Quantum-correlation breaking channels, broadcasting scenarios, and finite Markov chains

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(Received 28 August 2012; published 15 October 2012)

One of the classical results concerning quantum channels is the characterization of entanglement-breaking channels [M. Horodecki, P. W. Shor, and M. B. Ruskai, Rev. Math. Phys. **15**, 629 (2003)]. We address the question whether there exists a similar characterization on the level of quantum correlations which may go beyond entanglement. The answer is fully affirmative in the case of breaking quantum correlations down to the, so-called, QC (quantum-classical) type, while it is no longer true in the CC (classical-classical) case. The corresponding channels turn out to be measurement maps. Our study also reveals an unexpected link between quantum state and local correlation broadcasting and finite Markov chains. We present a possibility of broadcasting via non von Neumann measurements, which relies on the Perron-Frobenius theorem. Surprisingly, this is not the typical generalized controlled-NOT (C-NOT) gate scenario appearing naturally in this context.

DOI: 10.1103/PhysRevA.86.042319 PACS number(s): 03.67.Hk, 03.67.Mn, 03.65.Ta, 02.50.Ga

There is a well-known result concerning a characterization of entanglement-breaking channels [1,2]. The latter are defined as channels which turn any bipartite state (when applied to one subsystem) into a separable (nonentangled) one. The main result of Ref. [1] states that a channel Λ is entanglement breaking if and only if its Choi-Jamiołkowski state (i.e., its witness) $\mathbb{1} \otimes \Lambda(P_+)$ is a separable state [P_+ denotes the projector on the maximally entangled state, see Eq. (2)]. However, it is known that quantum correlations are more general than entanglement (see, e.g., Ref. [3] and references therein).

To our knowledge, the characterization from Ref. [1] has not yet been refined to a case when a channel breaks more general quantum correlations, i.e., transforms any state into a state that does not possess some type of quantum correlation (see, however, Ref. [4] where partial results were obtained). Here we show that such a refinement is indeed possible for channels mapping (when applied to one subsystem) any bipartite state into a, so-called, QC (quantum-classical) state. Such channels turn out to be quantum-to-classical measurement maps [5]. Moreover, we show that a similar statement does not hold in the case of a stronger requirement of fully breaking quantum correlations and transforming any bipartite state into a CC (classical-classical) form. In the latter case, which is even more intriguing than the QC one, the corresponding measurement maps are formed by commuting positive operator-valued measures (POVMs).

Our study of QC-type channels leads to an unintuitive and surprising connection between the broadcasting of quantum states [6] and correlations [5,7] on one side and finite Markov chains (see, e.g., Ref. [8]) on the other. The existence of a broadcastable state for a given QC-type channel is guaranteed by the fact that each finite Markov chain, described by a stochastic transition matrix [9], possesses by the Perron-Frobenius theorem a stationary distribution. In fact, it happens that there are maps that may broadcast full rank states and still have the broadcasting restricted only to a convex subset of a full commuting family. A similar conclusion works for the case of broadcasting of correlations.

Recall that a QC (or more precisely Q_AC_B) state is a bipartite state of the form

$$\sigma^{QC} = \sum_i p_i \sigma_i^A \otimes |e_i\rangle_B\langle e_i|, \tag{1}$$

where σ_i's are states on Alice's side, $\{e_i\}$ is an orthonormal basis on Bob's side (possibly different from the computational basis $\{|i\rangle\}$), and p_i's are probabilities. In the analogous way one defines a CQ (or more precisely C_AQ_B) state, where the classical part (projectors on the orthonormal basis) is located on Alice's side.

Throughout the work we always assume that Λ is a trace-preserving, completely positive map, i.e., a channel, and that

$$P_+ := |\psi_+\rangle\langle\psi_+| = \frac{1}{d}\sum_{i,j} |ii\rangle\langle jj| \tag{2}$$

is the projector on the maximally entangled state ψ_+ and $\{|ij\rangle\}$ is a fixed computational product basis. We prove the following.

Theorem 1. For any channel Λ its Choi-Jamiołkowski state $\mathbb{1} \otimes \Lambda(P_+)$ is a QC state if and only if $\mathbb{1} \otimes \Lambda(\varrho_{AB})$ is a QC state for any bipartite state ϱ_{AB}.

Proof. We propose to call the above type of channels *QC-type channels*. In order to set up the notation and methods (cf. Ref. [1]), we present a detailed proof. In one direction the implication is obvious. To prove it in the other one, assume that the state $\mathbb{1} \otimes \Lambda(P_+)$ is QC:

$$\mathbb{1} \otimes \Lambda(P_+) = \sum_i p_i \sigma_i \otimes |e_i\rangle\langle e_i|. \tag{3}$$

From the inversion formula for the Choi-Jamiołkowski isomorphism [10]

$$\Lambda(A) = d\mathrm{Tr}_A[W_\Lambda(A^T \otimes \mathbb{1})], \tag{4}$$

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where $W_\Lambda = \mathbb{1} \otimes \Lambda(P_+)$ and the transposition is defined in the computational basis $\{|i\rangle\}$, it follows that Eq. (3) is equivalent to

$$\Lambda(\varrho) = d \sum_i p_i \mathrm{Tr}\big(\varrho\sigma_i^T\big)|e_i\rangle\langle e_i|, \tag{5}$$

and hence

$$\mathbb{1} \otimes \Lambda(\varrho_{AB}) = d \sum_k p_k \mathrm{Tr}_B\big(\varrho_{AB}\mathbb{1} \otimes \sigma_k^T\big) \otimes |e_k\rangle\langle e_k| \tag{6}$$

for an arbitrary bipartite state ϱ_{AB}. We define unnormalized residual states as

$$\tilde{\varrho}_k^A := d\, p_k \mathrm{Tr}_B\big(\varrho_{AB}\mathbb{1} \otimes \sigma_k^T\big) \tag{7}$$

and their traces as

$$\tilde{p}_k := \mathrm{Tr}\tilde{\varrho}_k^A = d\, p_k \mathrm{Tr}_{AB}\big(\varrho_{AB}\mathbb{1} \otimes \sigma_k^T\big). \tag{8}$$

We show that $\sum_k \tilde{p}_k = 1$. From the assumption that Λ is trace-preserving, it follows that

$$\begin{aligned}\mathrm{Tr}_B[\mathbb{1} \otimes \Lambda(P_+)] &= \frac{1}{d}\sum_{i,j} |i\rangle\langle j|\, \mathrm{Tr}\Lambda(|i\rangle\langle j|)\\ &= \frac{1}{d}\sum_{i,j} |i\rangle\langle j|\, \mathrm{Tr}|i\rangle\langle j| = \frac{1}{d}\sum_i |i\rangle\langle i| = \frac{\mathbb{1}}{d}.\end{aligned} \tag{9}$$

On the other hand, the QC assumption (3) implies that

$$\mathrm{Tr}_B[\mathbb{1} \otimes \Lambda(P_+)] = \sum_k p_k\sigma_k, \tag{10}$$

and consequently

$$\sum_k p_k\sigma_k = \frac{\mathbb{1}}{d}. \tag{11}$$

Thus, the collection $\{dp_i\sigma_i\}$, or equivalently its transposition

$$E_i := dp_i\sigma_i^T, \tag{12}$$

forms a POVM, which together with Eq. (8) implies that

$$\sum_k \tilde{p}_k = d\mathrm{Tr}_{AB}\left(\varrho_{AB}\mathbb{1} \otimes \sum_k p_k\sigma_k^T\right) = \mathrm{Tr}\varrho_{AB} = 1. \tag{13}$$

Hence, Eq. (6) may be rewritten as

$$\mathbb{1} \otimes \Lambda(\varrho_{AB}) = \sum_k \tilde{p}_k\varrho_k^A \otimes |e_k\rangle\langle e_k|, \tag{14}$$

with $\varrho_k^A := \tilde{\varrho}_k^A/\mathrm{Tr}\tilde{\varrho}_k^A = \varrho_k^A/\tilde{p}_k$, which is a QC state. ■

We remark that Theorem 1 will not in general be true if one changes the QC state to a CQ one, keeping the form of the Choi-Jamiołkowski isomorphism. Indeed, if $\mathbb{1} \otimes \Lambda(P_+) = \sum_i p_i|e_i\rangle\langle e_i| \otimes \sigma_i$, then from Eq. (4) it follows that $\Lambda(\varrho) = d\sum_i p_i\langle e_i^*|\varrho|e_i^*\rangle\sigma_i$ and $\mathbb{1} \otimes \Lambda(\varrho_{AB}) = d\sum_i p_i \mathrm{Tr}_B(\varrho_{AB}\mathbb{1} \otimes |e_i^*\rangle\langle e_i^*|) \otimes \sigma_i$, which is in general a separable state but not a CQ or QC one. As an example, consider Λ^{CQ} as a von Neumann measurement in the standard basis on a qubit. Obviously, $\mathbb{1} \otimes \Lambda(P_+)$ is a CQ state, since it is CC. Now consider a two-qubit state ϱ_{AB} which is an unbiased mixture of the projectors corresponding to two vectors: $|\psi_+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ and $|+\rangle|0\rangle$ [here $|+\rangle := 1/\sqrt{2}(|0\rangle + |1\rangle)$]. Then $\mathbb{1} \otimes \Lambda(\varrho_{AB}) = 1/2\sum_{i=0,1}\varrho_i \otimes |i\rangle\langle i|$, where $\varrho_0 := 1/2(|+\rangle\langle +| + |0\rangle\langle 0|)$ and $\varrho_1 := |1\rangle\langle 1|$. But $[\varrho_0,\varrho_1] \neq 0$, breaking the necessary condition for $\mathbb{1} \otimes \Lambda(\varrho_{AB})$ to be a CQ state.

As expected from the general results of Ref. [1] on entanglement-breaking channels, Eqs. (5), (11), and (12) imply that the action of the QC-type channel Λ^{QC} consists of a POVM measurement followed by a state preparation, but the preparation is always done in the same orthonormal basis $\{e_i\}$

$$\Lambda(\varrho) = \sum_i \mathrm{Tr}(\varrho E_i)|e_i\rangle\langle e_i|. \tag{15}$$

The later plays the role of a classical register, so that every QC-type channel is in fact a quantum-to-classical measurement map [5]: $\Lambda(\varrho)$ gives the state of a measuring apparatus after the measurement of $\{E_i\}$ on a system in the state ϱ. In light of this observation, Theorem 1 states that a channel is a measurement map if and only if (iff) its Choi-Jamiołkowski state is a QC state.

A natural question arises if one can refine Theorem 1 even more to the so-called CC states, i.e., states of the form

$$\sigma^{\mathrm{CC}} = \sum_{i,j} p_{ij}^{AB}|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|, \tag{16}$$

where now $\{e_i\}$ and $\{f_j\}$ are orthonormal bases on Alice's and Bob's side, respectively, and p_{ij} is a classical joint probability distribution. It turns out that as stated, Theorem 1 does not specify down to such a case, because even if $\mathbb{1} \otimes \Lambda(P_+)$ is a CC state, $\mathbb{1} \otimes \Lambda(\varrho_{AB})$ is generically a QC state. To see this, assume that

$$\mathbb{1} \otimes \Lambda(P_+) = \sum_{i,j} p_{ij}|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|. \tag{17}$$

From the inversion formula (4) one then obtains that

$$\Lambda(\varrho) = \sum_j \mathrm{Tr}(\varrho E_j)|f_j\rangle\langle f_j|, \tag{18}$$

$$\mathbb{1} \otimes \Lambda(\varrho_{AB}) = \sum_j \mathrm{Tr}_B(\varrho_{AB}\mathbb{1} \otimes E_j) \otimes |f_j\rangle\langle f_j|, \tag{19}$$

where now

$$E_j := d\sum_i p_{ij}|e_i^*\rangle\langle e_i^*|, \tag{20}$$

and the complex conjugation e_i^* of the basis vectors e_i is defined in the computational basis $\{|i\rangle\}$.

Similarly to the QC case, the trace-preserving property of Λ implies that $\{E_j\}$ forms a POVM, $\sum_j E_j = \mathbb{1}$ [cf. Eqs. (9)–(11)]. However, in this case the POVM elements necessarily pairwise commute:

$$[E_j, E_{j'}] = 0, \tag{21}$$

since by Eq. (20) they correspond to a measurement in one fixed basis, but they need not form a von Neumann measurement, as in general E_j's may overlap:

$$E_jE_{j'} = \sum_i p_{ij}p_{ij'}|e_i^*\rangle\langle e_i^*| \neq \delta_{jj'}E_j. \tag{22}$$

What is quite important is that the POVM condition $\sum_j E_j = \mathbb{1}$ puts some constraints on p_{ij}:

$$\sum_{i,j} p_{ij}|e_i^*\rangle\langle e_i^*| = \frac{\mathbb{1}}{d} \Rightarrow p_i := \sum_j p_{ij} = \frac{1}{d}, \quad (23)$$

which in turn implies that the numbers

$$p^{\Lambda}_{j|i} := d p_{ij} \quad (24)$$

are in fact conditional probabilities: $\sum_j p^{\Lambda}_{j|i} = 1$ for any i. Thus, the matrix $P^{\Lambda} := [p^{\Lambda}_{j|i}]$ is a stochastic matrix [9] and

$$E_j = \sum_i p^{\Lambda}_{j|i}|e_i^*\rangle\langle e_i^*|. \quad (25)$$

From a probabilistic point of view, a stochastic matrix defines a finite Markov chain [8]: it provides transition probabilities between the sites. Hence, with every CC-type channel satisfying (17) there is an associated finite Markov chain and vice versa—with every d-site Markov chain and orthonormal bases $\{e_i\}$ and $\{f_i\}$ one can associate a CC-type channel through the formulas (18) and (25). In what follows we also associate a finite Markov chain with a general QC-type channel and investigate the consequences of the broadcasting of states and correlations.

The state (19) is obviously a QC state. It will be a CC state iff there exists a common basis $\{\tilde{e}_i\}$ such that

$$\frac{1}{p_j}\mathrm{Tr}_B(\varrho_{AB}\mathbb{1}\otimes E_j) = \sum_i p_{i|j}|\tilde{e}_i\rangle\langle\tilde{e}_i|, \quad (26)$$

for every j, where $p_j := \mathrm{Tr}(\varrho_{AB}\mathbb{1}\otimes E_j)$ and $p_{i|j} := (1/p_j)\langle\tilde{e}_i|\mathrm{Tr}_B(\varrho_{AB}\mathbb{1}\otimes E_j)|\tilde{e}_i\rangle$. Condition (26) means that all the Alice residual states, to which Bob steers via his measurement

$$\varrho_j^A := \frac{1}{p_j}\mathrm{Tr}_B(\varrho_{AB}\mathbb{1}\otimes E_j), \quad (27)$$

are simultaneously diagonalizable, or equivalently

$$[\varrho_j^A, \varrho_{j'}^A] = 0 \quad (28)$$

for all j, j' [cf. Eq. (19)].

The above failure of Theorem 1 for CC-type channels motivates us to introduce another characteristic of a channel. For an arbitrary channel Λ, we define a set $CC(\Lambda)$ of those bipartite states ϱ_{AB} which are mapped to a CC state by $\mathbb{1}\otimes\Lambda$:

$$CC(\Lambda) := \{\varrho_{AB} : \mathbb{1}\otimes\Lambda(\varrho_{AB}) - \text{CC state}\}. \quad (29)$$

Conditions (27) and (28) allow us to investigate $CC(\Lambda)$ for CC- and QC-type channels. We are able to state the following.

(1) Obviously $P_+ \in CC(\Lambda)$, by the very assumption (17), but it also contains mixtures of pure states with the following Schmidt decompositions:

$$\psi_{AB}(\vec{c};\tilde{e}) := \sum_i c_i|\tilde{e}_i\rangle_A\otimes|e_i^*\rangle_B, \quad (30)$$

where $\vec{c}\in\mathbb{R}_+^d$, $\sum_i c_i^2 = 1$, $\{\tilde{e}_i\}$ is some arbitrary basis, and $\{e_i^*\}$ is the fixed basis from Eq. (25). Indeed, the states (27) for $|\psi(\vec{c};\tilde{e})\rangle\langle\psi(\vec{c};\tilde{e})|$ read $p_j\varrho_j^A = \sum_i p^{\Lambda}_{j|i}c_i^2|\tilde{e}_i\rangle\langle\tilde{e}_i|$, from which there appears a stratified structure of convex sets generated by (30): mixing is allowed only within the states with the

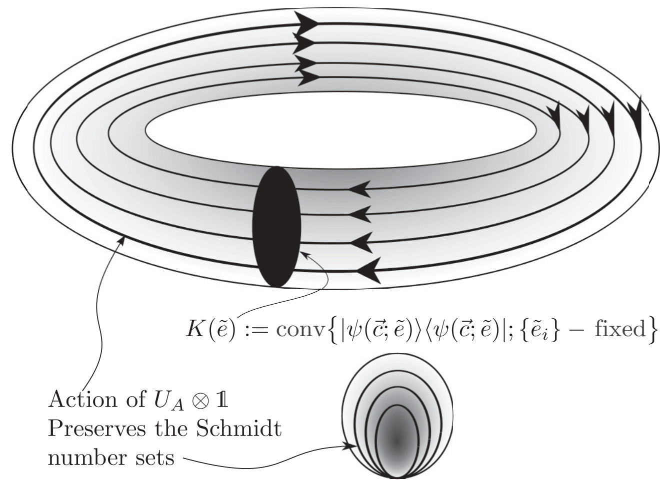

FIG. 1. Graphical representation of the set generated by vectors (30) as a solid torus. The cross section represents convex sets $K(\tilde{e})$, generated by mixing all the states $|\psi(\vec{c};\tilde{e})\rangle\langle\psi(\vec{c};\tilde{e})|$ with a fixed Alice's basis $\{\tilde{e}_i\}$: $\sum_{\vec{c}} p(\vec{c})|\psi(\vec{c};\tilde{e})\rangle\langle\psi(\vec{c};\tilde{e})|$. Each $K(\tilde{e})$ further contains a hierarchy of convex sets of states with Schmidt numbers [11] not greater than k, $k = 1,\ldots,d$. The action of $U_A\otimes\mathbb{1}$ connects different $K(\tilde{e})$'s and preserves the Schmidt number sets.

same, fixed $\{\tilde{e}_i\}$, thus generating convex subsets $K(\tilde{e})$. Partial unitaries $U_A\otimes\mathbb{1}$ transform between different $K(\tilde{e})$'s. Furthermore, inside each $K(\tilde{e})$ there is a hierarchy of convex sets with increasing Schmidt numbers [11]. This hierarchy is preserved by $U_A\otimes\mathbb{1}$. A schematic representation of this set is given in Fig. 1. Note that both ψ_+ and its local orbit $U_A\otimes U_B\psi_+$ are of the form (30), as $U_A\otimes U_B\psi_+ = (U_A U_B^T\otimes\mathbb{1})\psi_+$ and $U_A U_B^T$ is unitary. For a general QC-type channel, the states (30) (for an arbitrary $\{e_i^*\}$) will not be in its $CC(\Lambda^{\mathrm{QC}})$, since the residual states $p_j\varrho_j^A = \sum_{i,k} c_i c_k\langle e_i^*|E_j^{\mathrm{QC}}e_k^*\rangle|\tilde{e}_k\rangle\langle\tilde{e}_i|$ will not in general commute as E_j^{QC}'s do not.

(2) All CQ ($\mathrm{C}_A\mathrm{Q}_B$) states belong to $CC(\Lambda)$. Indeed, substituting into Eq. (19) an arbitrary $\mathrm{C}_A\mathrm{Q}_B$ state,

$$\varrho_{AB} = \sum_i p_i|\tilde{e}_i\rangle_A\langle\tilde{e}_i|\otimes\sigma_i^B, \quad (31)$$

we obtain from Eq. (27) that $\varrho_j^A = \sum_i(p_i/p_j)\mathrm{Tr}(\sigma_i^B E_j)|\tilde{e}_i\rangle\langle\tilde{e}_i|$. Since $\mathrm{Tr}(\sigma_i^B E_j) = p_{j|i}$ is the conditional probability of obtaining result j when measuring POVM $\{E_j\}$ in the state σ_i^B, from Bayes theorem $(p_i/p_j)\mathrm{Tr}(\sigma_i^B E_j) = p_{i|j}$ is the needed conditional probability [cf. Eq. (26)]. A schematic representation of the set of CQ states is given in Fig. 2. For a general QC-type channel, CQ states are also in its $CC(\Lambda^{\mathrm{QC}})$.

(3) Similarly to the set of all CC states, $CC(\Lambda)$ is not convex, which is easily seen from the bilinearity of the condition (28), but is star shaped with respect to the maximally mixed state $\mathbb{1}/d^2$: if $\varrho_{AB}\in CC(\Lambda)$, then

$$\tilde{\varrho}_{AB} := \lambda\varrho_{AB} + (1-\lambda)\frac{\mathbb{1}_A\otimes\mathbb{1}_B}{d^2} \in CC(\Lambda). \quad (32)$$

This follows immediately from (27), as $\tilde{p}_j\tilde{\varrho}_j^A = \lambda p_j\varrho_j^A + (1-\lambda)(\mathrm{Tr}E_j)\mathbb{1}/d^2$ and $\tilde{\varrho}_j^A$ pairwise commute iff ϱ_j^A do so. The same is true for $CC(\Lambda^{\mathrm{QC}})$ for a general CQ-type channel.

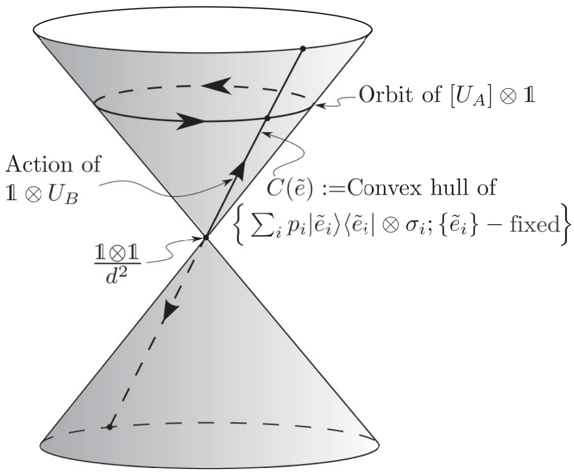

FIG. 2. Graphical representation of the set of CQ states as a conical surface. The generators of the cone represent convex subsets $C(\tilde{e})$, obtained by mixing all the states of the form $\sum_i p_i|\tilde{e}_i\rangle\langle\tilde{e}_i|\otimes\sigma_i$ with a fixed Alice's basis $\{\tilde{e}_i\}$. The local group $\mathbb{1}\otimes U_B$ acts along each such subset. Different subsets are connected by the action of $[U_A]\otimes\mathbb{1}$, where $[U_A]$ denotes the class of U_A modulo a permutation matrix [evidently the action of Alice's permutations conserve each $C(\tilde{e})$]. The whole set is star shaped with respect to $(\mathbb{1}\otimes\mathbb{1})/d^2$.

We do not know at this stage if the above conditions fully characterize $CC(\Lambda)$ for a given CC- or QC-type channel Λ and we postpone the question of its full characterization for future research. Note that in light of Theorem 1, the sets $CC(\Lambda)$ for QC- and CC-type channels possess an interesting interpretation: If we think of Alice and Bob as environment and system, respectively, then $CC(\Lambda)$ is the set of those initial system-environment states ϱ_{AB} that after the measurement, described by Theorem 1 by every Λ^{QC}, and tracing out the system lead to apparatus-environment states with no quantum correlations, i.e., the apparatus becomes quantumly decorrelated from the environment.

We now investigate if a QC-type channel Λ^{QC} can be used (after a modification) for state broadcasting [6]. We first study a relaxed scenario where we broadcast only eigenvalues, or in other words a classical probability distribution: For a given state ϱ_* we are looking for a broadcast state σ_{AB} such that $U_A\mathrm{Tr}_B\sigma_{AB}U_A^\dagger=\varrho_*=U_B\mathrm{Tr}_A\sigma_{AB}U_B^\dagger$ for some unitaries U_A and U_B. We call such a relaxed broadcasting *spectrum broadcasting* and the usual state broadcasting in the sense of Ref. [6]—*full broadcasting*. We prove the following.

Theorem 2. For any QC-type channel Λ^{QC} and any orthonormal basis $\{\phi_j\}$ there exists at least one state $\varrho_*(\phi)$, diagonal in $\{\phi_j\}$, which is N-copy spectrum broadcastable using Λ^{QC}. The state $\varrho_*(e)$, diagonal in the channel's basis $\{e_j\}$ [cf. Eq. (3)], is also N-copy fully broadcastable.

Proof. By Theorem 1 and Eq. (15) every QC-type channel is a quantum-to-classical measurement map. A sufficient condition for spectrum broadcastability of the state

$$\varrho(\phi):=\sum_j\lambda_j(\phi)|\phi_j\rangle\langle\phi_j| \tag{33}$$

is then that its eigenvalues $\vec{\lambda}(\phi)$ are preserved by the measurement, i.e.,

$$\mathrm{Tr}[\varrho(\phi)E_i]=\lambda_i(\phi) \tag{34}$$

for every i. This is equivalent to the following eigenvalue problem,

$$\sum_j p_{i|j}(\phi)\lambda_j(\phi)=\lambda_i(\phi), \tag{35}$$

for a $d\times d$ stochastic matrix

$$P(\phi):=[p_{i|j}(\phi)],\quad p_{i|j}(\phi):=\langle\phi_j|E_i\phi_j\rangle. \tag{36}$$

That this is a stochastic matrix, or equivalently a matrix of conditional probabilities, follows from the fact that E_i's form a POVM by Eqs. (11) and (12):

$$\sum_i p_{i|j}(\phi)=\langle\phi_j|\left(\sum_i E_i\right)\phi_j\rangle=\langle\phi_j|\phi_j\rangle=1 \tag{37}$$

for every j. By the celebrated Perron-Frobenius theorem [9] the above eigenvalue problem (35) has at least one non-negative, normalized solution $\vec{\lambda}_*(\phi)$, from which we construct through Eq. (33) the desired state $\varrho_*(\phi)$. Moreover, this solution is unique iff the matrix $P(\phi)=[p_{i|j}(\phi)]$ is *primitive*, i.e., is irreducible and possesses exactly one eigenvector of the maximum modulus (equal to 1 in our case), which in turn is equivalent to that all the entries of the (d^2-2d+2)-th power of $P(\phi)$ are nonzero [9]. We now construct from Λ^{QC} a new channel [cf. Eq. (15)]:

$$\Lambda^{(N)}(\varrho):=\sum_i\mathrm{Tr}(\varrho E_i)|e_i\rangle\langle e_i|\otimes\cdots\otimes|e_i\rangle\langle e_i|, \tag{38}$$

which by condition (34) N-copy spectrum broadcasts the state $\varrho_*(\phi)$ (or equivalently N-copy broadcasts its eigenvalues).

Since the basis $\{\phi_j\}$ above is arbitrary, we obtain from the Perron-Frobenius theorem that there exists a spectrum-broadcastable state in any basis (the states in different bases can be equal though, e.g., when the bases differ only by a permutation). For the basis $\{e_i\}$, associated with Λ^{QC} by the QC condition (3), the corresponding state $\varrho_*(e)$ will be a fixed point of Λ^{QC}: $\Lambda^{\mathrm{QC}}(\varrho_*(e))=\varrho_*(e)$ by Eqs. (15) and (34). Thus $\Lambda^{(N)}(\varrho_*(e))=\sum_j\lambda_{*j}(e)|e_j\rangle\langle e_j|\otimes\cdots\otimes|e_j\rangle\langle e_j|$ is a full N-copy broadcast state of $\varrho_*(e)$. ■

All the above obviously applies to CC-type channels, as a subclass of QC-type ones. However, as already mentioned, with any CC-type channel Λ there is a naturally associated stochastic matrix $p^\Lambda_{j|i}$ through Eqs. (17) and (24), without the need for an additional basis [$\{e_i^*\}$ of Eq. (25) plays its role]. The corresponding solution $\vec{\lambda}^\Lambda_*\equiv\vec{\lambda}_*(e^*)$ of Eq. (35), $\sum_i p^\Lambda_{j|i}\lambda^\Lambda_{*i}=\lambda^\Lambda_{*j}$, and the state $\varrho^\Lambda_*\equiv\varrho_*(e^*)$ are now intrinsic characteristics of the channel. Note that for an arbitrary basis $\{\phi_j\}$, Eq. (35) reads

$$\sum_{i,k}p^\Lambda_{j|i}|U_{ik}|^2\lambda_k(\phi)=\lambda_j(\phi), \tag{39}$$

where $\phi_j=:Ue_j^*$ and $|U_{ik}|^2:=|\langle e_i^*|Ue_k^*\rangle|^2$ is a doubly stochastic matrix. By the Birkhoff theorem every such a matrix is a convex combination of at most d^2-2d+2 distinct permutation matrices P_σ, $\sigma \in \mathfrak{S}_d$ [9], and hence

$$p_{i|j}(\phi) = \sum_{\sigma\in\mathfrak{S}_d} p_\sigma \sum_k p^{\Lambda}_{i|k}(P_\sigma)_{kj} = \sum_{\sigma\in\mathfrak{S}_d} p_\sigma p^{\Lambda}_{i|\sigma^{-1}(j)}, \quad (40)$$

while for a general QC-type channel there will also be a "coherent" part:

$$p_{i|j}(U\phi) = \sum_{\sigma\in\mathfrak{S}_d} p_\sigma p_{\ i|\sigma^{-1}(j)}(\phi) + \sum_{k\neq l} U^*_{kj} U_{lj} \langle\phi_k|E_i|\phi_l\rangle. \quad (41)$$

The existence of a fully broadcastable state(s) $\varrho_*(e)$ for any QC-type channel is in some way surprising, as the measurements described by such channels are in general not von Neumann measurements, but rather POVMs [cf. Eq. (15)]. The existence of a whole family of spectrum-broadcastable states is perhaps even more surprising. Note, however, that spectrum broadcastability is a far weaker condition than full state broadcasting. By the same reason, although the broadcasting channel $\Lambda^{(N)}$ is the same for every basis, it depends only on Λ, we do not contradict the no-go theorem for state broadcasting from Ref. [6].

From a probabilistic point of view, the existence of (spectrum-)broadcastable states follows from the fact that one can associate a finite Markov process with the problem through Eq. (36), and by the Perron-Frobenius theorem each such process possesses a stationary distribution. The (spectrum-)broadcastable states are constructed precisely from this distribution.

Let us continue the above analysis and study the implications of the ergodic theorem for finite Markov chains [9]: For a stochastic matrix P, there exists a limit $P^\infty := \lim_{r\to\infty} P^r$ iff P is primitive. The limit is given by

$$P^\infty_{ij} = \lambda_{*i} 1_j, \quad (42)$$

where λ_{*i} is the stationary distribution (Perron vector) of P [cf. Eq. (35)] and $\vec{1} := (1,\ldots,1)$. Note that the limiting matrix elements are the same for each column index i: Asymptotically the probability for the process to be at site j does not depend on the initial site i. As a consequence, the limiting distribution of the process $p^\infty_i := \sum_j P^\infty_{ij} p_j$ does not depend on the initial distribution p_j:

$$\sum_j P^\infty_{ij} p_j = \lambda_{*i}. \quad (43)$$

Consider now the rth power of a QC-type channel Λ:

$$\Lambda^r(\varrho) = \sum_{i,j} P(e)^{r-1}_{ij} \mathrm{Tr}(\varrho E_j)|e_i\rangle\langle e_i|, \quad (44)$$

where $P(e)$ is defined by Eq. (36). By the ergodic Theorem, the limit $\lim_{r\to\infty}\Lambda^r =: \Lambda^\infty$ exists iff the matrix $P(e)$ is primitive. By Eqs. (42) and (44), Λ^∞ is then a constant channel, analogously to Eq. (43),

$$\Lambda^\infty(\varrho) = \varrho_*(e) \quad (45)$$

for any state ϱ. Indeed, $\Lambda^\infty(\varrho) = \sum_{i,j} P(e)^\infty_{ij}\mathrm{Tr}(\varrho E_j)|e_i\rangle\langle e_i| = (\mathrm{Tr}\varrho)\sum_i \lambda_{*i}|e_i\rangle\langle e_i| = \varrho_*(e)$ [cf. Eq. (33)]. As a consequence, Λ^∞ breaks all correlations: $\mathbb{1}\otimes\Lambda^\infty(\varrho_{AB}) = \varrho_B \otimes \varrho_*$, $\varrho_B := \mathrm{Tr}_B\varrho_{AB}$.

An interesting situation arises when Eq. (35) has more than one solution, i.e., when a QC-type channel Λ^{QC} (spectrum-)broadcasts [12] more than one state. Probabilistically, this means that the Markov process, corresponding to Λ^{QC} and a context $\{\phi_i\}$ through Eq. (36), possesses more than one stationary distribution. This happens when the process splits into two or more disconnected processes. Algebraically this means that the transition matrix $P(\phi) = [p_{i|j}(\phi)]$ is, modulo a column permutation, a direct sum of two or more primitive stochastic matrices:

$$P_{d\times d}(\phi) = P^{(1)}_{k\times k}(\phi) \oplus P^{(2)}_{(d-k)\times(d-k)}(\phi). \quad (46)$$

According to the Perron-Frobenius theorem, each of the blocks has a unique Perron vector, $\vec{\lambda}^{(1)}_*(\phi)$ or $\vec{\lambda}^{(2)}_*(\phi)$, respectively (each of them is normalized). Clearly, any d-dimensional vector of the form $\vec{\lambda}_* = p\vec{\lambda}^{(1)}_* \oplus (1-p)\vec{\lambda}^{(2)}_*$ is again an eigenvalue-1 eigenvector of $P(\phi)$ for any $p\in[0,1]$. We denote the corresponding states by $\varrho^{(1)}_*(\phi) := \mathrm{diag}[\lambda^{(1)}_1,\ldots,\lambda^{(1)}_k, 0,\ldots,0]$ and $\varrho^{(2)}_*(\phi) := \mathrm{diag}[0,\ldots,0,\lambda^{(2)}_{k+1},\ldots,\lambda^{(2)}_d]$. This is an example of the case where any state from the convex combination $p\varrho^{(1)}_* + (1-p)\varrho^{(2)}_*$ can be (spectrum-)broadcasted. Clearly, this example generalizes to more than a binary combination of states if the matrix $P(\phi)$ decomposes into more than two components: if the number of terms (degeneracy) in Eq. (46) is D, there exists a D-dimensional simplex of states (spectrum-)broadcastable by Λ^{QC} [cf. Eq. (38)]. The most degenerate case is of course when $D = d$, i.e., when the transition matrix $P(\phi) = \mathbb{1}$, so that the Markov process is trivial—there are no transitions between the sites, which happens when the POVM is in fact a von Neumann measurement in $\{\phi_i\}$: $E_i = |\phi_i\rangle\langle\phi_i|$.

One can continue the above analysis and consider local broadcasting of correlations. From the general no-local-broadcasting theorem from Ref. [5], we know that the only locally broadcastable states are the CC ones. Let us thus consider a family of CC states, built from the stationary solutions $\varrho^{(m)}_*(\phi)$ corresponding to a degenerate transition matrix $P(\phi)$:

$$\begin{aligned}\varrho_{*AB}(\pi;\phi) &:= \sum_{m,n=1}^{D} \pi_{mn}\varrho^{(m)}_*(\phi)\otimes\varrho^{(n)}_*(\phi)\\ &= \sum_{i,j=1}^{d}\sum_{m,n=1}^{D} \pi_{mn}\lambda^{(m)}_{*i}\lambda^{(n)}_{*j}|\phi_i\rangle\langle\phi_i|\otimes|\phi_j\rangle\langle\phi_j|.\end{aligned} \quad (47)$$

Applying to $\varrho_{*AB}(\pi;\phi)$ the product channel $\Lambda^{(N)}\otimes\Lambda^{(N)}$, where $\Lambda^{(N)}$ is defined in Eq. (38), one achieves a local N-copy (spectrum-)broadcasting [12] of the classical correlations: $[\Lambda^{(N)}\otimes\Lambda^{(N)}]\varrho_{*AB}(\pi;\phi) = \sigma_{A_1\ldots A_N B_1\ldots B_N}(\pi;\phi)$ and all the bipartite reductions $\sigma_{A_r B_r}(\pi;\phi)$ are (unitary equivalent)equal to $\varrho_{*AB}(\pi;\phi)$. We present a concrete example of this broadcasting scheme in the Appendix, Eqs. (A1) and (A2), while a version with two different channels is studied in what follows.

Let us now assume that two different channels, Λ_A and Λ_B, satisfy the assumptions of Theorem 1 on Alice's and Bob's side, respectively, i.e.,

$$\Lambda_A \otimes \mathbb{1}(P_+) = \sum_i p_i^A |e_i\rangle_A\langle e_i| \otimes \sigma_i^B, \tag{48}$$

$$\mathbb{1} \otimes \Lambda_B(P_+) = \sum_j p_j^B \sigma_j^A \otimes |f_j\rangle_B\langle f_j|. \tag{49}$$

Then one easily proves the following.

Corollary 1. If $\Lambda_A \otimes \mathbb{1}(P_+)$ and $\mathbb{1} \otimes \Lambda_B(P_+)$ are $\mathrm{C}_A\mathrm{Q}_B$ and $\mathrm{Q}_A\mathrm{C}_B$ states, respectively, then $\Lambda_A \otimes \Lambda_B(\varrho_{AB})$ is a CC state for any state ϱ_{AB}.

Proof. Indeed, from the proof of Theorem 1 it follows that Λ_A and Λ_B are measurement maps [cf. Eq. (15)] on Alice's and Bob's sides respectively, defined by POVM elements

$$E_i^A := dp_i^A\left(\sigma_i^B\right)^T, \quad E_j^B := dp_j^B\left(\sigma_j^A\right)^T. \tag{50}$$

Thus

$$\begin{aligned}\Lambda_A \otimes \Lambda_B(\varrho_{AB}) &= (\Lambda_A \otimes \mathbb{1})(\mathbb{1} \otimes \Lambda_B)\varrho_{AB} \\ &= \sum_{i,j} \mathrm{Tr}_A\left[E_i^A \mathrm{Tr}_B\left(\varrho_{AB}\mathbb{1} \otimes E_j^B\right)\right]|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j| \\ &= \sum_{i,j} \mathrm{Tr}\left(\varrho_{AB} E_i^A \otimes E_j^B\right)|e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|. \end{aligned} \tag{51}$$

■

The analysis of state broadcasting may be repeated in the present scenario as well. Since $\mathbb{1}_{AB} \otimes \Lambda_{A'}^{\mathrm{CQ}} \otimes \Lambda_{B'}^{\mathrm{QC}}(P_+^{ABA'B'}) = [\mathbb{1}_A \otimes \Lambda_{A'}^{\mathrm{CQ}}(P_+^{AA'})] \otimes [\mathbb{1}_B \otimes \Lambda_{B'}^{\mathrm{QC}}(P_+^{BB'})]$, the channel $\Lambda_A^{\mathrm{CQ}} \otimes \Lambda_B^{\mathrm{QC}}$ is of a $\mathrm{Q}_{A'B}\mathrm{C}_{AB'}$ type. From Theorem 2 it then immediately follows that for any basis $\{\phi_\alpha^{AB}\}$, $\alpha = 1,\ldots,d_Ad_B$, of $\mathcal{H}_A \otimes \mathcal{H}_B$ (the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ need not be the same now) there exists a state $\varrho_{*AB}(\phi^{AB})$, built from a stationary distribution of the stochastic matrix (36)

$$P^{AB}(\phi^{AB})_{\alpha\beta} := \left\langle\phi_\beta^{AB}\middle|E_i^A \otimes E_j^B\phi_\beta^{AB}\right\rangle, \tag{52}$$

$\alpha := (ij)$, and locally (spectrum-)broadcastable through $\Lambda_A^{(N)} \otimes \Lambda_B^{(N)}$ [cf. Eq. (38)]. Note that the basis $\{\phi_\alpha^{AB}\}$ need not be a product one in general.

However, for a product basis $\phi_\alpha^{AB} \equiv \phi_i^A \otimes \phi_j^B$ one can say more. The matrix $P^{AB}(\phi^{AB})$ is then a product as well: $P^{AB}(\phi^{AB}) = P^A(\phi^A) \otimes P^B(\phi^B)$ and $P^{AB}(\phi^{AB})$ is primitive iff both $P^A(\phi^A)$ and $P^B(\phi^B)$ are, i.e., $\Lambda_A^{(N)}$ and $\Lambda_B^{(N)}$ spectrum broadcast only one state each. In such a case, the product state $\varrho_{*AB}(\phi^{AB}) = \varrho_{*A}(\phi^A) \otimes \varrho_{*B}(\phi^B)$ is the only state that can be spectrum broadcasted and there is no local broadcasting of classical correlations—the spectrum of $\varrho_{*AB}(\phi^{AB})$ is a product, $\lambda_{*ij}(\phi^{AB}) = \lambda_{*i}(\phi^A)\lambda_{*j}(\phi^B)$. If, however, at least one channel spectrum broadcasts more than one state, then there exists a family of locally spectrum-broadcastable correlated CC states, built analogously as in Eq. (47): $\varrho_{*AB}(\pi;\phi^A,\phi^B) := \sum_{m,n=1}^{D_A,D_B} \pi_{mn}\varrho_{*A}^{(m)}(\phi^A) \otimes \varrho_{*B}^{(n)}(\phi^B)$. A concrete example of such a situation is presented in the Appendix, Eqs. (A3) and (A4). When it comes to local full state broadcasting, by Theorem 2 it is guaranteed for $\varrho_{*AB}(e,f)$, which is a CC state in the bases $\{e_i\}$ and $\{f_j\}$ [cf. Eqs. (48) and (49)], in accordance with the general results of Ref. [5]. Again, if both matrices $P^A(e)$ and $P^B(f)$ are primitive, $\varrho_{*AB}(e,f)$ is a product state with no correlations. However, if at least one $P^A(e)$ or $P^B(f)$ is not primitive, by the above construction there will be a family of locally broadcastable correlated CC states $\varrho_{*AB}(\pi;e,f)$.

Before we conclude, let us digress on the nature of some multipartite QC states. We assume that, e.g., Bob holds two (possibly different) subsystems and that the joint state is $\mathrm{Q}_A\mathrm{C}_{BB'}$, that is,

$$\varrho_{ABB'} = \sum_\alpha p_\alpha \sigma_\alpha^A \otimes |e_\alpha\rangle_{BB'}\langle e_\alpha|, \tag{53}$$

where $\{e_\alpha\}$ is a basis in $\mathcal{H}_B \otimes \mathcal{H}_{B'}$, labeled by α. It is not necessarily a product basis—for the definition of a $\mathrm{Q}_A\mathrm{C}_{BB'}$ state it is enough that it is orthonormal. What is interesting is that simultaneously forcing both reductions $\varrho_{AB} := \mathrm{Tr}_{B'}\varrho_{ABB'}$ and $\varrho_{AB'} := \mathrm{Tr}_{B}\varrho_{ABB'}$ to be $\mathrm{Q}_A\mathrm{C}_B$ and $\mathrm{Q}_A\mathrm{C}_{B'}$, respectively,

$$\varrho_{AB} = \sum_i \lambda_i \varrho_i^A \otimes |e_i\rangle_B\langle e_i|, \tag{54}$$

$$\varrho_{AB'} = \sum_{i'} \pi_{i'}\tau_{i'}^A \otimes |f_{i'}\rangle_{B'}\langle f_{i'}|, \tag{55}$$

does not force $\varrho_{ABB'}$ to be $\mathrm{Q}_{AB}\mathrm{C}_{B'}$ and $\mathrm{Q}_{AB'}\mathrm{C}_B$ simultaneously (we may label such a class by $\mathrm{Q}_A\mathrm{C}_B\mathrm{C}_{B'}$); i.e., $\{e_\alpha\}$ in Eq. (53) still need not be a product basis. As a simple example consider $\mathcal{H}_B = \mathcal{H}_{B'} = \mathbb{C}^2$, and $\{e_\alpha\}_{\alpha=1,\ldots,4}$—the Bell basis. Then obviously both reductions ϱ_{AB} and $\varrho_{AB'}$ are product, $1/2(\sum_\alpha p_\alpha\sigma_\alpha) \otimes \mathbb{1}$, and hence trivially $\mathrm{Q}_A\mathrm{C}_B$ and $\mathrm{Q}_A\mathrm{C}_{B'}$, but the whole state $\varrho_{ABB'}$ is not $\mathrm{Q}_A\mathrm{C}_B\mathrm{C}_{B'}$.

In some sense a converse of the above observation is also true: there exist $\mathrm{Q}_A\mathrm{C}_{BB'}$ states with a product basis on BB', which are nevertheless not $\mathrm{Q}_A\mathrm{C}_B\mathrm{C}_{B'}$, or, equivalently, both reductions $\mathrm{Tr}_B\varrho_{ABB'}$ and $\mathrm{Tr}_{B'}\varrho_{ABB'}$ are not $\mathrm{Q}_A\mathrm{C}_B$ and $\mathrm{Q}_A\mathrm{C}_{B'}$ respectively. As an example of such a state consider $\mathcal{H}_B = \mathcal{H}_{B'} = \mathbb{C}^3$, and choose as $\{e_\alpha\}_{\alpha=1,\ldots,9}$ in Eq. (53) the "nonlocality without entanglement" $3 \otimes 3$ basis from Ref. [13]. Then both $\mathrm{Tr}_B\varrho_{ABB'}$ and $\mathrm{Tr}_{B'}\varrho_{ABB'}$ will contain an overcomplete set on the B and B' side, respectively.

In conclusion, we have provided a refinement of the characterization of entanglement-breaking channels from Ref. [1] to more general quantum correlations and connected it to measurement maps, quantum state and correlation broadcasting, and finite Markov chains. We have considered two classes of channels— (i) the ones that break quantum correlations by turning them into the QC form and (ii) the ones that fully break quantum correlations by turning them into CC ones. We have shown that a channel belongs to the first class iff it turns a maximally entangled state into a QC state or equivalently it is represented by a measure-and-prepare scheme, where the outcomes of a POVM measurement are followed by a preparation of states from some specific orthonormal basis. In other words, it is a quantum-to-classical measurement map (i.e., it gives the state of the apparatus after tracing the system).

Surprisingly, a similar question in the case of the second class of channels becomes even more interesting: the analogy to entanglement-breaking channels now fails and one cannot characterize the channels from the second class only by their actions on the maximally entangled state. However, a characterization from a different perspective seems possible. First of all, it turns out that the POVMs, constituting the channels, are mutually commuting and arise from a stochastic matrix, thus making a connection to finite Markov chains. Second, the set of bipartite states that are mapped into the CC form is more complicated.

Our analysis of the ability to broadcast quantum states and correlations by QC-type channels reveals an interesting application of the Perron-Frobenius theorem. The existence of a family of spectrum-broadcastable states and at least one fully broadcastable state, even if the POVM measurement is not of the von Neumann type, follows from the fact that each finite Markov process possesses a stationary distribution. This broadcasting scheme, albeit in general substantially weaker than the standard broadcasting of, e.g., Refs. [5,6], surprisingly goes beyond the simple C-NOT scenario. The connection between broadcasting and finite Markov chains is, to our knowledge, quite unexpected and will be the subject of further research.

In fact, perfect broadcasting operations applied so far have corresponded to a scenario where to a given input CC state $\varrho_{AB} = \sum_{i,j} p_{ij}|i\rangle\langle i| \otimes |j\rangle\langle j|$ one locally applies the generalized C-NOT gates $U|i\rangle|j\rangle := |i\rangle|i \oplus j\rangle$. Application of the Perron-Frobenius theorem presented in this work goes beyond this simple scenario.

We believe that the current work opens new perspectives for analysis of the measurement problem and state and correlation broadcasting. Especially interesting is the possibility to study quantum decoherence in terms of broadcasting.

ACKNOWLEDGMENTS

We thank D. Chruściński, A. Grudka, M. Horodecki, M. Kuś, A. Rutkowski, and K. Życzkowski for discussions and remarks; F. G. S. L. Brandão for a question on the CQ limitation of Theorem 1; and J. Wehr for providing useful information on Markov chains. This work is supported by the ERC Advanced Grant QOLAPS and the National Science Centre project Maestro DEC-2011/02/A/ST2/00305. J.K.K. acknowledges the financial support of the QCS and TOQUATA Projects and would like to thank the National Quantum Information Centre (KCIK) in Gdańsk and the Center for Theoretical Physics of The Polish Academy of Sciences for hospitality.

APPENDIX

Consider the following example. Let

$$P^{(1)} := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \tag{A1}$$

for some fixed basis $\{\phi_i\}$ and let $P^{(2)}$ be an arbitrary irreducible bistochastic matrix on $\mathbb{R}^3$, say

$$P^{(2)} := \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{3}{8} & 0 & \frac{5}{8} \\ \frac{1}{2} & \frac{5}{8} & 0 \end{bmatrix} \tag{A2}$$

for the same basis. Since we know that any matrix $A \in \mathcal{M}_{d\times d}(\mathbb{R})$ with non-negative elements is irreducible iff $(\mathbb{1} + A)^{d-1}$ has all elements non-negative, we may easily check that both matrices are irreducible. The unique Perron vector of $P^{(1)}$ is just $\vec{\lambda}^{(1)} = [\frac{1}{3},\frac{1}{6},\frac{1}{2}]^T$. The unique eigenvector of the irreducible bistochastic matrix is of course $\vec{\lambda}^{(2)} = [\frac{1}{3},\frac{1}{3},\frac{1}{3}]^T$. Consider now the stochastic matrix $P := P^{(1)} \oplus P^{(2)}$ on $\mathbb{R}^6$ Then any state of the form $\varrho_{*AB}(\pi) = \sum_{m,n=1}^{2} \pi_{mn}\varrho_*^{(m)} \otimes \varrho_*^{(n)}$ with $\varrho_*^{(1)} := \mathrm{diag}[\frac{1}{3},\frac{1}{6},\frac{1}{2}]$ and $\varrho_*^{(2)} := \mathrm{diag}[\frac{1}{3},\frac{1}{3},\frac{1}{3}]$ can be spectrum broadcasted and full broadcasted by the product of the channels $\Lambda^{(N)}$, defined in Eq. (38).

An even simpler example with two different channels can be constructed to illustrate spectrum broadcasting and full broadcasting of correlations. Namely, consider two bistochastic matrices of the form

$$P^{A} := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix} \tag{A3}$$

and

$$P^{B} := \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \end{bmatrix} \tag{A4}$$

for some basis $\{\phi_i\}$. They are clearly reducible. Finding their Perron vectors and defining $\varrho_{*AB}(\pi) := \sum_{m,n=1}^{2} \pi_{mn}\varrho_*^{(m)} \otimes \varrho_{*B}^{(j)}$ as $\varrho_{*A}^{(1)} := \mathrm{diag}[0,\frac{1}{2},\frac{1}{2}]$, $\varrho_{*A}^{(2)} := [1,0,0]$ and $\varrho_{*B}^{(1)} := \mathrm{diag}[\frac{1}{2},0,\frac{1}{2}]$, $\varrho_{*B}^{(2)} := [0,1,0]$, we see that $\varrho_{*AB}(\pi)$ is locally broadcastable by the map $\Lambda_A^{(N)} \otimes \Lambda_B^{(N)}$, where Λ_A are Λ_B are defined again by Eq. (38).

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