Cent. Eur. J. Phys. • 10(5) • 2012 • 1116-1124
DOI: 10.2478/s11534-012-0098-8

VERSITA

Central European Journal of **Physics**

Generation of the vorticity mode by sound in a vibrationally relaxing gas

Research Article

Anna Perelomova*, Pawel Wojda†

Gdansk University of Technology, Faculty of Applied Physics and Mathematics,
ul. Narutowicza 11/12, 80-233 Gdansk, Poland

Received 05 September 2011; accepted 18 June 2012

Abstract: The procedure of derivation of a new dynamical equation governing the vorticity mode that is generated by sound, is discussed in detail. It includes instantaneous quantities and does not require averaging over sound period. The resulting equation applies to both periodic and aperiodic sound as the origin of the vorticity mode. Under certain conditions, the direction of streamlines of the vorticity mode may be inverted as compared with that in a fluid with standard attenuation. This reflects an anomalous absorption of sound, when transfer of momentum of the vorticity mode into momentum of sound occurs. The theory is illustrated by a representative example of the generation of vorticity in a vibrationally relaxing gas in the field of periodic weakly diffracting acoustic beam.

PACS (2008): 43.35.Fj, 43.25.Ed, 43.25.Nm

Keywords: non-equilibrium gas • relaxation • nonlinear acoustics

1. Introduction. Basic equations and starting points

The theory of nonlinear sound propagation and phenomena associated with it in a thermoviscous nonlinear flow has achieved undoubted success. The secondary nonlinear phenomena induced by sound are of great importance in medical and technical applications. These effects involve acoustic streaming, i.e., nonlinear generation of the vorticity mode by periodic sound, and acoustic heating, i.e., nonlinear generation of the entropy mode by it. Extensive reviews on this subject exist in the literature [1–3].

Larger streaming velocities are obtained for compressible fluids and are more evident in a gas than in a liquid [4]. Nonlinear effects of aperiodic sound, including pulses or wave packets, are also of great importance in medical and technical applications. The nonlinear generation of the vorticity mode in the thermoviscous flow occurs independently on the type of sound, periodic or not. The procedure of determination of every branch of acoustic and non-acoustic modes in Newtonian fluids, and further derivation of individual dynamic equations governing every mode in a weakly nonlinear flow, has been proposed and applied by one of the authors in analysis of nonlinear interaction of sound and non-acoustic modes [5, 6].

It may be readily applied in studies of flows over non-Newtonian fluids, like Bingham plastics [7]. The method

*E-mail: anpe@mif.pg.gda.pl (Corresponding Author)
†E-mail: pwojda@mif.pg.gda.pl

makes it possible to distribute the nonlinear terms between dynamic equations of different modes correctly.

The interest on acoustics of non-Newtonian fluids constantly grows. This is due to new applications of sound in different areas, starting from shear-thinning and shear-thickening fluids and ending by thixotropy fluids with very complex fluid characteristics, which are common in food products [8].

This study considers a gas flow in some sense similar to flows in the Maxwell relaxing fluids. It considers relaxation of excited vibrational degrees of freedom of gas molecules. The important feature of these gases (widely used in lasers), making their dynamics special, is that excitation may take place reversibly or not. In spite of different physical content, some chemically reacting gases possess similar features. Sound amplifies instead of being attenuated during propagation over medium in irreversible (i.e., non-equilibrium) regime in both cases. The properties of nonlinear interaction of sound with non-acoustic modes also look different in these two regimes. The hydrodynamics of non-equilibrium fluids is one of the new fields of modern hydrodynamics. It is now passing through the stage of formulating the fundamental equations and of revealing new physical effects. A number of problems related to the effects of nonlinearity and rate processes in gases with internal relaxation have been studied previously. In this context, the contributions due to Chu, Parker, Clarke and McChesney are worth mentioning [9–11].

This paper is devoted to the nonlinear generation of the vorticity mode by sound in vibrationally relaxing gas. The theoretical possibility of anomalous streaming in a vibrationally excited molecular gas, was first discovered by Molevich [12]. The mean flow of the vortex motion may occur in the opposite direction compared to that in an equilibrium gas. The possibility to estimate vortex motion is important in closed vessels occupied by gaseous mixtures which are used in lasers. The bulk vortex motion transports perturbations in temperature. In turn, this affects the conditions of heat exchange between internal and translational degrees of freedoms of the molecules in a gas. The bulk motion alters the propagation of a sound beam. Along with cooling, it may lead to anomalous self-focusing of a beam [12]. The large-scale vortex motion is of importance in the atmospheres of Venus and Mars, consisting predominately of CO_2 molecules. They are natural laser objects.

In Sec. 3.2 we derive the instantaneous equation governing the vorticity mode. As an example, in Sec. 3.3 we illustrate the latter by applying it to a weakly diffracting periodic sound beam as the origin of streaming. Our starting point is the linear determination of modes as specific types of gas motion whose steady, but non-equilibrium, state is maintained by pumping energy into the vibrational degrees of freedom at a time rate I and a heat withdrawal from the translational degrees of freedom at a time rate Q (both I and Q refer to unit mass) (Sec. 2). The effects of standard attenuation on both sound and vorticity mode will be briefly discussed in the Concluding remarks. The relaxation equation for the vibrational energy per unit mass should complete the system of conservation equations in the differential form,

$$\frac{d\varepsilon}{dt} = -\frac{\varepsilon - \varepsilon_{eq}(T)}{\tau} + I. \tag{1}$$

The equilibrium value of the vibrational energy at the given temperature T is denoted by $\varepsilon_{eq}(T)$, and $\tau(\rho, T)$ denotes the vibrational relaxation time. The equilibrium quantity $\varepsilon_{eq}(T)$ equals in the case of a system of harmonic oscillators,

$$\varepsilon_{eq}(T) = \frac{\hbar\Omega}{m\left(\exp(\hbar\Omega/k_B T) - 1\right)}, \tag{2}$$

where m is the mass of a molecule, $\hbar\Omega$ is the magnitude of the vibrational quantum and k_B is the Boltzmann constant. Eq.(2) is valid for temperatures above those where one can neglect anharmonic effects [13, 14]. The mass, momentum and energy equations describing the flow over a vibrationally relaxing gas read:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0,$$

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p, \tag{3}$$

$$\rho\left[\frac{\partial(e+\varepsilon)}{\partial t} + (\vec{v}\cdot\vec{\nabla})(e+\varepsilon)\right] + p(\vec{\nabla}\cdot\vec{v}) = \chi\Delta T\rho(I-Q),$$

where $\vec{v}$ denotes velocity of a fluid, ρ and p are density and pressure, e is internal energy per unit mass of translation motion of molecules. Besides Eq. (2), two thermodynamic functions $e(p,\rho)$, $T(p,\rho)$ complement the system (3). The standard attenuation of a gas including first, second viscosity and thermal conductivity is not considered by Eqs. (3). Acoustic streaming in standard attenuating media is well studied, and the objective of this study is to concentrate on nonlinear phenomena originated exclusively from the excitation of internal degrees of freedom of the gas molecules. The validity of this limitation will be briefly discussed in the Concluding Remarks. The thermodynamics of an ideal gases gives the equalities

$$e(p,\rho) = \frac{p}{(\gamma-1)\rho} = \frac{R}{\mu(\gamma-1)}T(p,\rho), \tag{4}$$

where γ is the isentropic exponent without account for vibrational degrees of freedom, R is the universal gas constant, and μ is the molar mass of a gas.

2. Modes in a linear flow and their decomposition

2.1. Equations governing a linear flow

Let us consider a linear two-dimensional flow in the plane OXY. Every quantity q represents a sum of an unperturbed value q_0 and its variation q', ($|q'| \ll |q_0|$). Following Molevich, we assume that stationary quantities ε_0, T_0, P_0, ρ_0 are maintained by a transverse pumping. The background quantities are constant in the plane OXY, and $I_0 = Q_0$, $I' = Q' = 0$. The governing equations of momentum, energy balance and continuity may be readily rearranged into

$$\frac{\partial v'_x}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0,$$

$$\frac{\partial v'_y}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} = 0,$$

$$\frac{\partial p'}{\partial t} + \gamma p_0 \left(\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y}\right) - (\gamma-1)\rho_0 \frac{\varepsilon'}{\tau} + (\gamma-1)\rho_0 T_0 \Phi_1 \left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0}\right) = 0, \quad (5)$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \left(\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y}\right) = 0,$$

$$\frac{\partial \varepsilon'}{\partial t} + \frac{\varepsilon'}{\tau} - T_0\Phi_1 \left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0}\right) = 0,$$

where

$$\Phi_1 = \left(\frac{C_v}{\tau} + \frac{\varepsilon - \varepsilon_{eq}}{\tau^2}\frac{d\tau}{dT}\right)_0 \quad (6)$$

is the quantity evaluated at p_0, T_0, and $C_v = d\varepsilon_{eq}/dT$. The series expansion of the equations of state (4) were used to express the perturbations of internal translational energy per unit mass and temperature in terms of perturbations in pressure and density,

$$e' = \frac{p_0}{(\gamma-1)\rho_0}\left(\frac{p'}{p_0} - \frac{\rho'}{\rho}\right) = \frac{R}{\mu(\gamma-1)}T'. \quad (7)$$

The last equation in the set (5) is actually the linearized version of Eq. (1):

$$\frac{\partial \varepsilon'}{\partial t} + \frac{\varepsilon'}{\tau} = \left(\frac{C_v}{\tau} + \frac{\varepsilon - \varepsilon_{eq}}{\tau^2}\frac{d\tau}{dT}\right)_0 T' = T_0\Phi_1\left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0}\right). \quad (8)$$

The relaxation time in the most important cases may be thought as a function of temperature according to Landau and Teller, $\tau(T) = \alpha \exp(\beta T^{-1/3})$ for some constants α and β [13, 14].

2.2. Dispersion relations

Studies of fluid motion of infinitely-small magnitude, usually start by representing all perturbations as a sum of planar waves, where $\overline{q}\exp(i\omega t)$ is the Fourier-transforms of any perturbation q':

$$q'(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{q}(k_x,k_y) \cdot \exp i(\omega t - k_x x - k_y y) dk_x dk_y + cc. \quad (9)$$

The dispersion equation is algebraic of the fifth order:

$$\omega^2\left(\omega^3 - i\left(\frac{1}{\tau} + \frac{\gamma(\gamma-1)T_0}{c^2}\Phi_1\right)\omega^2 - c^2\widetilde{\Delta}\omega + i\left(\frac{c^2}{\tau} + (\gamma-1)T_0\Phi_1\right)\widetilde{\Delta}\right) = 0, \quad (10)$$

where $\widetilde{\Delta} = k^2 = k_x^2 + k_y^2$ and $c = \sqrt{\gamma\frac{T_0}{\mu}} = \sqrt{\gamma\frac{p_0}{\rho_0}}$ denotes the linear sound speed in an ideal uniform gas. It is the velocity of sound of large frequency as compared with the inverse characteristic duration of relaxation, i.e., the "frozen" sound velocity. Eq. (10) determines two acoustic types of motion (they represent wave motion) and three non-wave modes.

The first simplifying, but physically founded condition on the way of establishing the acoustic roots, relates to the domain of acoustic frequencies large compared with the inverse duration of vibrational relaxation,

$$\frac{1}{\omega_{1,2}\tau} \approx \frac{1}{c\sqrt{\widetilde{\Delta}}\tau} \equiv \delta \ll 1. \quad (11)$$

The approximate roots of the dispersion equation for both acoustic branches under this condition are well-known [12, 14]:

$$\omega_1 = c\left(\sqrt{\widetilde{\Delta}} - iB\right), \quad \omega_2 = -c\left(\sqrt{\widetilde{\Delta}} + iB\right), \quad (12)$$

where

$$B = -\frac{(\gamma-1)^2 T_0}{2c^3}\Phi_1. \tag{13}$$

We will consider wave processes associated with sound, so its attenuation ($B < 0$, or amplification, $B > 0$) on the wavelength is small, $|B| \ll k$. The non-wave roots of the dispersive equation, estimated without the above limitations $\omega\tau \gg 1$ and $|B| \ll k$, are:

$$\begin{aligned} \omega_3 &= i\left(\frac{1}{\tau} + \frac{(\gamma-1)(\gamma + c^2\widetilde{\Delta}\tau^2)T_0}{c^2(1+c^2\widetilde{\Delta}\tau^2)}\Phi_1\right), \\ \omega_4 &= 0, \quad \omega_5 = 0. \end{aligned} \tag{14}$$

The mode determined by the third root of the dispersion equation, comes from the vibrational relaxation. One of the zero-frequency modes, labeled as fourth, exists in any planar flow of a fluid and it represents the thermal, or entropy mode. The fifth root determines the vorticity mode. It appears as one of the possible types of motion in flows exceeding one dimension. The last three roots manifest slow varying and stationary, non-wave motion of a gas.

2.3. Determination of modes

The dispersion relations (12), (14) uniquely determine the specific modes of a flow. The modes are in fact relations of perturbations in dynamic variables. They are different for the different modes. The total perturbation vector is a sum of all eigenvectors. In Fourier space, it takes the form

$$\overline{\psi} = \sum_{n=1}^{5} \overline{\psi}_n.$$

The relations for both acoustic branches determined by ω_1 and ω_2, are represented by $\overline{\psi}_1$, $\overline{\psi}_2$:

$$\overline{\psi}_i = \begin{pmatrix} \overline{v}_{x,i} \\ \overline{v}_{y,i} \\ \overline{p}'_i \\ \overline{\varepsilon}'_i \\ \overline{\rho}'_i \end{pmatrix} = \begin{pmatrix} \frac{\omega_i k_x}{\rho_0 \widetilde{\Delta}} \\ \frac{\omega_i k_y}{\rho_0 \widetilde{\Delta}} \\ c^2 + \frac{2Bc^3}{i\omega_i + A} \\ -\frac{2Bc^3}{(\gamma-1)\rho_0(i\omega_i + A)} \\ 1 \end{pmatrix} \overline{\rho}'_i,$$

$$A = \left(\frac{1}{\tau} + \frac{\gamma B c}{(\gamma-1)}\right), \quad i = 1, 2. \tag{15}$$

The third mode is determined by relationships

$$\overline{\psi}_3 = \begin{pmatrix} \overline{v}_{x,3} \\ \overline{v}_{y,3} \\ \overline{p}'_3 \\ \overline{\varepsilon}'_3 \\ \overline{\rho}'_3 \end{pmatrix} = \begin{pmatrix} \frac{\omega_3 k_x}{\rho_0 \widetilde{\Delta}} \\ \frac{\omega_3 k_y}{\rho_0 \widetilde{\Delta}} \\ \frac{\omega_3^2}{\widetilde{\Delta}} \\ \frac{1}{\rho_0(\gamma-1)}\left(c^2 - \frac{\omega_3^2}{\widetilde{\Delta}}\right) \\ 1 \end{pmatrix} \overline{\rho}'_3, \tag{16}$$

and the fourth (i.e., the entropy) mode is given by the vector $\overline{\psi}_4$ below

$$\overline{\psi}_4 = \begin{pmatrix} \overline{v}_{x,4} \\ \overline{v}_{y,4} \\ \overline{p}'_4 \\ \overline{\varepsilon}'_4 \\ \overline{\rho}'_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\tau T_0}{\rho_0}\Phi_1 \\ 1 \end{pmatrix} \overline{\rho}'_4. \tag{17}$$

All four first modes represent the potential velocity field,

$$\vec{\nabla} \times \vec{v}_i = \vec{0}, \ i = 1, \ldots 4. \tag{18}$$

The velocity of the vortex mode is solenoidal,

$$\vec{\nabla} \cdot \vec{v}_5 = 0, \ p'_5 = 0, \ \varepsilon'_5 = 0, \ \rho'_5 = 0. \tag{19}$$

Eqs. (18), (19) represent actually a certain way of application of the Helmholtz vector decomposition theorem, which enables us to decompose potential and solenoidal vector fields. The relations of specific perturbations in the (x, y) space follow from Eqs. (15) - (17) and dispersion relations, Eqs. (12), (14). The linear flow may be uniquely decomposed into individual modes at any time. That may be achieved by the use of a set of matrix projectors.

The matrix projectors were derived and exploited by one of the authors in some problems of nonlinear hydrodynamics in media with standard absorption [5, 6], and in studies of some dispersive media as well [16, 17].

For example, in order to decompose the vorticity part from the overall velocity vector, it is sufficient to apply the operator $\overline{P}_v$ on the vector of the Fourier-transforms of velocity components:

$$\overline{P}_v \begin{pmatrix} \overline{v}_x \\ \overline{v}_y \end{pmatrix} = \frac{1}{\widetilde{\Delta}} \begin{pmatrix} k_y^2 & -k_x k_y \\ -k_x k_y & k_x^2 \end{pmatrix} \begin{pmatrix} \overline{v}_x \\ \overline{v}_y \end{pmatrix} = \begin{pmatrix} \overline{v}_{x,5} \\ \overline{v}_{y,5} \end{pmatrix}. \tag{20}$$

Operating in the (x, y) space P_v satisfies the equality

$$P_v \Delta = \begin{pmatrix} \frac{\partial^2}{\partial y^2} & -\frac{\partial^2}{\partial x \partial y} \\ -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} \end{pmatrix}, \tag{21}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the Laplacian operator acting on the (x, y) space. It corresponds to the factor $-\widetilde{\Delta}$ in the space of Fourier transforms.

2.4. Quasi-planar sound

In order to simplify the mathematical content and meaning of the physically interesting case of quasi-planar sound propagating in the direction of the OX axis, let assume that all acoustic perturbations vary much faster in the OX direction than along OY: $k_x \gg k_y$. That is a standard assumption in the theory of acoustics which studies acoustic beams [2, 18]. It allows us to expand the relations for sound perturbations in power series of the small parameter $\mu = k_y^2/k_x^2$. For a progressive sound beam in the positive OX direction, the leading-order relationships (within accuracy up to terms involving Φ_1^1, μ^1 and δ^1), take the form:

$$\begin{pmatrix} v'_{x,1}(x,y,t) \\ v'_{y,1}(x,y,t) \\ p'_1(x,y,t) \\ \varepsilon'_1(x,y,t) \\ \rho'_1(x,y,t) \end{pmatrix} = \begin{pmatrix} \frac{c}{\rho_0} + \frac{c}{2\rho_0}\frac{\partial^2}{\partial y^2}\int dx \int dx - \frac{Bc}{\rho_0}\int dx \\ \frac{c}{\rho_0}\frac{\partial}{\partial y}\int dx \\ c^2 - 2Bc^2 \int dx \\ \frac{2Bc^2}{(\gamma-1)\rho_0}\int dx \\ 1 \end{pmatrix} \rho'_1. \tag{22}$$

In view of relations (22), the equation governing an acoustic excess density in the wave propagating in the positive OX direction, to leading order is

$$\frac{\partial \rho'_1}{\partial t} + c\frac{\partial \rho'_1}{\partial x} + \frac{c}{2}\frac{\partial^2}{\partial y^2}\int \rho'_1 dx - cB\rho'_1 = 0. \tag{23}$$

We are able, obviously, to suggest the model propagation equation (23) follows not only from relations (22), but also from the dispersion relations (12), taking into account that $\sqrt{\widetilde{\Delta}} \approx k_x + \frac{1}{2}\frac{k_y^2}{k_x}$. The model propagation equation (23) not only from relations (22), but also from the dispersion relations (12), taking into account, that $\sqrt{\widetilde{\Delta}} \approx k_x + \frac{1}{2}\frac{k_y^2}{k_x}$.

3. Dynamic equations of sound beam and vorticity in a weakly nonlinear flow

3.1. Accounting for the second-order nonlinear terms in the hydrodynamic system

Taking into account the leading-order nonlinear terms of the second order in Eqs. (1), (4) yield the series:

$$\frac{d\varepsilon'}{dt} = -\frac{\varepsilon'}{\tau} + T_0\left(\frac{1}{\tau^2}\frac{d\tau}{dT}\right)_0 \varepsilon'\left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0}\right) + T_0\Phi_1\left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0} + \frac{\rho'^2}{\rho_0^2} - \frac{p'\rho'}{p_0\rho_0}\right) + T_0\Phi_2\left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0}\right)^2,$$

$$\Phi_2 = T_0\left(-\frac{1}{\tau^2}C_v\frac{d\tau}{dT} - \frac{(\varepsilon_0 - \varepsilon_{eq})}{\tau^3}\left(\frac{d\tau}{dT}\right)^2 + \frac{1}{2\tau}\frac{dC_v}{dT} + \frac{(\varepsilon_0 - \varepsilon_{eq})}{2\tau^2}\frac{d^2\tau}{dT^2}\right)_0, \tag{24}$$

$$T' = T_0\left(\frac{p'}{p_0} - \frac{\rho'}{\rho_0} + \frac{\rho'^2}{\rho_0^2} - \frac{p'\rho'}{p_0\rho_0}\right).$$

The governing dynamic system with account for quadratic nonlinear terms differs from (5) by the quadratic right-hand side:

$$\frac{\partial v'_x}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = -\left(v'_x\frac{\partial}{\partial x} + v'_y\frac{\partial}{\partial y}\right)v'_x + \frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial x},$$

$$\frac{\partial v'_y}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial y} = -\left(v'_x\frac{\partial}{\partial x} + v'_y\frac{\partial}{\partial y}\right)v'_y + \frac{\rho'}{\rho_0^2}\frac{\partial p'}{\partial y},$$

$$
\begin{aligned}
&\frac{\partial p'}{\partial t}+\gamma p_0\left(\frac{\partial v'_x}{\partial x}+\frac{\partial v'_y}{\partial y}\right)-(\gamma-1)\rho_0\frac{\varepsilon'}{\tau}+(\gamma-1)\rho_0T_0\Phi_1\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)=\\
&-\left(v'_x\frac{\partial}{\partial x}+v'_y\frac{\partial}{\partial y}\right)p'-\gamma p'\left(\frac{\partial v'_x}{\partial x}+\frac{\partial v'_y}{\partial y}\right)+(\gamma-1)\rho'\left(\frac{\varepsilon'}{\tau}-T_0\Phi_1\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)\right)\\
&-(\gamma-1)\rho_0\left(T_0\left(\frac{1}{\tau^2}\frac{d\tau}{dT}\right)_0\varepsilon'\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)+T_0\Phi_1\left(\frac{\rho'^2}{\rho_0^2}-\frac{p'\rho'}{p_0\rho_0}\right)+T_0\Phi_2\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)^2\right),
\end{aligned}
\tag{25}
$$

$$
\frac{\partial \rho'}{\partial t}+\rho_0\left(\frac{\partial v'_x}{\partial x}+\frac{\partial v'_y}{\partial y}\right)=-\left(v'_x\frac{\partial}{\partial x}+v'_y\frac{\partial}{\partial y}\right)\rho'-\rho'\left(\frac{\partial v'_x}{\partial x}+\frac{\partial v'_y}{\partial y}\right),
$$

$$
\begin{aligned}
&\frac{\partial \varepsilon'}{\partial t}+\frac{\varepsilon'}{\tau}-T_0\Phi_1\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)=T_0\left(\frac{1}{\tau^2}\frac{d\tau}{dT}\right)_0\varepsilon'\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)+T_0\Phi_1\left(\frac{\rho'^2}{\rho_0^2}-\frac{p'\rho'}{p_0\rho_0}\right)+\\
&T_0\Phi_2\left(\frac{p'}{p_0}-\frac{\rho'}{\rho_0}\right)^2-\left(v'_x\frac{\partial}{\partial x}+v'_y\frac{\partial}{\partial y}\right)\varepsilon'.
\end{aligned}
$$

3.2. Nonlinear equations governing sound and vorticity mode

The linear projection is fruitful in investigations of weakly nonlinear interactions of different modes [5, 6, 16, 17]. We still fix linear relations of perturbations in accordance to Eqs. (15) - (17), (19) and will consider every field perturbation as a sum of perturbations of different modes.

The main idea is to decompose the equations governing different modes by applying the corresponding projector on the system that includes weakly nonlinear terms (25). Application of projector on the linear part of the equations results in a reduction of all other modes there, and the nonlinear terms become distributed between the equations in the proper way. The approximate solution of the final dynamic equations depends on the contribution of every mode in the overall field perturbation.

The problems of generation of non-acoustic types of motion by intense sound are of major importance. The solution of these problems gives hope to change the parameters of a medium (its temperature, the mean flow in it, etc.) remotely. We consider intense sound, progressive in the positive direction, as compared to all other modes. That means that the characteristic amplitude of the velocity associated with the first branch of sound, in the considered domain, is much greater than that of the other modes:

$$
max|v_1|\gg max|v_n|,\ n=2,..,5. \tag{26}
$$

We will only keep terms corresponding to the rightwards progressive sound in the nonlinear terms in all formulas below. In view of relations specific for sound, the governing leading-order equations for an acoustic excess density ($\rho_a\equiv\rho'_1$) takes the forms, the first being general, and the second describing sound beam,

general

$$
\frac{\partial \rho_a}{\partial t}+c\sqrt{\Delta}\rho_a-cB\rho_a+\frac{1}{2}\left(\gamma\rho_a\vec{\nabla}\vec{v_a}+(\vec{v_a}\vec{\nabla})\rho_a\right)=0,
\tag{27}
$$

quasi-planar

$$
\frac{\partial \rho_a}{\partial t}+c\frac{\partial \rho_a}{\partial x}+\frac{c}{2}\frac{\partial^2}{\partial y^2}\int\rho_a dx-cB\rho_a+\frac{(\gamma+1)c}{2\rho_0}\frac{\partial \rho_a}{\partial x}\rho_a=0.
$$

In order to decompose the dynamic equations for the velocity of the vorticity mode, it is sufficient to apply the matrix operator P_v (Eq. (20)) on the momentum equation (two first equations from the system (25)).

As a result, all terms, corresponding to the potential velocity vector become reduced in the linear part. We only keep acoustic terms in the right-hand side of equations.

Application of P_v yields the dynamic equation for the vorticity mode in the field of intense sound, in two equivalent forms,

$$
\begin{aligned}
\frac{\partial \vec{v}_v}{\partial t}&=-\frac{1}{\rho_0}P_v\left(\rho_a\frac{\partial \vec{v}_a}{\partial t}\right),\\
\frac{\partial \vec{\Omega}}{\partial t}&=-\frac{1}{\rho_0}\vec{\nabla}\times\left(\rho_a\frac{\partial \vec{v}_a}{\partial t}\right),
\end{aligned}
\tag{28}
$$

where $\vec{\Omega}$ is the vorticity of a flow, $\vec{\Omega} = \vec{\nabla} \times \vec{v}_v$ with $\vec{v}_v$ replacing $\vec{v}_5$.
Eqs. (28) are valid for all sound frequencies, not only those large compared with $1/\tau$, and for any geometry of sound wave. Moreover, they decompose the solenoidal part of overall velocity in any fluid flow. The difference is in relations linking acoustic perturbations ρ_a and $\vec{v}_a$ in different media. Accounting for Eqs. (15), (22), one may rearrange equations (28) in the case of high-frequency sound, general (the first one) and concerning the quasi-planar geometry of a beam (the second one):

general

$$\frac{\partial \vec{v}_v}{\partial t} = \frac{2Bc^3}{\rho_0^2} P_v \left(\rho_a \int \vec{\nabla} \rho_a dt \right),$$
$$\frac{\partial \vec{\Omega}}{\partial t} = \frac{2Bc^3}{\rho_0^2} \left(\vec{\nabla} \rho_a \times \int \vec{\nabla} \rho_a dt \right), \tag{29}$$

quasi-planar

$$\frac{\partial \vec{v}_v}{\partial t} = -\frac{2Bc^2}{\rho_0^2} P_v \left(\rho_a \int \vec{\nabla} \rho_a dx \right),$$
$$\frac{\partial \vec{\Omega}}{\partial t} = \vec{F}_a = -\frac{2Bc^2}{\rho_0^2} \left(\vec{\nabla} \rho_a \times \int \vec{\nabla} \rho_a dx \right).$$

$\vec{F}_a$ represents the acoustic source of the vorticity mode. While Eqs. (28) do not reveal that the necessary condition of vorticity to be generated is $B \neq 0$, Eqs. (29), accounting for relations between perturbations in density and velocity in the sound wave, include acoustic force proportional to B. In order to make progress in deriving the equations describing the effects of quasi-planar sound, we may use the main term in the approximate series expansion of P_v with respect to μ, following from (20):

$$P_v \approx \begin{pmatrix} \frac{\partial^2}{\partial y^2} \int dx \int dx & -\frac{\partial}{\partial y} \int dx \\ -\frac{\partial}{\partial y} \int dx & 1 \end{pmatrix}. \tag{30}$$

3.3. The vorticity mode generated by periodic sound

Eqs. (29) apply to periodic as well as aperiodic sound. They refer to instantaneous quantities. The only limitations which were used in order to simplify the calculations, are $\omega\tau \gg 1$ and $|B| \ll k$.

The difficulty of the solution of (27), (29) is obvious in view of the nonlinearity in both equations. A simple estimation of acoustic source in the case of periodic sound may be taken as a solution of the linearized version of Eq. (27), valid in the case of the sound of infinitely-small magnitude. We assume also very weak transversal diffraction of a beam.
Under these simplifying conditions, the periodic solution of Eq. (27) in an unbounded volume of a gas, takes the form

$$\rho_a' = M\rho_0 \sin(\omega(t - x/c)) \exp(Bx - Y^2), \tag{31}$$

where $Y = y/L$ denotes the dimensionless coordinate, L is the characteristic transversal width of the sound beam, and M is the acoustic Mach number. Substitution of an excess density (31) in the last equation in (29), in view of $\omega\tau \gg 1$ and definition of A (Eqs. (15)), yields

$$\begin{aligned} \frac{\partial \Omega_z}{\partial t} &= -\frac{4M^2Bc^2\omega^2}{(A^2+\omega^2)L} Y \exp\left(2\left(Bx - Y^2\right)\right) \\ &\approx -\frac{4M^2Bc^2}{L} Y \exp\left(2\left(Bx - Y^2\right)\right). \end{aligned} \tag{32}$$

This example belongs to the field of interest of acoustic streaming in its classical meaning, because the acoustic force in the right-hand side of Eq. (32) represents the mean field. In its derivation, however, we did not use the averaging over sound period at any stage.
If $B < 0$ (that corresponds to the standard attenuation in a fluid), the sign of acoustic force in the right-hand side of Eq. (32) is positive for $x \geq 0$ and it decreases with x. The production of vorticity occurs similarly as in the case of a fluid with standard attenuation. If $B = 0$, there is no nonlinear generation of vorticity by sound. If $B > 0$, the acoustic source is negative and decreases as x increases. That reflects the inversion in generation of the vortex flow comparatively with the equilibrium regime when $B < 0$. In particular, direction of streamlines become opposite.

4. Concluding remarks

The novelty of this study is the analysis of nonlinear phenomena differing from those in a Newtonian fluid.
The main result of this study are instantaneous equations governing the vorticity mode, its velocity or vorticity, Eqs. (29). They are valid at any time for any type of sound, periodic or aperiodic.
The only limitations used to simplify the evaluations, are domain of high-frequency sound $\omega\tau \gg 1$ and its weak attenuation or amplification over the wavelength, $|B| \ll k$.

Our conclusions are valid in temporally and spatially confined domains, where sound remains dominant with respect to other modes (vorticity, entropy, and vibrational). In standard thermoviscous fluid, sound always attenuates. This also takes place in the relaxing gas if $B < 0$.

The reason for the acceleration of the acoustic wave (when $B > 0$) is special in thermodynamics of relaxation processes in a gas. Sound "takes energy" from the background, making it cooler [19, 20]. Streamlines in standard attenuating unbounded fluids are directed according to the course of sound.

If $B > 0$, the direction of streamlines becomes opposite. It is useful to estimate B for a typical laser mixture $CO_2 : N_2 : He = 1 : 2 : 3$ at normal conditions $p_0 = 1atm = 101325\ Pa$, $T = 300\ K$.

The density of this mixture is $\rho_0 = 0,76\ kg \cdot m^{-3}$, $\tau = 5 \cdot 10^{-5} s$, $\frac{T}{\tau}\frac{d\tau}{dT} = -3,4$. The value of B depends on the pumping intensity $I \approx (\varepsilon - \varepsilon_{eq})/\tau$ in accordance to Eq. (13); the threshold quantity is $I_{th} \cdot \rho_0 = 1,5 \cdot 10^6 W \cdot m^{-3}$. If $I \cdot \rho_0 = 10^9 W \cdot m^{-3}$, then $B = 3,3\ m^{-1}$, and if $I \cdot \rho_0 = 10^8\ W \cdot m^{-3}$, then $B = 0,3\ m^{-1}$. For intensities less than the threshold quantity, B takes negative values, varying from $B = -0,005\ m^{-1}$ for zero I till $B = 0$ for $I = I_{th}$.

Attenuation, or amplification of sound considered in this study, and nonlinear generation of the vorticity mode by it, occur exclusively due to relaxation processes. In real systems, the influence of inhomogeneity of the nonequilibrium gas is imposed on the effects discussed above, and this must be taken into account in concrete calculations, though strongly complicates mathematical description [21].

This study does not take into account the thermal and viscous (standard, Newtonian) attenuation of a gas. The terms reflecting these phenomena should complement the momentum and energy equations in the system (3). They originate from the stress tensor and the energy flux associated with thermal conductivity. The larger frequency stipulates the larger thermal and viscous attenuation. Dispersion relations for both acoustic branches and the vorticity mode depend on viscosity. For the vorticity mode, it takes the form

$$\omega_5 = i\frac{\eta\tilde{\Delta}}{\rho_0}, \qquad (33)$$

where η is the shear viscosity. That changes the equation governing vorticity in the field of sound,

$$\frac{\partial\vec{\Omega}}{\partial t} - \frac{\eta\Delta\vec{\Omega}}{\rho_0} = \frac{2Bc^3}{\rho_0^2}\left(\vec{\nabla}\rho_a \times \int \vec{\nabla}\rho_a dt\right). \qquad (34)$$

The equation that governs sound includes the term depending on the total diffusivity of sound b due to the first, second viscosities and thermal conductivity:

$$\frac{\partial\rho_a}{\partial t} + c\sqrt{\Delta}\rho_a - cB\rho_a - \frac{b}{2}\Delta\rho_a + \frac{1}{2}\left(\gamma\rho_a\vec{\nabla}\vec{v_a} + (\vec{v_a}\vec{\nabla})\rho_a\right) = 0. \qquad (35)$$

Eqs. (34), (35) are more complex to solve than (27), (29). Standard attenuation can be neglected in comparison with the non-equilibrium effects at frequencies $\omega \ll (\tau\tau_0)^{-1/2}$, where τ_0 is the average time of the molecular free path.

On the other hand, $\omega\tau \gg 1$. For O_2 at room temperature τ equals $10^8\tau_0$, so that the condition of validity reads: $10^4 \gg \omega\tau \gg 1$.

That belongs to the domain $\omega\tau_0 \ll 1$ ($\omega\tau \ll 10^8$). One can derive the conservation equations in the differential form (3) from the gas kinetic Boltzmann equation only under condition $\omega\tau_0 \ll 1$.

Otherwise, the starting point should be the Boltzmann equation. It may be concluded that results are applicable to some classes of relaxation processes; although conceptually different, these processes are described by similar equations. The analogy with gases where a chemical reaction occurs, reversible or not, may be noted readily [15–17].

References

[1] W.L. Nyborg, In: W.P. Manson (Ed.), Physical Acoustics, Vol. II. part B (Academic Press, New York, 1965) 265

[2] O.V. Rudenko, S.I. Soluyan, Theoretical foundations of nonlinear acoustics (Plenum, New York, 1977)

[3] M.J. Lighthill, J. Sound. Vib. 61, 391 (1978)

[4] Q. Qi, J. Acoust. Soc. Am. 94, 1090 (1993)

[5] A. Perelomova, Acta Acust. 89, 754 (2003)

[6] A. Perelomova, Phys. Lett. A 357, 42 (2006)

[7] A. Perelomova, P. Wojda, Cent. Eur. J. Phys. 9, 1135 (2011)

[8] A.A. Collyer, Phys. Educ. 9, 38 (1974)

[9] B.T. Chu, Weak nonlinear waves in nonequilibrium flows, In: P.P. Wegener (Ed.), Nonequilibrium flows, Vol. 1 (Marcel Dekker, New York, 1970)

[10] D.F. Parker, Phys. Fluids 15, 256 (1972)

[11] J.F. Clarke, A. McChesney, Dynamics of relaxing gases (Butterworth, London, 1976)

[12] N.E. Molevich, Acoust. Phys. 48, 209 (2002)

[13] Ya.B. Zeldovich, Yu.P. Raizer, Physics of shock waves and high-temperature hydrodynamic phenomena (Academic Press, New York, 1966)

[14] A.I. Osipov, A.V. Uvarov, Sov. Phys. Usp. 35, 903 (1992)

[15] N.E. Molevich, Acoust. Phys.+ 49, 229 (2003)
[16] A. Perelomova, Can. J. Phys. 88, 29 (2010)
[17] A. Perelomova, Acta Acust. 96, 43 (2010)
[18] M. Hamilton, V. Khokhlova, O.V. Rudenko, J. Acoust. Soc. Am. 101, 1298 (1997)
[19] N.E. Molevich, Acoust. Phys.+ 47, 102 (2001)
[20] A. Perelomova, Cent. Eur. J. Chem. 88, 293 (2010)
[21] A.V. Koltsova, A.I. Osipov, A.V. Uvarov, Sov. Phys. Acoust.+ 40, 969 (1994)