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Homoclinic orbits for a class of singular second order Hamiltonian systems in $\mathbb{R}^3$

Research Article

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Abstract: We consider a conservative second order Hamiltonian system $\ddot{q} + \nabla V(q) = 0$ in $\mathbb{R}^3$ with a potential V having a global maximum at the origin and a line $l \cap \{0\} = \emptyset$ as a set of singular points. Under a certain compactness condition on V at infinity and a strong force condition at singular points we study, by the use of variational methods and geometrical arguments, the existence of homoclinic solutions of the system.

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1. Introduction

In this work we shall study the existence of homoclinic orbits for a class of conservative second order Hamiltonian systems

$$\ddot{q} + \nabla V(q) = 0, \tag{1}$$

where $q \in \mathbb{R}^3$. We will suppose that a potential V satisfies the following conditions:

(V_1) there is a line l in $\mathbb{R}^3$ such that $l \cap \{0\} = \emptyset$ and $V \in C^1(\mathbb{R}^3 \setminus l, \mathbb{R})$,

(V_2) $\lim_{x \to l} V(x) = -\infty$,

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(V_3) there exists a neighbourhood $\mathcal{N}$ of the line l and a function $U \in C^1(\mathcal{N}\setminus l, \mathbb{R})$ such that $|U(x)| \to \infty$ as $x \to l$ and $|\nabla U(x)| \leq -V(x)$ for all $x \in \mathcal{N}\setminus l$,

(V_4) $V(x) \leq 0$ and $V(x) = 0$ if and only if $x = 0$,

(V_5) there is a constant $V_0 < 0$ such that $\limsup_{|x|\to\infty} V(x) \leq V_0$.

Here and subsequently, $x \to l$ stands for $d(x, l) = \inf\{|x-y| : y \in l\} \to 0$ and $|\cdot| \colon \mathbb{R}^3 \to \mathbb{R}$ is the norm in $\mathbb{R}^3$ induced by the standard inner product.

Condition (V_3) governs the rate at which $-V(x) \to \infty$ as $x \to l$. This condition was introduced in [6] by the physicist William B. Gordon. If V satisfies (V_3) then ∇V is called a strong force. Condition (V_4) implies that the origin is a critical point of V, and condition (V_5) guarantees that the critical point, 0, is a global maximum of V.

Definition 1.1.
A solution $q\colon \mathbb{R} \to \mathbb{R}^3$ of (1) is said to be *homoclinic* (to 0) if $q(\pm\infty) = \lim_{t\to\pm\infty} q(t) = 0$ and $\dot{q}(\pm\infty) = 0$.

Let E be given by

$$E = \left\{ q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^3) : \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt < \infty \right\}.$$

It is known that E equipped with the norm

$$\|q\|_E = \left(\int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt + |q(0)|^2 \right)^{1/2}$$

is a Hilbert space. We will consider the families of paths that omit l defined as follows:

$$\Lambda = \{q \in E : q(\mathbb{R}) \cap l = \emptyset\}, \qquad \Omega = \{q \in \Lambda : q(\pm\infty) = 0\}.$$

For $q \in \Lambda$, set

$$I(q) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\dot{q}(t)|^2 - V(q(t)) \right) dt. \tag{2}$$

We will minimize the action functional I to prove the existence of a homoclinic solution winding around l. To this end, we will define a rotation (winding) number of $q \in \Omega$ around l.

Let Π be the plane perpendicular to the line l and such that $0 \in \Pi$. Let us introduce the cylindrical coordinate system in $\mathbb{R}^3$ with the reference plane Π and the height axis l. We will denote by P the intersection point of Π and l. Then P is the pole and the half-line $P0$ is the polar axis. If $\{\vec{P0}, \vec{PR}\}$ is an orthogonal positively oriented basis of Π then a positive direction of the height axis l is determined by $\vec{P0} \times \vec{PR}$. In this coordinate system, for all $q \in \Lambda$ one has

$$q(t) = \big(r(t)\cos\varphi(t),\ r(t)\sin\varphi(t),\ z(t)\big),$$

where $r(t)$ is a distance of $q(t)$ from l, $\varphi(t)$ is a polar angle and $z(t)$ is a distance of $q(t)$ from Π. There is no uniqueness of a function φ. However, if q is continuous then we can assume that r, φ and z are continuous, too.

Definition 1.2.
For each $q \in \Omega$ we can determine the rotation number $\mathrm{rot}(q)$ as follows:

$$\mathrm{rot}(q) = \frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}.$$

This definition is independent of the choice of a function φ.

Set $\varepsilon_0 = |P|/3$. From now on, $B_\varepsilon(x)$ stands for an open ball in $\mathbb{R}^3$ of radius $\varepsilon > 0$, centered at a point $x \in \mathbb{R}^3$.

Remark 1.3.

Let $0 < \varepsilon \leq \varepsilon_0$. Assume that $q \in \Omega$ and there is $T \in \mathbb{R}$ such that $q(T) \in B_\varepsilon(0)$. Then, by $\mathrm{rot}(q\restriction_{(-\infty,T]})$ and $\mathrm{rot}(q\restriction_{[T,\infty)})$ we mean the rotation numbers of appropriate paths in Ω that arise from $q\restriction_{(-\infty,T]}$ and $q\restriction_{[T,\infty)}$ resp., by connecting $q(T)$ to 0 by a line segment. It is justified by elementary homotopy arguments that

$$\mathrm{rot}(q) = \mathrm{rot}\big(q\restriction_{(-\infty,T]}\big) + \mathrm{rot}\big(q\restriction_{[T,\infty)}\big).$$

Moreover, if $q([T,\infty)) \subset B_\varepsilon(0)$ then

$$\mathrm{rot}(q) = \mathrm{rot}\big(q\restriction_{(-\infty,T]}\big).$$

Remark 1.4.

If $q_1, q_2 \in \Omega$ and there exist $T_1, T_2 \in \mathbb{R}$ and $0 < \varepsilon \leq \varepsilon_0$ such that $T_1 < T_2$, $q_1((-\infty, T_1]) \cup q_2((-\infty, T_1]) \subset B_\varepsilon(0)$, $q_1([T_2,\infty)) \cup q_2([T_2,\infty)) \subset B_\varepsilon(0)$ and $q_1(t) = q_2(t)$ for all $t \in [T_1, T_2]$, then $\mathrm{rot}(q_1) = \mathrm{rot}(q_2)$.

Set $\Omega^\pm = \{q \in \Omega : \pm\mathrm{rot}(q) > 0\}$ and define

$$\omega^\pm = \inf\{I(q) : q \in \Omega^\pm\}. \tag{3}$$

The main result of this paper is the following.

Theorem 1.5.

If $V\colon \mathbb{R}^3 \setminus l \to \mathbb{R}$ satisfies (V_1)–(V_5), then $\omega^\pm > 0$ and there exists $q^\pm \in \Omega^\pm$ such that $I(q^\pm) = \omega^\pm$. Moreover, $q^\pm$ is a classical homoclinic solution of (1).

Let us remark that if $q \in \Omega^+$, then $p(t) = q(-t) \in \Omega^-$ and $I(q) = I(p)$. It follows that $\omega^+ = \omega^- \equiv \omega$ and for each minimizer $q \in \Omega^+$ for $(3)^+$, $p(t) = q(-t)$ is a minimizer for $(3)^-$. Note also that if $q \in \Omega^\pm$, then $q + s\psi \in \Omega^\pm$ for $s \in \mathbb{R}$ small enough and $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^3)$. From this we conclude that if $q \in \Omega^\pm$ is a minimizer of the action integral I on $\Omega^\pm$, then

$$\frac{d}{ds} I(q + s\psi)\big|_{s=0} = 0 = \int_{-\infty}^{\infty} \big((\dot q(t), \dot\psi(t)) - (\nabla V(q(t)), \psi(t))\big)\, dt,$$

and, in consequence, q is a weak solution of (1). Arguments similar to those in [10] show that q is a classical solution of (1).

The case of singular Hamiltonian systems is rather important, due to the fact that potentials arising in physics have infinitely deep wells. It seems there are not many works on homoclinics for singular Hamiltonian systems involving strong forces.

In 1996, in the article [13], P.H. Rabinowitz investigated a nonautonomous planar second order Hamiltonian system $\ddot q + V_q(t, q) = 0$. He assumed that $V\colon \mathbb{R} \times (\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{R}$ possesses a global maximum at the origin and the singularity at a point ξ. Moreover, V is periodic with respect to a real variable t. He proved that there exist at least two homoclinic solutions: at least one of a positive rotation and at least one of a negative rotation. In [4], by the extra assumption about the existence of a minimal noncontractible periodic orbit around ξ due to S. Bolotin [2], P. Caldiroli and L. Jeanjean established that if V does not depend on a time variable, then for each $k \in \mathbb{Z}$ there exists a homoclinic solution of rotation k, cf. [1, 5]. In [3] M. Borges considered a planar second order Hamiltonian system with a potential having a global maximum at the origin and two strong force singularities at points ξ_1 and ξ_2. By the additional assumptions that $V\colon \mathbb{R}^2 \setminus \{\xi_1, \xi_2\} \to \mathbb{R}$ is of class C^2 and the second derivative of V at 0 is negative definite, she found homoclinic solutions winding around each singularity and around both singularities, periodic solutions and heteroclinic solutions joining 0 to periodic solutions. For $n > 2$ and $V = V(q)$, the existence of homoclinic solutions under slightly stronger assumptions than (V_1)–(V_5) was shown by K. Tanaka in [14]. Finally, in [12] Rabinowitz proved the existence of so-called multibump homoclinic solutions for a family of singular Hamiltonian systems in $\mathbb{R}^2$ which are subjected to almost periodic forcing in time, cf. [11].

Our paper extends the result of [13] for the case $n = 3$ and a line as a set of singular points. The work is organized as follows. In Section 2 we discuss certain properties of the action integral. Section 3 contains a proof of Theorem 1.5.

2. Some properties of the action integral

In this section we present some properties of the action functional I given by (2). Define

$$\alpha_\varepsilon = \inf\{-V(x) : x \notin B_\varepsilon(0)\},$$

where $0 \le \varepsilon \le \varepsilon_0$. By ($V_2$), ($V_4$) and ($V_5$) it follows that $\alpha_\varepsilon > 0$.

Lemma 2.1.
Suppose that $q \in \Lambda$ and $q(t) \notin B_\varepsilon(0)$ for each $t \in \bigcup_{i=1}^k [r_i, s_i]$, where $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ for $i \neq j$. Then

$$I(q) \ge \sqrt{2\alpha_\varepsilon} \sum_{i=1}^{k} |q(r_i) - q(s_i)|.$$

The proof of Lemma 2.1 is the same as that of [8, Lemma 2.1] or [10, Lemma 3.6].

Lemma 2.2.
If $q \in \Lambda$ and $I(q) < \infty$, then $q \in L^\infty(\mathbb{R}, \mathbb{R}^3)$.

Lemma 2.3.
If $q \in \Lambda$ and $I(q) < \infty$, then $q(\pm\infty) = 0$.

We can easily prove these two lemmas by the use of Lemma 2.1. For more details we refer the reader to [8, Corollary 2.2 and Lemma 2.4] and to [10, Remark 3.10 and Proposition 3.11].

Proposition 2.4.
If $\{q_m\}_{m\in\mathbb{N}} \subset \Lambda$ is a sequence such that $\{I(q_m)\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, then $\{q_m\}_{m\in\mathbb{N}}$ possesses a subsequence that converges weakly in E, and hence strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^3)$.

The proof is similar to the proof of [9, Proposition 2.4]. We briefly sketch it.

Proof. It suffices to show that $\{q_m\}_{m\in\mathbb{N}}$ is bounded in E. By assumption, there is a constant $M > 0$ such that $0 \le I(q_m) \le M$ for all $m \in \mathbb{N}$. Using (2) we get $\|\dot{q}_m\|^2_{L^2(\mathbb{R},\mathbb{R}^3)} \le 2M$ for all $m \in \mathbb{N}$. By Lemma 2.2, $q_m \in L^\infty(\mathbb{R}, \mathbb{R}^3)$ for all $m \in \mathbb{N}$. From Lemma 2.3 it follows that $q_m(\pm\infty) = 0$ for all $m \in \mathbb{N}$. Finally, from Lemma 2.1 we deduce that $\{q_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}, \mathbb{R}^3)$. □

Lemma 2.5.
If $q \in \Lambda$ and $q(t) \in \mathcal{N}$ for all $t \in [\sigma, \mu]$, then

$$|U(q(\mu))| \le |U(q(\sigma))| + \left(\int_\sigma^\mu (-V(q(t)))\, dt\right)^{1/2} \left(\int_\sigma^\mu |\dot{q}(t)|^2 dt\right)^{1/2}.$$

The proof of this lemma is the same as that of the inequality [13, (2.21), p. 271]. Applying the above inequality and (2), for $q \in \Lambda$ such that $q(t) \in \mathcal{N}$ for all $t \in [\sigma, \mu]$ we get

$$|U(q(\mu))| \leq |U(q(\sigma))| + \sqrt{2}\, I(q). \tag{4}$$

Proposition 2.6.
Let $\{q_m\}_{m\in\mathbb{N}} \subset \Lambda$ be a sequence such that $\{I(q_m)\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Then there is $r > 0$ such that $d(q_m(t), l) > r$ for all $t \in \mathbb{R}$ and $m \in \mathbb{N}$.

The above proposition is analogous to [9, Proposition 2.6]. However, a slight modification of the proof is needed.

Proof. By Lemma 2.3, $q_m(\pm\infty) = 0$ for each $m \in \mathbb{N}$. From Lemma 2.1 we deduce that $\{q_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}, \mathbb{R}^3)$. Hence there exists $r_0 > 0$ such that $q_m(t) \in B_{r_0}(0)$ for all $t \in \mathbb{R}$ and $m \in \mathbb{N}$. If $B_{r_0}(0) \cap \mathcal{N} = \emptyset$, then $r = d(B_{r_0}(0), l)$.

Consider the case $B_{r_0}(0) \cap \mathcal{N} \neq \emptyset$. On the contrary, suppose that there exists a sequence $\{q_m(\mu_m)\}_{m\in\mathbb{N}}$ such that $q_m(\mu_m) \to l$ as $m \to \infty$. Fix $0 < \delta \leq \varepsilon_0$ such that $\{S \in \Pi : |S - P| \leq \delta\} \times \{Z \in l : |Z - P| \leq r_0\} \subset \mathcal{N}$. There is $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, $d(q_m(\mu_m), l) < \delta$. For each $m \geq m_0$ there exists $\sigma_m < \mu_m$ such that $q_m(\sigma_m) \in \{S \in \Pi : |S - P| = \delta\} \times \{Z \in l : |Z - P| \leq r_0\}$ and $q_m(t) \in \{S \in \Pi : |S - P| < \delta\} \times \{Z \in l : |Z - P| < r_0\}$ for all $t \in (\sigma_m, \mu_m)$. Then, by (4), for all $m \geq m_0$,

$$|U(q_m(\mu_m))| \leq |U(q_m(\sigma_m))| + \sqrt{2}\, I(q_m).$$

As $\{U(q_m(\sigma_m))\}_{m\in\mathbb{N}}$ and $\{I(q_m)\}_{m\in\mathbb{N}}$ are bounded, we get $\{U(q_m(\mu_m))\}_{m\in\mathbb{N}}$ is bounded, too. On the other hand, by (V_3), we obtain $|U(q_m(\mu_m))| \to \infty$ as $m \to \infty$, a contradiction. □

3. The proof of Theorem 1.5

Let $\{q_m\}_{m\in\mathbb{N}} \subset \Omega^-$ be a sequence such that

$$\lim_{m\to\infty} I(q_m) = \omega^-.$$

From Proposition 2.4 it follows that there is $Q \in E$ such that going to a subsequence if necessary, $q_m \rightharpoonup Q$ in E, and hence $q_m \to Q$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^3)$. By Proposition 2.6 we conclude that $Q \in \Lambda$.

Remark 3.1.
For all $T_1, T_2 \in \mathbb{R}$ such that $T_1 < T_2$, a functional given by

$$E \ni q \mapsto \int_{T_1}^{T_2} \left(\frac{1}{2} |\dot{q}(t)|^2 - V(q(t)) \right) dt$$

is weakly lower semi-continuous.

Hence for each $k \in \mathbb{N}$,

$$\int_{-k}^{k} \left(\frac{1}{2} |\dot{Q}(t)|^2 - V(Q(t)) \right) dt \leq \liminf_{m\to\infty} \int_{-k}^{k} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt \leq \lim_{m\to\infty} I(q_m) = \omega^-.$$

Letting $k \to \infty$ we get

$$I(Q) \leq \omega^-. \tag{5}$$

By Lemma 2.3, $Q(\pm\infty) = 0$. Thus $Q \in \Omega$.

Fix $0 < \varepsilon \leq \varepsilon_0$. Since $q_m(-\infty) = 0$, there is $\tau_m \in \mathbb{R}$ such that $q_m(\tau_m) \in \partial B_\varepsilon(0)$ and $q_m(t) \in B_\varepsilon(0)$ for all $t < \tau_m$. Note that if $q \in \Lambda$ and $\tau \in \mathbb{R}$, then $\tau q(t) = q(t-\tau) \in \Lambda$ and $I(\tau q) = I(q)$. Therefore we can assume that $\tau_m = 0$ for each $m \in \mathbb{N}$. Consequently, $q_m(0) \in \partial B_\varepsilon(0)$ and $q_m(t) \in B_\varepsilon(0)$ for all $t < 0$ and $m \in \mathbb{N}$. Hence $|Q(t)| \leq \varepsilon$ for all $t \leq 0$. Moreover, $Q \neq 0$, which implies $I(Q) > 0$.

An indirect argument will be employed to obtain $Q \in \Omega^-$. Suppose that $\mathrm{rot}(Q) \geq 0$.

Lemma 3.2.

For each $\eta > 0$ there is $0 < r \leq \varepsilon_0$ such that for all $x, y \in B_r(0)$ and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2} |x-y|^2 - V(p_{x,y}(t)) \right) dt < \eta,$$

where $p_{x,y}(t) = (T+1-t)x + (t-T)y$ for each $t \in [T, T+1]$.

The proof of this lemma is immediate.

Fix $\eta > 0$. By Lemma 3.2, there is $0 < \delta \leq \varepsilon_0$ such that for all $x, y \in B_\delta(0)$ and $T \in \mathbb{R}$,

$$\int_T^{T+1} \left(\frac{1}{2} |x-y|^2 - V(p_{x,y}(t)) \right) dt < \frac{\eta}{2}.$$

Choose $T > 0$ such that $Q([T, \infty)) \subset B_\delta(0)$ and

$$\int_{-\infty}^{T} \left(\frac{1}{2} |\dot{Q}(t)|^2 - V(Q(t)) \right) dt > I(Q) - \frac{\eta}{4}.$$

Since $q_m \to Q$ uniformly on $[0, T+1]$, there is $m_0 \in \mathbb{N}$ such that $q_m([T, T+1]) \subset B_\delta(0)$ and $\mathrm{rot}\big(q_m\restriction_{(-\infty,T]}\big) = \mathrm{rot}(Q)$ for all $m \geq m_0$.

By Remark 3.1, there is $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$,

$$\int_{-\infty}^{T} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt > \int_{-\infty}^{T} \left(\frac{1}{2} |\dot{Q}(t)|^2 - V(Q(t)) \right) dt - \frac{\eta}{4}.$$

For $m \geq m_0$, let

$$u_m(t) = \begin{cases} 0 & \text{if} \quad t \leq T, \\ (t-T)\, q_m(T+1) & \text{if} \quad t \in [T, T+1], \\ q_m(t) & \text{if} \quad t \geq T+1. \end{cases}$$

Since $\mathrm{rot}(q_m) < 0$ and $\mathrm{rot}\big(q_m\restriction_{(-\infty,T]}\big) = \mathrm{rot}(Q) \geq 0$, we get

$$\mathrm{rot}(u_m) = \mathrm{rot}\big(q_m\restriction_{[T+1,\infty)}\big) < 0.$$

Thus $u_m \in \Omega^-$. Furthermore, for $m \geq \max\{m_0, m_1\}$,

$$I(q_m) - I(u_m) = \int_{-\infty}^{T+1} \left(\frac{1}{2} |\dot{q}_m(t)|^2 - V(q_m(t)) \right) dt - \int_{T}^{T+1} \left(\frac{1}{2} |\dot{u}_m(t)|^2 - V(u_m(t)) \right) dt > I(Q) - \eta,$$

and so

$$I(q_m) > I(u_m) + I(Q) - \eta.$$

Passing to a limit we obtain

$$\omega^- = \lim_{m\to\infty} I(q_m) \geq \liminf_{m\to\infty} I(u_m) + I(Q) - \eta \geq \omega^- + I(Q) - \eta.$$

Letting $\eta \to 0^+$,

$$\omega^- \geq \omega^- + I(Q) > \omega^-,$$

a contradiction. Therefore $\mathrm{rot}(Q) < 0$, and, in consequence, $Q \in \Omega^-$. From (5) and $(3)^-$ it follows that $I(Q) = \omega^-$. To complete the proof of Theorem 1.5 we have to show that $\dot{Q}(\pm\infty) = 0$. For this purpose, we use the inequality [7, (28)].

Fact 3.3.
If $q\colon \mathbb{R} \to \mathbb{R}^n$ is a continuous mapping such that $\dot{q} \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, then for each $t \in \mathbb{R}$,

$$|q(t)| \leq \sqrt{2}\left(\int_{t-1/2}^{t+1/2} (|q(s)|^2 + |\dot{q}(s)|^2)\, ds\right)^{1/2}.$$

Using this inequality we get

$$|\dot{Q}(t)| \leq \sqrt{2}\left(\int_{t-1/2}^{t+1/2} (|\dot{Q}(s)|^2 + |\ddot{Q}(s)|^2)\, ds\right)^{1/2}$$

for all $t \in \mathbb{R}$. As Q is a classical solution of (1), we have

$$|\dot{Q}(t)|^2 \leq 2\int_{t-1/2}^{t+1/2} |\dot{Q}(s)|^2 ds + 2\int_{t-1/2}^{t+1/2} |\nabla V(Q(s))|^2 ds.$$

Let $\eta > 0$. By (V_4), there is a constant $L_1 > 0$ such that if $|s| > L_1$, then $|\nabla V(Q(s))|^2 < \eta/4$. Since $Q \in \Omega^- \subset E$, there is a constant $L_2 > 0$ such that

$$\int_{t-1/2}^{t+1/2} |\dot{Q}(s)|^2 ds < \frac{\eta}{4}$$

for $|t| > L_2$. Put $L = \max\{L_1, L_2\}$. By the above, if $|t| > L + 1/2$, then $|\dot{Q}(t)|^2 < \eta$. Hence $\dot{Q}(t) \to 0$ as $t \to \pm\infty$.

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