf \{ I(q) : q \in Z_1 \}.$$

We may now take the sequence $\eta_n = 1/n$, where $n \in \mathbb{N}$, and repeat our construction. In this way we find a decreasing sequence $\{r_n\}_{n\in\mathbb{N}}$ going to 0, an increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ going to ∞, an increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers $m_n = m(T_n)$ and a sequence $\{p_n\}_{n\in\mathbb{N}}$ given by

$$p_n(t) = q^{T_n}_{m_n}(t) = \begin{cases} x_1 & \text{if} \quad t < T_n, \\ (T_n+1-t)x_1 + (t-T_n)\, q_{m_n}(T_n+1) & \text{if} \quad t \in [T_n, T_n+1], \\ q_{m_n}(t) & \text{if} \quad t > T_n+1. \end{cases}$$

Lemma 3.3.
The sequence $\{p_n\}_{n\in\mathbb{N}}$ is a minimizing sequence of the functional I restricted to Z_1, i.e. $\lim_{n\to\infty} I(p_n) = z_1$.

Lemma 3.4.
The following equality holds: $z = I(Q_1) + z_1$.

For the proofs of these lemmas we refer the reader to Appendix. If $z_1 = 0$ then $I(Q_1) = z$, Q_1 is a solution of (1) joining x_0 to y_0 and $\mathrm{rot}(Q_1) = M$. Consider the case $z_1 > 0$. By shifting each p_n appropriately in time, we can assume that $p_n(0) \in \partial B_\varepsilon(x_1)$ and $|p_n(t) - x_1| < \varepsilon$ for all $n \in \mathbb{N}$ and $t < 0$. From Proposition 2.4 it follows that there is $Q_2 \in E$ such that, passing to a subsequence if necessary, $p_n \rightharpoonup Q_2$ in E and $p_n \to Q_2$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. Lemma 2.12 implies Q_2 is a homoclinic or heteroclinic solution of (1). Obviously, $Q_2(0) \in \partial B_\varepsilon(x_1)$ and $|Q_2(t) - x_1| \le \varepsilon$ for all $t < 0$. Thus $Q_2(-\infty) = x_1$. By Remark 2.9, $I(Q_2) \le z_1$.

Set $x_2 = Q_2(\infty)$. Define

$$Z_2 = \left\{ q \in \Lambda : q(-\infty) = x_2, \, q(\infty) = y_0, \, \mathrm{rot}(q) = M - \mathrm{rot}(Q_1) - \mathrm{rot}(Q_2) \right\}, \qquad z_2 = \inf \{ I(q) : q \in Z_2 \}.$$

Replacing $\{q_m\}_{m\in\mathbb{N}}$ by $\{p_n\}_{n\in\mathbb{N}}$ we can now proceed analogously as above. Next we continue by induction. Lemma 2.1 makes it obvious that l is finite. □

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Note that Theorem 1.4 is a direct consequence of the above lemma. To this end, consider two families $\Gamma_{-1}, \Gamma_0 \in \mathcal{F}$ defined as follows:

$$\Gamma_i = \big\{q \in \Lambda : q(-\infty) = a,\, q(\infty) = b,\, \mathrm{rot}(q) = i\big\}, \qquad i = -1, 0.$$

Set

$$\gamma_i = \inf\{I(q) : q \in \Gamma_i\}, \qquad i = -1, 0.$$

Without loss of generality we can assume that $\gamma_{-1} \le \gamma_0$. Let $\{q_m\}_{m\in\mathbb{N}} \subset \Gamma_{-1}$ be a minimizing sequence. By the shadowing chain lemma, there is a heteroclinic solution $q \in \Gamma_{-1}$ of (1) with $I(q) = \gamma_{-1}$. Moreover, if $\{p_m\}_{m\in\mathbb{N}} \subset \Gamma_0$ is a minimizing sequence then there is a heteroclinic solution $p \in \Lambda$ of (1) joining a to b. Now two cases are possible. If $\mathrm{rot}(p) \ne -1$ then p and q are two geometrically distinct heteroclinic solutions of (1). Assume that $\mathrm{rot}(p) = -1$. It might happen that, with accuracy up to a reparametrization, $p = q$. However, by the shadowing chain lemma, the chain containing p possesses at least two elements, and Theorem 1.4 is proved.

Note that we are actually able to obtain even more precise information. Define

$$\Omega_a = \{q \in \Lambda : q(\pm\infty) = a,\, \mathrm{rot}(q) = 1\}, \qquad \Omega_b = \{q \in \Lambda : q(\pm\infty) = b,\, \mathrm{rot}(q) = 1\}.$$

Let ω_a and ω_b be the corresponding infima. The shadowing chain lemma now implies the following.

Fact 3.5.

If $\gamma_0 - \gamma_{-1} < \min\{\omega_a, \omega_b\}$ then (1) possesses at least two heteroclinic solutions p and q in Λ such that $\mathrm{rot}(p) \ne \mathrm{rot}(q)$.

Appendix

For the convenience of the reader this section will be devoted to the proof of two technical lemmas of Section 3. We follow the notation used in the proof of Lemma 3.2.

Proof of Lemma 3.3. Suppose the lemma were false, i.e.

$$\liminf_{n\to\infty} I(p_n) > z_1.$$

Then we could find $\{u_n\}_{n\in\mathbb{N}} \subset Z_1$ such that $I(u_n) \to z_1$, $n \to \infty$. By Remark 2.10, we can assume that for each $n \in \mathbb{N}$, $u_n(T_n+1) \in \partial B_{r_n}(x_1)$ and $|u_n(t) - x_1| < r_n$ for all $t < T_n + 1$. Let $\{v_n\}_{n\in\mathbb{N}}$ be given by

$$v_n(t) = \begin{cases} q_{m_n}(t) & \text{if } t < T_n, \\ (T_n+1-t)\, q_{m_n}(T_n) + (t-T_n)\, u_n(T_n+1) & \text{if } t \in [T_n, T_n+1], \\ u_n(t) & \text{if } t > T_n+1. \end{cases}$$

We see at once that $v_n(-\infty) = x_0$, $v_n(\infty) = y_0$ and $\mathrm{rot}(v_n) = M$. Hence $\{v_n\}_{n\in\mathbb{N}} \subset Z$. Furthermore,

$$I(v_n) < I_{-\infty}^{T_n+1}(q_{m_n}) + \frac{1}{n} + I(u_n)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get

$$\liminf_{n\to\infty} I(v_n) \le \liminf_{n\to\infty} I_{-\infty}^{T_n+1}(q_{m_n}) + z_1.$$

We also have

$$\liminf_{n\to\infty} I_{-\infty}^{T_n+1}(q_{m_n}) = z - \liminf_{n\to\infty} I(p_n) < z - z_1.$$

Consequently,

$$\liminf_{n\to\infty} I(v_n) < z,$$

which is impossible. □

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Proof of Lemma 3.4. The proof will be divided into two steps.

Step 1. We first prove that $z \leq I(Q_1) + z_1$. Fix $\eta > 0$. From Lemma 2.11 it follows that there is $0 < \delta \leq R$ such that for all $x, y \in B_\delta(x_1)$ and for every $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2} |y - x|^2 - V(l_{x,y}(t)) \right) dt < \eta.$$

Choose $T > 0$ such that $Q_1([T, \infty)) \subset B_\delta(x_1)$. By the definition of infimum, there is $q \in Z_1$ such that $I(q) < z_1 + \eta$. Applying Remark 2.10 we can assert that $q((-\infty, T+1]) \subset B_\delta(x_1)$. Define

$$u(t) = \begin{cases} Q_1(t) & \text{if } t \leq T, \\ (T+1-t)\, Q_1(T) + (t-T)\, q(T+1) & \text{if } t \in [T, T+1], \\ q(t) & \text{if } t \geq T+1. \end{cases}$$

We check at once that $u(-\infty) = x_0$, $u(\infty) = y_0$ and $\operatorname{rot}(u) = M$. Hence $u \in Z$. By the above,

$$z \leq I(u) < I(Q_1) + z_1 + 2\eta.$$

Letting $\eta \to 0^+$ we get $z \leq I(Q_1) + z_1$.

Step 2. We now show that $z \geq I(Q_1) + z_1$. Fix $\eta > 0$. Lemma 2.11 implies there is $0 < \delta \leq R$ such that for all $x \in B_\delta(x_1)$ and $T \in \mathbb{R}$,

$$\int_{T-1}^{T} \left(\frac{1}{2} |x - x_1|^2 - V(l_{x_1,x}(t)) \right) dt < \eta,$$

where $l_{x_1,x}(t) = (T-t)x_1 + (t-T+1)x$ for each $t \in [T-1, T]$. Choose $T > 0$ such that $Q_1([T, \infty)) \subset B_\delta(x_1)$ and $I_{-\infty}^T(Q_1) > I(Q_1) - \eta$. Let $\{u_m\}_{m \in \mathbb{N}}$ be defined by

$$u_m(t) = \begin{cases} x_1 & \text{if } t \leq T-1, \\ (T-t)x_1 + (t-T+1)\, q_m(T) & \text{if } t \in [T-1, T], \\ q_m(t) & \text{if } t \geq T. \end{cases}$$

Then $u_m(-\infty) = x_1$ and $u_m(\infty) = y_0$. Moreover, there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we have $\operatorname{rot}(u_m) = M - \operatorname{rot}(Q_1)$ and

$$I(u_m) < I_T^\infty(q_m) + \eta.$$

From Remark 2.9 we conclude that there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$I_{-\infty}^T(q_m) > I_{-\infty}^T(Q_1) - \eta.$$

In consequence, for $m \in \mathbb{N}$ large enough, $I(q_m) > I(Q_1) + z_1 - 3\eta$. Letting $m \to \infty$, and next $\eta \to 0^+$, we get $z \geq I(Q_1) + z_1$. □

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