



## Proper gradient otopies

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### ABSTRACT

We prove that the inclusion of the space of proper gradient local maps into the space of proper local maps induces a bijection between the sets of the respective otopy classes of these maps.

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## 0. Introduction

J.C. Becker and D.H. Gottlieb have introduced an extremely useful generalization of the concept of homotopy called *otopy* (see for instance [3,4,7]). The main advantage of using otopies is that otopy relates maps with not necessarily the same domain (the *local maps* of Definition 1.1). Furthermore, otopy theory turns out to be fruitful in equivariant degree theory (see [2,6]).

Our main result is the following theorem: the inclusion of the space of proper gradient local maps into the space of all proper local maps induces a bijection between the sets of connected components of these spaces i.e. between the respective otopy classes of local maps. We expected this result to be true (see [1, Remark 2.2]), but were not able to prove at that moment due to many technical difficulties. The result may be regarded as a version of Parusiński's Theorem (see [9] for details) and as a special (simplest) case of the following conjecture: the above inclusion is a (weak) homotopy equivalence.

It is worth pointing out that the advantage of using proper local maps instead of all local maps is that the space of proper local maps is a “very nice” metrizable space. In fact, it is homeomorphic to the space of based continuous maps of the  $n$ -sphere into itself.

However, the proof of our main result is more difficult compared to that concerning all local maps presented in [1].

The paper is arranged as follows. Section 1 presents some preliminaries from otopy theory for local maps. Section 2 contains a discussion of our result with some comments. This result is proved in Section 4. In Section 3 we introduce canonical, elementary and standard maps. Sections 5, 6 and 7 contain proofs of key lemmas needed in Section 4. Finally, Appendix A presents three technical facts used in Section 5.

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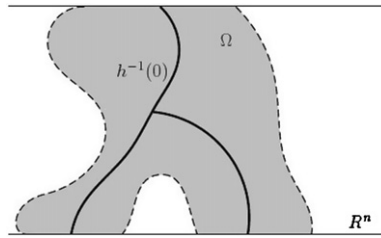


Fig. 1. Domain of the otopy and its zeros.

1. Preliminaries

The notation  $A \Subset B$  means that  $A$  is a compact subset of  $B$ .

**Definition 1.1.** A continuous map  $f : U \rightarrow \mathbb{R}^n$  is called a *local map* if

- $U$  is an open subset of  $\mathbb{R}^n$ ,
- $f^{-1}(0) \Subset U$ .

We will often write such maps as pairs  $(f, U)$ , pointing out their domains.

The set-theoretic union of two local maps  $f$  and  $g$  with disjoint domains will be denoted by  $f \sqcup g$ . Recall that a local map  $(f, U)$  is called *gradient* if there is a  $C^1$ -function  $\varphi : U \rightarrow \mathbb{R}$  such that  $f = \nabla\varphi$ . The function  $\varphi$  is called the *potential* of  $f$ . A local map is *proper* if preimages of compact subsets are compact. In the sequel, we will also use the following equivalent characterization of proper local maps: for every sequence  $\{x_k\} \subset U$  with no accumulation point in  $U$  we have  $\lim_{k \rightarrow \infty} \|f(x_k)\| = \infty$ .

We will consider the set of all local maps, denoted by  $\mathcal{F}(n)$ , and the following subsets:

$$\begin{aligned} \mathcal{F}_\nabla(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is gradient}\}, \\ \mathcal{P}(n) &:= \{f \in \mathcal{F}(n) \mid f \text{ is proper}\}, \\ \mathcal{P}_\nabla(n) &:= \mathcal{F}_\nabla(n) \cap \mathcal{P}(n). \end{aligned}$$

Immediately from the above definitions we obtain the following commutative diagram of inclusions:

$$\begin{array}{ccc} \mathcal{P}_\nabla(n) \hookrightarrow & \mathcal{P}(n) & \\ \downarrow & \downarrow & \\ \mathcal{F}_\nabla(n) \hookrightarrow & \mathcal{F}(n) & \end{array} \tag{1.1}$$

Let  $I = [0, 1]$ .

**Definition 1.2.** A continuous map  $h : \Omega \rightarrow \mathbb{R}^n$  is called an *otopy* if

- $\Omega$  is an open subset of  $\mathbb{R}^n \times I$ ,
- $h^{-1}(0) \Subset \Omega$ ,

see Fig. 1.

Given an otopy  $(h, \Omega)$  we can define for each  $t \in I$  sets  $\Omega_t = \{x \in \mathbb{R}^n \mid (x, t) \in \Omega\}$  and maps  $h_t : \Omega_t \rightarrow \mathbb{R}^n$  with  $h_t(x) = h(x, t)$ . Note that from the above  $h_t$  may be the empty map.

**Definition 1.3.** If  $(h, \Omega)$  is an otopy, we say that  $(h_0, \Omega_0)$  and  $(h_1, \Omega_1)$  are *otopic* (written  $h_0 \sim h_1$  or  $(h_0, \Omega_0) \sim (h_1, \Omega_1)$ ).

**Remark 1.** Of course, otopy gives an equivalence relation on  $\mathcal{F}(n)$ . The set of otopy classes of local maps will be denoted by  $\mathcal{F}[n]$ . Observe that if  $(f, U)$  is a local map and  $V$  is an open subset of  $U$  such that  $f^{-1}(0) \subset V$ , then  $(f, U)$  and  $(f|_V, V)$  are otopic. In particular, if  $f^{-1}(0) = \emptyset$  then  $(f, U)$  is otopic to the empty map.

Apart from the usual otopies, we will consider otopies that satisfy some additional conditions, namely:

- *gradient* i.e.  $h(x, t) = \nabla_x \chi(x, t)$  for some not necessarily continuous function  $\chi$  such that  $\chi_t$  is  $C^1$  for each  $t \in I$ ,
- *proper* i.e.  $h$  is proper,
- *proper gradient*.

The sets of respective otopy classes in  $\mathcal{F}_\nabla(n)$ ,  $\mathcal{P}(n)$ ,  $\mathcal{P}_\nabla(n)$  will be denoted by  $\mathcal{F}_\nabla[n]$ ,  $\mathcal{P}[n]$ ,  $\mathcal{P}_\nabla[n]$ . We will abbreviate *proper gradient otopy* to *pg-otopy*.

**Remark 2.** In [1] we use the stronger definition of the gradient otopy. However, it is easy to see that the results of [1] are not affected if in [1] we replace the stronger definition with the above weaker one. Moreover, the replacement of the definition is mainly motivated by the following expectation: paths in  $\mathcal{P}_\nabla(n)$  should bijectively correspond to proper gradient otopies (see below for the definition of topology in  $\mathcal{P}_\nabla(n)$ ).

Let  $\Sigma^n = \mathbb{R}^n \cup \{*\}$  denote the one-point compactification of  $\mathbb{R}^n$ . It is a pointed space with base point  $*$ . We will write  $\mathcal{M}_* \Sigma^n$  for the set of pointed continuous maps from  $\Sigma^n$  to  $\Sigma^n$ . With every map  $f \in \mathcal{M}_* \Sigma^n$  one associates a proper local map  $(f \upharpoonright_{f^{-1}(R^n)}, f^{-1}(R^n))$ . Conversely, if  $(f, U) \in \mathcal{P}(n)$ , then the map  $f^+ : \Sigma^n \rightarrow \Sigma^n$  given by

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in U, \\ * & \text{otherwise,} \end{cases}$$

is continuous. Using this observation we see that the map

$$\begin{aligned} \mu : \mathcal{P}(n) &\rightarrow \mathcal{M}_* \Sigma^n, \\ \mu((f, U)) &= f^+ \end{aligned}$$

is a bijection. Since  $\mathcal{M}_* \Sigma^n$  is equipped with the supremum metric,  $\mathcal{P}(n)$  also has the metric structure induced by the pullback.

**2. Main result**

The diagram (1.1) induces the following commutative diagram of maps between sets of otopy classes (all the maps are induced by inclusions).

$$\begin{array}{ccc} \mathcal{P}_\nabla[n] & \xrightarrow{\alpha} & \mathcal{P}[n] \\ \downarrow \beta & & \downarrow \gamma \\ \mathcal{F}_\nabla[n] & \xrightarrow{\delta} & \mathcal{F}[n] \end{array} \tag{2.1}$$

Let us formulate the main result of this paper.

**Theorem A.** *All the maps in the diagram (2.1) are bijections.*

**Remark 3.** It is worth pointing out that our result includes a version of Parusiński’s Theorem: the maps  $\alpha$  and  $\delta$  are bijections. However, our proof makes no appeal to the original proof of Parusiński.

It is clear from the topological degree theory [e.g. 5, Ch. II and Ch. IV] that all the maps in the following commutative diagram are bijections.

$$\begin{array}{ccc} \mathcal{M}_*[\Sigma^n] & \longleftarrow \mathcal{P}[n] \longrightarrow & \mathcal{F}[n] \\ & \searrow \text{deg} \quad \downarrow \text{deg} \quad \swarrow \text{deg} & \\ & & \mathbb{Z} \end{array}$$

Consequently, the map  $\gamma$  in the diagram (2.1) is bijective. But in [1] we proved that  $\text{deg} : \mathcal{F}_\nabla[n] \rightarrow \mathbb{Z}$  is a bijection, so it is clear from diagram (2.2)

$$\begin{array}{ccccc}
 & & \mathcal{F}[n] & & \\
 & \nearrow & \downarrow \text{deg} & \nwarrow & \\
 \mathcal{F}_{\nabla}[n] & \xrightarrow{\text{deg}} & \mathbb{Z} & \xleftarrow{\text{deg}} & \mathcal{P}[n] \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & \mathcal{P}_{\nabla}[n] & & 
 \end{array}
 \tag{2.2}$$

that all the solid arrows represent bijections. Therefore the true difficulty in proving Theorem A lies in the following:

**Theorem B.**  $\text{deg} : \mathcal{P}_{\nabla}[n] \rightarrow \mathbb{Z}$  is bijective.

We will see that, in fact, only injectivity causes a problem.

### 3. Canonical, elementary and standard maps

We shall consider the open unit ball  $B := \{x \in \mathbb{R}^n \mid \sum_i x_i^2 < 1\}$  and the open unit cube  $C := \{x \in \mathbb{R}^n \mid \max_i |x_i| < 1\}$ . Let  $|c_i| = 1$  for  $i = 1, \dots, n$ . If  $\varphi(x) = \frac{\sum_{i=1}^n c_i x_i^2}{1 - \sum_{i=1}^n x_i^2}$ , then  $\nabla\varphi : B \rightarrow \mathbb{R}^n$  is called *ball-canonical* and if  $\varphi(x) = \sum_{i=1}^n \frac{c_i x_i^2}{1 - x_i^2}$ , then  $\nabla\varphi : C \rightarrow \mathbb{R}^n$  is called *cube-canonical*. We say that two maps ball-canonical and cube-canonical are *of the same type* if both of them have the same sequence of coefficients  $\{c_i\}_1^n$ . Moreover, if  $\varphi(x) = \sum_{i=1}^n \frac{x_i^2}{1 - x_i^2}$ , then  $\nabla\varphi : C \rightarrow \mathbb{R}^n$  is called *plus-elementary* and if  $\varphi(x) = \frac{-x_1^2}{1 - x_1^2} + \sum_{i=2}^n \frac{x_i^2}{1 - x_i^2}$ , then  $\nabla\varphi$  is called *minus-elementary*. The center of domains of all above maps is the origin of  $\mathbb{R}^n$ , but in the sequel we will use the same terminology for translations of the these maps to any point. Note that such translations are evidently proper gradient otopies. Finally, for  $l \in \mathbb{N}$  a finite disjoint union of  $l$  plus-elementary (resp. minus-elementary) maps is called *l-standard* (resp. *(-l)-standard*).

### 4. Proof of Theorem B

The proof of Theorem B is based on the following lemmas, which will be proved in the next three sections. In fact, it is easy to see that Theorem B is an immediate consequence of Corollary 4.5.

**Lemma 4.1.** Any proper gradient local map is pg-otopic to a finite disjoint union of ball-canonical maps.

**Lemma 4.2.** Any ball-canonical map is pg-otopic to cube-canonical one of the same type.

**Lemma 4.3.** Any two cube-canonical maps with different coefficients at only one position annihilate i.e. their union is pg-otopic to the empty map.

**Corollary 4.4.** Any cube-canonical map is pg-otopic to some elementary map.

**Proof.** Let  $\varphi$  be cube-canonical and  $1 < k \leq n$ . By Lemma 4.3  $\nabla\varphi$  is pg-otopic to  $\nabla\varphi \sqcup \nabla\psi \sqcup \nabla\xi$ , where  $\nabla\psi$  and  $\nabla\xi$  are also cube-canonical,  $\psi$  has only the first coefficient and  $\xi$  has only the first and  $k$ -th coefficient different from those of  $\varphi$ . Since  $\nabla\varphi$  and  $\nabla\psi$  also annihilate,  $\nabla\varphi$  is pg-otopic to  $\nabla\xi$ . That means that using a proper gradient otopy we are able to replace a given cube-canonical map with one having coefficients changed on exactly two positions: first and  $k$ -th. Repeating the above procedure it is possible in the end to obtain an elementary map.  $\square$

**Corollary 4.5.** For each  $l \in \mathbb{Z}$  any proper gradient local map of degree  $l$  is pg-otopic to an  $l$ -standard map.

**Proof.** Let  $\nabla\varphi \in \mathcal{P}_{\nabla}(n)$  with  $\text{deg}\nabla\varphi = l$ . By Lemma 4.1  $\nabla\varphi$  is pg-otopic to a finite union of ball-canonical maps. From Lemma 4.2 we can replace all of them with cube-canonical maps. Similarly, by Corollary 4.4 we can replace cube-canonical maps with elementary maps. Finally by Lemma 4.3 pairs of elementary maps of different signs annihilate, so we obtain an  $l$ -standard map.  $\square$

### 5. Proof of Lemma 4.1

Let  $(f, U) \in \mathcal{P}_{\nabla}(n)$  with  $f = \nabla\varphi$ . The proof will be divided into 5 steps. Each of them represents a proper gradient otopy. In the first we replace the initial potential by a Morse function. Next, in the Step 2, we replace the Morse function by a locally quadratic potential. Step 3 shows how to blow up the domain  $U$  to obtain big enough neighborhoods of critical

points. In Step 4 we restrict the domain to the disjoint union of unit balls. Finally, Step 5 transforms matrices in the formulas to the diagonal form.

**Step 1.** We deform the potential  $\varphi$  to a Morse function  $\varphi_M$ . By density of the Morse functions and openness of the proper vector fields (see [8]) we can choose a Morse function  $\varphi_M$  such that the straight-line homotopy of potentials  $(1-t)\varphi + t\varphi_M$  induces a proper gradient homotopy.

**Step 2.** Let us denote by  $B_r(p)$  the open  $r$ -ball around  $p$  and let  $B_r := B_r(0)$ .

We deform the potential  $\varphi_M$  to some potential  $\psi$  satisfying

- $\text{Crit}(\varphi_M) = \text{Crit}(\psi)$ ,
- there is  $\epsilon > 0$  such that for each  $p \in \text{Crit}(\psi)$  the map  $\psi$  is a nondegenerate quadratic form around  $p$  i.e.

$$\psi \upharpoonright_{B_\epsilon(p)}(x) = (x-p)^T A_p (x-p)$$

for some nondegenerate symmetric matrix  $A_p$ ,

- $B_\epsilon(p) \cap B_\epsilon(q) = \emptyset$  for  $p \neq q$ ,
- $\text{cl } B_\epsilon(p) \subset U$

via homotopy being proper on gradients.

Let  $x=0$  be a critical point of  $\varphi_M$ . We have

$$\varphi_M(x) = \frac{1}{2}x^T H_0 \varphi_M x + R(x),$$

where  $R(x)$  is  $C^2$ -function such that  $R(x) = o(\|x\|^2)$ . Note that it implies that  $\|\nabla R(x)\| = o(\|x\|)$ .

There exists a  $C^\infty$ -function  $\eta : [0, \infty) \rightarrow [0, 1]$  satisfying

- $\eta(x) := \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \geq 2, \end{cases}$
- $|\eta'(x)| < 2$  for all  $x \in [0, \infty)$ .

Define  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  by the formula  $\lambda(x) := \eta(\|x\|/\epsilon)$ . Observe that  $\|\nabla \lambda(x)\| < 2/\epsilon$  for  $x \in \mathbb{R}^n$ . Set

$$\chi(x, t) := \frac{1}{2}x^T H_0 \varphi_M x + (1 - t\lambda(x))R(x)$$

for  $(x, t) \in U \times I$ . Obviously, the homotopy  $\chi$  does not change  $\varphi_M$  outside  $B_{2\epsilon}$ . Finally, set  $\psi(x) := \chi(x, 1)$ . We will show that there is  $\epsilon > 0$  such that  $x=0$  is the only zero of  $\nabla \psi$  in  $B_{2\epsilon}$ . Let  $c := \min_{\|x\|=1} \|H_0 \varphi_M\|$ . We choose  $\epsilon > 0$  such that in  $B_{2\epsilon}$ ,

- $|R(x)| < \frac{c}{16} \|x\|^2$ ,
- $\|\nabla R(x)\| < \frac{c}{4} \|x\|$ .

Then for  $x \in B_{2\epsilon}$ ,

$$\begin{aligned} \|\nabla \psi(x)\| &= \|H_0 \varphi_M(x) + \nabla(1 - \lambda(x)) \cdot R(x) + (1 - \lambda(x)) \cdot \nabla R(x)\| \\ &\geq c \cdot \|x\| - \frac{2c}{8}\epsilon \|x\| - \frac{c}{4}\|x\| = \frac{c}{2}\|x\| \end{aligned}$$

which is our claim.

Above we have defined  $\chi$  in case of one critical point ( $x=0$ ). Of course, we can define  $\chi$  in the similar way near each point of  $\text{Crit}(\varphi_M)$ .

**Step 3.** If  $\epsilon$  from Step 2 is less than 1 we define

- $\Omega_t := \frac{1}{(1-t)+t\epsilon} U$ ,
- $\alpha(x, t) := \psi[((1-t) + t\epsilon)x]$ .

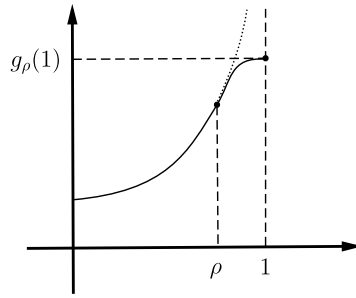


Fig. 2. Function  $g_\rho$ .

Observe that for  $p \in \text{Crit}(\psi)$ ,

$$\alpha_1(x) = \psi(\epsilon x) = \left(x - \frac{p}{\epsilon}\right)^T A'_p \left(x - \frac{p}{\epsilon}\right)$$

on  $B_1(p/\epsilon)$ , where  $A'_p = \epsilon^2 A_p$ .

**Step 4.** We construct a proper gradient otopy  $\nabla_x \beta$  between  $\nabla \alpha_1 = \nabla \beta_0$  and  $\nabla \beta_1$ , where  $\beta_1$  is defined not on the whole  $U$ , but only on unit balls around critical points of  $\alpha_1$ . Let  $g_1 : [0, 1) \rightarrow [1, \infty)$  be given by  $g_1(x) = \frac{1}{1-x^2}$ . Consider the family of auxiliary functions  $g_\rho : [0, 1] \rightarrow [1, \infty)$  indexed by the real parameter  $\rho \in [0, 1)$  and uniquely determined by the conditions (see Fig. 2)

- $g_\rho \upharpoonright_{[0, \rho]} \equiv g_1 \upharpoonright_{[0, \rho]}$ ,
- $g_\rho$  is a quadratic polynomial on  $[\rho, 1]$ ,
- $g_\rho$  is  $C^1$ -function such that  $g'_\rho(1) = 0$ .

We define otopy  $(\nabla_x \beta, \Omega)$  by

- $\Omega_t := \begin{cases} U & \text{if } t \in [0, 1), \\ \bigcup_{p \in \text{Crit}(\alpha_1)} B_1(p) & \text{if } t = 1, \end{cases}$
- $\beta(x, t) := \begin{cases} g_t(\|x - p\|) \cdot \alpha_1(x) & \text{if } t \in [0, 1] \text{ and } x \in B_1(p), \\ g_t(1) \cdot \alpha_1(x) & \text{otherwise.} \end{cases}$

Observe that

- $\beta_0 = \alpha_1$ ,
- $\beta_1(x) = \frac{1}{1-\|x-p\|^2} (x-p)^T A'_p (x-p)$  for  $x \in B_1(p)$ ,
- $(\nabla_x \beta, \Omega)$  is proper by Corollary A.2 applied to each ball  $B_1(p)$  and the family of functions  $g_\rho$ .

**Step 5.** It is enough to consider one fixed ball centered at  $p = 0$ . Let us define otopy  $(\nabla_x \gamma, \Omega)$  by

- $\Omega := B_1 \times I$ ,
- $\gamma(x, t) := \frac{1}{1-\|x\|^2} (x)^T A_t(x)$ , where  $A_t$  is a path in  $GL_n(\mathbb{R}) \cap \text{Symm}(n)$  connecting  $A$  to some diagonal matrix with  $\pm 1$  on the diagonal.

Observe that  $(\nabla_x \gamma, \Omega)$  is proper by Corollary A.3.  $\square$

### 6. Proof of Lemma 4.2

Suppose that  $\nabla \varphi$  is ball-canonical (see Section 3). There is no loss of generality in assuming that the potential  $\varphi : B \rightarrow \mathbb{R}$  has the form

$$\varphi(x, y) := \frac{\sum_{i=1}^m x_i^2 - \sum_{j=1}^l y_j^2}{1 - \sum_{i=1}^m x_i^2 - \sum_{j=1}^l y_j^2},$$

where  $m + l = n$ . Set

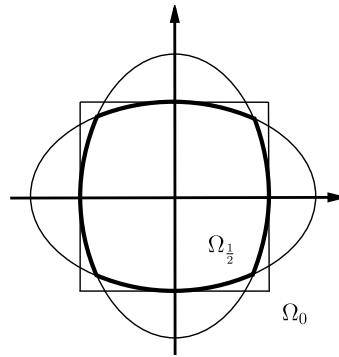


Fig. 3. Transformation of the domain  $\Omega_t$ .

$$S(x, y, t) := 1 - t \sum_{i=1}^m x_i^2 - t \sum_{j=1}^l y_j^2,$$

$$F_i(x, y, t) := S(x, y, t) - (1-t)x_i^2 \quad \text{for } 1 \leq i \leq m,$$

$$G_j(x, y, t) := S(x, y, t) - (1-t)y_j^2 \quad \text{for } 1 \leq j \leq l.$$

Define

- $\Omega := \{(x, y, t) \mid t \in I, F_i > 0, G_j > 0 \text{ for all } i, j\}$ ,
- $\chi(x, y, t) := \sum_{i=1}^m \frac{x_i^2}{F_i} - \sum_{j=1}^l \frac{y_j^2}{G_j}$ .

Observe that

$$\chi_0(x, y) = \sum_{i=1}^m \frac{x_i^2}{1-x_i^2} - \sum_{j=1}^l \frac{y_j^2}{1-y_j^2} \quad \text{and} \quad \chi_1 = \varphi.$$

The otopy  $(\nabla_x \chi, \Omega)$  connects the cube-canonical  $\chi_0$  with the ball-canonical  $\chi_1$ .

**Example.** Let  $m = l = 1$ . Then

- $\Omega_t = \{(x, y) \mid 1 - x^2 - ty^2 > 0, 1 - tx^2 - y^2 > 0\}$ ,
- $\chi(x, y, t) = \frac{x^2}{1-x^2-ty^2} - \frac{y^2}{1-tx^2-y^2}$ .

So  $\chi_0(x, y) = \frac{x^2}{1-x^2} - \frac{y^2}{1-y^2}$  (cube-canonical) and  $\chi_1(x, y) = \frac{x^2-y^2}{1-x^2-y^2}$  (ball-canonical), see Fig. 3.

What is left is to show that  $(\nabla_x \chi, \Omega)$  is proper i.e. if the sequence  $\{(w_k, t_k)\} \subset \Omega$  has no accumulation point in  $\Omega$ , then  $\|\nabla_x \chi(w_k, t_k)\| \rightarrow \infty$ . So assume that  $\{(w_k, t_k)\} \subset \Omega$  has no accumulation point in  $\Omega$ . In particular,

$$\min_{i,j} \{F_i(w_k, t_k), G_j(w_k, t_k)\} \rightarrow 0 \quad \text{with } k \rightarrow \infty.$$

Note that the proof will be completed by showing the following claim.

**Claim.** If  $\min_{i,j} \{F_i, G_j\} \leq 1/2$ , then

$$\|\nabla_x \chi(w_k, t_k)\| \geq \frac{1}{2(m+1)(l+1)} \cdot \frac{1}{\min_{i,j} \{F_i, G_j\}}.$$

We first observe that if  $\min_{i,j} \{F_i, G_j\} \leq 1/2$ , then

$$\forall_j y_j^2 < \frac{1}{2(m+1)l} \implies x_r^2 := \max_i \{x_i^2\} \geq \frac{1}{2(m+1)}. \quad (6.1)$$



Conversely, suppose that

$$\forall_j y_j^2 < \frac{1}{2(m+1)l} \quad \text{and} \quad \forall_i x_i^2 < \frac{1}{2(m+1)}.$$

Then  $\sum_i x_i^2 + \sum_j y_j^2 < 1/2$  and hence, by the definition of  $F_i$  and  $G_j$ ,  $F_i > 1/2$  and  $G_j > 1/2$  for all  $i, j$ , contrary to our assumption.

Similarly,

$$\forall_i x_i^2 < \frac{1}{2(l+1)m} \implies y_s^2 := \max_j \{y_j^2\} \geq \frac{1}{2(l+1)}. \tag{6.2}$$

Let us compute

$$\begin{aligned} \frac{\partial \chi}{\partial x_r} &= 2x_r \left( \sum_i \frac{tx_i^2}{F_i^2} - \sum_j \frac{ty_j^2}{G_j^2} + \frac{S}{F_r^2} \right) := 2x_r A_r, \\ \frac{\partial \chi}{\partial y_s} &= 2y_s \left( \sum_i \frac{tx_i^2}{F_i^2} - \sum_j \frac{ty_j^2}{G_j^2} - \frac{S}{G_s^2} \right) := 2y_s B_s. \end{aligned}$$

Now we only need to consider the following three cases (by (6.1) and (6.2) at least one of them holds). Since  $|x_i|, |y_j| \leq 1$ , we will use the inequalities  $|x_i| \geq x_i^2$  and  $|y_j| \geq y_j^2$ .

**Case 1.**  $\boxed{\forall_j y_j^2 < \frac{1}{2(m+1)l}}$  By (6.1)  $x_r^2 := \max_i \{x_i^2\} \geq \frac{1}{2(m+1)}$  and so  $x_r^2 > \sum_j y_j^2$ , which implies  $F_r := \min_{i,j} \{F_i, G_j\}$ . This gives

$$\sum_i \frac{tx_i^2}{F_i^2} - \sum_j \frac{ty_j^2}{G_j^2} \geq \sum_i \frac{tx_i^2}{F_i^2} - \frac{tx_r^2}{F_r^2} \geq 0$$

and hence finally

$$\left| \frac{\partial \chi}{\partial x_r} \right| = |2x_r||A_r| \geq \frac{1}{m+1} \cdot \frac{S}{F_r^2} \geq \frac{1}{m+1} \cdot \frac{1}{\min_{i,j} \{F_i, G_j\}}.$$

**Case 2.**  $\boxed{\forall_i x_i^2 < \frac{1}{2(l+1)m}}$  Analogous reasoning shows that

$$\left| \frac{\partial \chi}{\partial y_s} \right| \geq \frac{1}{l+1} \cdot \frac{1}{\min_{i,j} \{F_i, G_j\}}.$$

**Case 3.**  $\boxed{\exists_{r,s} x_r^2 := \max_i \{x_i^2\} \geq \frac{1}{2(l+1)m} \text{ and } y_s^2 := \max_j \{y_j^2\} \geq \frac{1}{2(m+1)l}}$  By definition,

$$A_r - B_s = \frac{S}{F_r^2} + \frac{S}{G_s^2} \geq \frac{1}{F_r} + \frac{1}{G_s} \geq \frac{1}{\min_{i,j} \{F_i, G_j\}}.$$

Hence  $\max\{|A_r|, |B_s|\} \geq \frac{1}{2\min_{i,j} \{F_i, G_j\}}$  and, in consequence,

$$\begin{aligned} \max \left\{ \left| \frac{\partial \chi}{\partial x_r} \right|, \left| \frac{\partial \chi}{\partial y_s} \right| \right\} &= \max \{ 2|x_r||A_r|, 2|y_s||B_s| \} \\ &\geq \frac{1}{2(m+1)(l+1)} \cdot \frac{1}{\min_{i,j} \{F_i, G_j\}}. \end{aligned}$$

Combining all the above cases we immediately get our claim.  $\square$

### 7. Proof of Lemma 4.3

To simplify notation we restrict ourselves to showing Lemma 4.3 for two elementary maps of different signs. Furthermore, the otopy parameter  $t$  will run the interval  $[-1, 1]$  instead of the usual  $[0, 1]$  to get nice formulas. Let us define the domain



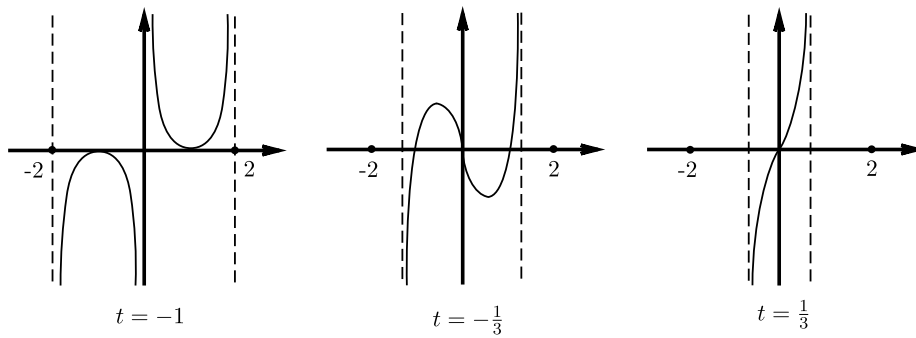


Fig. 4. Annihilation of elementary maps on potentials.

- $\Omega_t := \begin{cases} \{(x, t) \mid 0 < |x_1| < 2, |x_i| < 1 \text{ for } i > 1\} & \text{if } t = -1, \\ \{(x, t) \mid |x_1| < 1 - t, |x_i| < 1 \text{ for } i > 1\} & \text{if } t \in (-1, 1), \end{cases}$
- $\Omega := \bigcup_{t \in [-1, 1]} \Omega_t.$

and the otopy potential (see Fig. 4),

$$\chi(x, t) := \begin{cases} \frac{-(x_1-t)^2}{1-(x_1-t)^2} + \sum_{i>1} \frac{x_i^2}{1-x_i^2} + \operatorname{sgn}(t+1) \frac{t^2}{1-t^2} & \text{for } -1+t < x_1 \leq 0, \\ \frac{(x_1-t)^2}{1-(x_1-t)^2} + \sum_{i>1} \frac{x_i^2}{1-x_i^2} - \operatorname{sgn}(t+1) \frac{t^2}{1-t^2} & \text{for } 0 \leq x_1 < 1-t. \end{cases}$$

Note that  $\Omega_1 = \emptyset$ . It is easy to see that  $(\nabla_x \chi, \Omega)$  is a proper gradient otopy.  $\square$

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**Appendix A**

For a matrix  $A$ , let  $\mu(A) := \min_{\|x\|=1} \|Ax\|$ . Note that if  $A \in GL_n(\mathbb{R})$ , then  $\mu(A) > 0$ . Let us denote by  $\operatorname{Sym}(n)$  the set of  $n \times n$  real symmetric matrices.

**Proposition A.1.** *If*

- $A \in \operatorname{Sym}(n)$ ,
- $g \in C^1([0, 1], \mathbb{R}_+)$  with  $g' \geq 0$ ,
- $\varphi : B_1 \rightarrow \mathbb{R}$  is given by  $\varphi(x) := g(\|x\|) \cdot x^T Ax$ ,

then  $\|\nabla \varphi(x)\| \geq 2g(\|x\|)\mu(A)\|x\|$ .

**Proof.** Since

$$\nabla \varphi(x) = g'(\|x\|) \cdot \frac{x^T Ax}{\|x\|} \cdot x + g(\|x\|) \cdot 2Ax := w_1 + w_2,$$

we have

$$w_1 \cdot w_2 = 2g'(\|x\|)g(\|x\|) \frac{(x^T Ax)^2}{\|x\|} \geq 0.$$

Hence

$$\|\nabla \varphi(x)\| \geq \|w_2\| \geq 2g(\|x\|)\mu(A)\|x\|. \quad \square$$

**Corollary A.2.** *Let  $A \in GL_n(\mathbb{R}) \cap \operatorname{Sym}(n)$ . Assume that we have a family of functions  $\{g_\rho\}_{\rho \in I}$  such that*

- $g_\rho \in C^1([0, 1], \mathbb{R}_+)$ ,

- $g'_\rho \geq 0$ ,
- $\lim_{\substack{\rho \rightarrow 1 \\ x \rightarrow 1}} g_\rho(x) = \infty$ .

Let  $\varphi_\rho : B_1 \rightarrow \mathbb{R}$  be given by  $\varphi_\rho(x) := g_\rho(\|x\|) \cdot x^T A x$ . Then

$$\lim_{\substack{\rho \rightarrow 1 \\ \|x\| \rightarrow 1}} \|\nabla \varphi_\rho(x)\| = \infty.$$

**Corollary A.3.** *If*

- $A : I \rightarrow GL_n(\mathbb{R}) \cap \text{Symm}(n)$  *is continuous,*
- $\Omega := B_1 \times I$ ,
- $\chi : \Omega \rightarrow \mathbb{R}$  *is given by*  $\chi(x, t) := \frac{1}{1 - \|x\|^2} x^T A_t x$ ,

*then*  $(\nabla_x \chi, \Omega)$  *is a proper gradient otopy.*

**Proof.** Assume that a sequence  $\{(x_n, t_n)\} \subset \Omega$  has no accumulation points in  $\Omega$  (i.e.  $\|x_n\| \rightarrow 1$ ). Let  $\mu := \min_{t \in I} \mu(A_t)$ . Of course,  $\mu > 0$ . There is no loss of generality in assuming that  $\|x_n\| \geq 1/2$ . By Proposition A.1

$$\|\nabla_x \chi(x_n, t_n)\| \geq \frac{1}{1 - \|x_n\|^2} \mu(A_{t_n}) \geq \frac{1}{1 - \|x_n\|^2} \mu \rightarrow \infty,$$

which completes the proof.  $\square$

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