

## ALL GRAPHS WITH PAIRED-DOMINATION NUMBER TWO LESS THAN THEIR ORDER

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**Abstract.** Let  $G = (V, E)$  be a graph with no isolated vertices. A set  $S \subseteq V$  is a paired-dominating set of  $G$  if every vertex not in  $S$  is adjacent with some vertex in  $S$  and the subgraph induced by  $S$  contains a perfect matching. The paired-domination number  $\gamma_p(G)$  of  $G$  is defined to be the minimum cardinality of a paired-dominating set of  $G$ . Let  $G$  be a graph of order  $n$ . In [*Paired-domination in graphs*, Networks **32** (1998), 199–206] Haynes and Slater described graphs  $G$  with  $\gamma_p(G) = n$  and also graphs with  $\gamma_p(G) = n - 1$ . In this paper we show all graphs for which  $\gamma_p(G) = n - 2$ .

**Keywords:** paired-domination, paired-domination number.

**Mathematics Subject Classification:** 05C69.

### 1. INTRODUCTION

All graphs considered in this paper are finite, undirected, without loops, multiple edges and isolated vertices. Let  $G = (V, E)$  be a graph with the vertex set  $V$  and the edge set  $E$ . Then we use the convention  $V = V(G)$  and  $E = E(G)$ . Let  $G$  and  $G'$  be two graphs. If  $V(G) \subseteq V(G')$  and  $E(G) \subseteq E(G')$  then  $G$  is a *subgraph* of  $G'$  (and  $G'$  is a *supergraph* of  $G$ ), written as  $G \subseteq G'$ . The number of vertices of  $G$  is called the *order* of  $G$  and is denoted by  $n(G)$ . When there is no confusion we use the abbreviation  $n(G) = n$ . Let  $C_n$  and  $P_n$  denote the cycle and the path of order  $n$ , respectively. The *open neighborhood* of a vertex  $v \in V$  in  $G$  is denoted  $N_G(v) = N(v)$  and defined by  $N(v) = \{u \in V : vu \in E\}$  and the *closed neighborhood*  $N[v]$  of  $v$  is  $N[v] = N(v) \cup \{v\}$ . For a set  $S$  of vertices the *open neighborhood*  $N(S)$  is defined as the union of open neighborhoods  $N(v)$  of vertices  $v \in S$ , the *closed neighborhood*  $N[S]$  of  $S$  is  $N[S] = N(S) \cup S$ . The *degree*  $d_G(v) = d(v)$  of a vertex  $v$  in  $G$  is the number of edges incident to  $v$  in  $G$ ; by our definition of a graph, this is equal to  $|N(v)|$ .

A *leaf* in a graph is a vertex of degree one. A *subdivided star*  $K_{1,t}^*$  is a star  $K_{1,t}$ , where each edge is subdivided exactly once.

In the present paper we continue the study of paired-domination. Problems related to paired-domination in graphs appear in [1–5]. A set  $M$  of independent edges in a graph  $G$  is called a *matching* in  $G$ . A *perfect matching*  $M$  in  $G$  is a matching in  $G$  such that every vertex of  $G$  is incident to an edge of  $M$ . A set  $S \subseteq V$  is a *paired-dominating set*, denoted PDS, of a graph  $G$  if every vertex in  $V - S$  is adjacent to a vertex in  $S$  and the subgraph  $G[S]$  induced by  $S$  contains a perfect matching  $M$ . Therefore, a paired-dominating set  $S$  is a dominating set  $S = \{u_1, v_1, u_2, v_2, \dots, u_k, v_k\}$  with matching  $M = \{e_1, e_2, \dots, e_k\}$ , where  $e_i = u_i v_i$ ,  $i = 1, \dots, k$ . Then we say that  $u_i$  and  $v_i$  are *paired* in  $S$ . Observe that in every graph without isolated vertices the end-vertices of any maximal matching form a PDS. The *paired-domination number* of  $G$ , denoted  $\gamma_p(G)$ , is the minimum cardinality of a PDS of  $G$ . We will call a set  $S$  a  $\gamma_p(G)$ -set if  $S$  is a paired-dominating set of cardinality  $\gamma_p(G)$ . The following statement is an immediate consequence of the definition of PDS.

**Observation 1.1** ([4]). *If  $u$  is adjacent to a leaf of  $G$ , then  $u$  is in every PDS.*

Haynes and Slater [4] show that for a connected graph  $G$  of order at least six and with minimum degree  $\delta(G) \geq 2$ , two-thirds of its order is the bound for  $\gamma_p(G)$ .

**Theorem 1.2** ([4]). *If a connected graph  $G$  has  $n \geq 6$  and  $\delta(G) \geq 2$ , then*

$$\gamma_p(G) \leq 2n/3.$$

Henning in [5] characterizes the graphs that achieve equality in the bound of Theorem 1.2.

In [4] the authors give the solutions of the graph-equations  $\gamma_p(G) = n$  and  $\gamma_p(G) = n - 1$ , where  $G$  is a graph of order  $n$ .

**Theorem 1.3** ([4]). *A graph  $G$  with no isolated vertices has  $\gamma_p(G) = n$  if and only if  $G$  is  $mK_2$ .*

Let  $\mathcal{F}$  be the collection of graphs  $C_3$ ,  $C_5$ , and the subdivided stars  $K_{1,t}^*$ . Now, we can formulate the following statements.

**Theorem 1.4** ([4]). *For a connected graph  $G$  with  $n \geq 3$ ,  $\gamma_p(G) \leq n - 1$  with equality if and only if  $G \in \mathcal{F}$ .*

**Corollary 1.5** ([4]). *If  $G$  is a graph with  $\gamma_p(G) = n - 1$ , then  $G = H \cup rK_2$  for  $H \in \mathcal{F}$  and  $r \geq 0$ .*

In the present paper we consider the graph-equation

$$\gamma_p(G) = n - 2, \tag{1.1}$$

where  $n \geq 4$  is the order of a graph  $G$ .

Our aim in this paper is to find all graphs  $G$  satisfying (1.1). For this purpose we need the following definition and statements.



**Definition 1.6.** A subgraph  $G$  of a graph  $G'$  is called a *special subgraph* of  $G'$ , and  $G'$  is a *special supergraph* of  $G$ , if either  $V(G) = V(G')$  or the subgraph  $G'[V(G') - V(G)]$  has a perfect matching.

It is clear that if  $V(G) = V(G')$  then the concepts “subgraph” and “special subgraph” are equivalent. Now we can formulate the following fact.

**Fact 1.7.** Let  $G$  be a special subgraph of  $G'$ .

- a) If  $S$  is a PDS in  $G$  then  $S' = S \cup (V(G') - V(G))$  is a PDS in  $G'$ .  
 b) If  $\gamma_p(G) = n - r$  then  $\gamma_p(G') \leq n' - r$ , where  $n = |V(G)|$ ,  $n' = |V(G')|$  and  $0 \leq r \leq n - 2$ .

*Proof.* a) Assume that

$$S = \{u_1, v_1, u_2, v_2, \dots, u_t, v_t\} \quad \text{and} \quad V(G') - V(G) = \{u_{t+1}, v_{t+1}, \dots, u_k, v_k\},$$

where  $u_i$  and  $v_i$  are paired in  $S$  (for  $i = 1, \dots, t$ ) and in  $V(G') - V(G)$  (for  $i = t+1, \dots, k$ ). Hence  $M = \{e_1, e_2, \dots, e_k\}$ , where  $e_i = u_i v_i$ , for  $i = 1, \dots, k$ , is a perfect matching in  $G'[S']$ . By definition of a PDS and by  $V(G) - S = V(G') - S'$  we obtain the statement of a).

b) Let  $S$  be a  $\gamma_p$ -set in  $G$ , thus  $|V(G) - S| = r$ . It follows from a) that  $S' = S \cup (V(G') - V(G))$  is a PDS in  $G'$ . Moreover, we have the equality

$$|S'| = n' - |V(G') - S'| = n' - |V(G) - S| = n' - r.$$

Therefore we obtain  $\gamma_p(G') \leq |S'| = n' - r$ . □

Now assume that  $G$  is a connected graph of order  $n \geq 4$  satisfying (1.1). Let  $S = \{u_1, v_1, u_2, v_2, \dots, u_k, v_k\}$  be a  $\gamma_p(G)$ -set with a perfect matching  $M = \{e_1, e_2, \dots, e_k\}$ , where  $e_i = u_i v_i$  for  $i = 1, 2, \dots, k$ , and  $V - S = \{x, y\}$ . By letting  $\alpha(S)$  denote the minimum cardinality of a subset of  $S$  which dominates  $V - S$ , i.e.

$$\alpha(S) = \min\{|S'| : S' \subseteq S, V - S \subseteq N(S')\}.$$

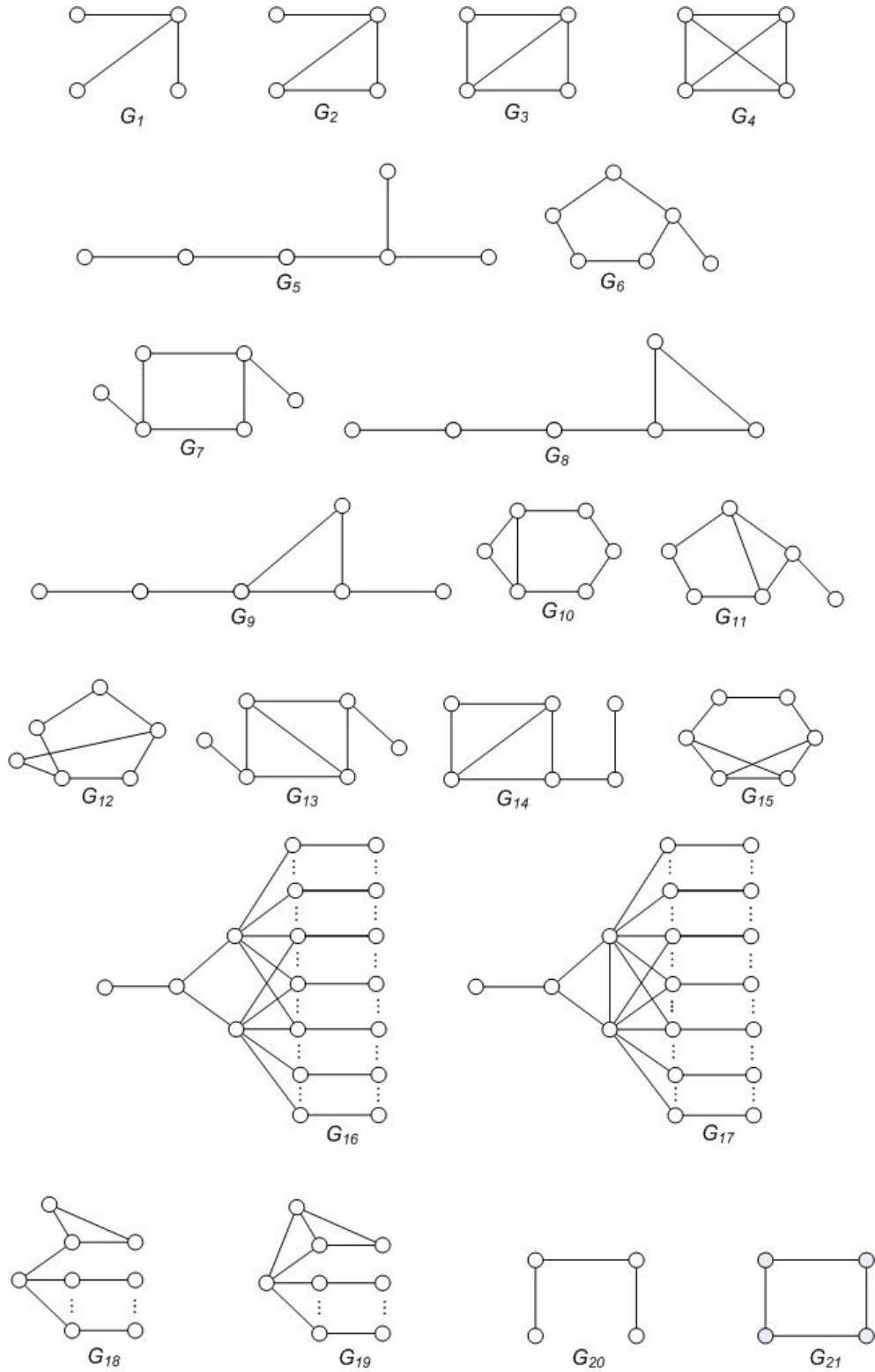
Let  $S_i$  be any set of size  $\alpha(S)$  such that  $S_i \subseteq S$  and  $V - S \subseteq N(S_i)$ . For  $S$ ,  $M$  and  $S_i$  we define a graph  $H$  as follows:

$$V(H) = V(G) \quad \text{and} \quad E(H) = M \cup \{uv : u \in S_i, v \in \{x, y\}\}.$$

It is clear that  $H$  is a spanning forest of  $G$ ; we denote it as  $G_{sf}(S, M, S_i)$ .

## 2. THE MAIN RESULT

The main purpose of this paper is to construct all graphs  $G$  of order  $n$  for which  $\gamma_p(G) = n - 2$ . At first consider the family  $\mathcal{G}$  of graphs in Fig. 1. We shall show that only the graphs in family  $\mathcal{G}$  are connected and satisfy condition (1.1).



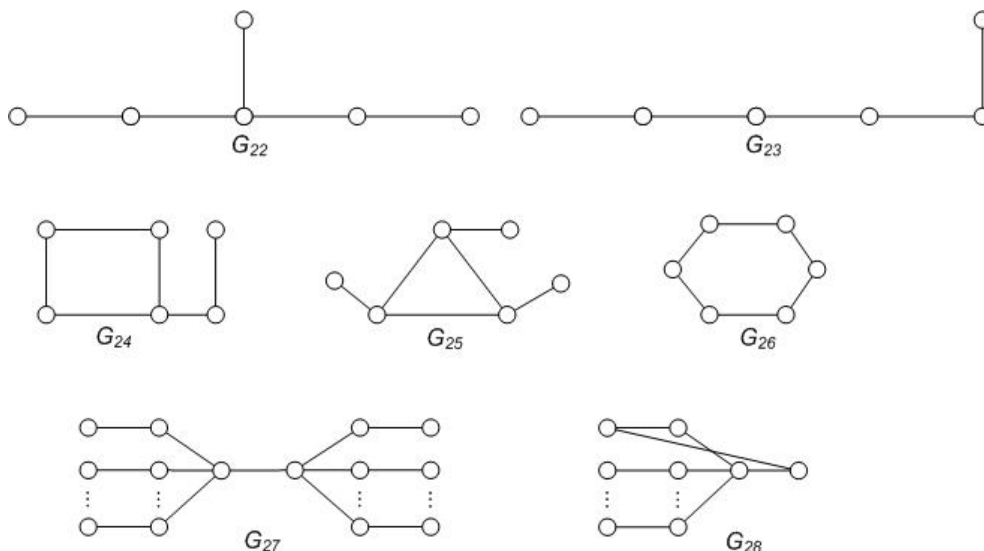


Fig. 1. Graphs in family  $\mathcal{G}$

**Theorem 2.1.** Let  $G$  be a connected graph of order  $n \geq 4$ . Then  $\gamma_p(G) = n - 2$  if and only if  $G \in \mathcal{G}$ .

*Proof.* Our aim is to construct all connected graphs  $G$  for which (1.1) holds. Let  $G$  be a connected graph of order  $n \geq 4$  satisfying (1.1). We shall prove that  $G \in \mathcal{G}$ .

Let us consider the following cases.

*Case 1.* There exists a  $\gamma_p(G)$ -set  $S$  such that  $\alpha(S) = 1$ .

*Case 1.1.*  $k = 1$ . Then we have the graphs shown in Fig. 2. It is clear that the graphs  $H_i$  satisfy (1.1) and  $H_i = G_i$  for  $i = 1, 2, 3, 4$ .

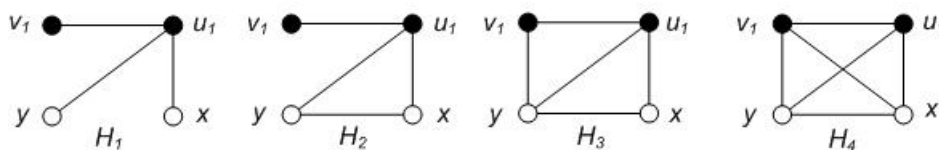
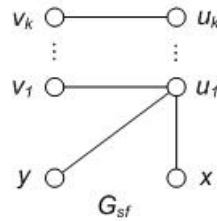


Fig. 2. The graphs for Case 1.1.

Figure 2 illustrates the graphs  $H_i$ , where the shaded vertices form a  $\gamma_p$ -set. We shall continue to use this convention in our proof.

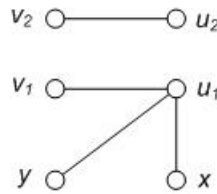
At present for  $k \geq 2$  we shall find all connected graphs  $G$  satisfying (1.1) and having a  $\gamma_p(G)$ -set  $S$  with  $\alpha(S) = 1$ . It is clear that in Case 1 any graph  $G_{sf}(S, M, S_i)$  is independent of the choice of  $S, M$  and  $S_i$ , so we can write  $G_{sf}(S, M, S_i) = G_{sf}$ . The spanning forest  $G_{sf}$  consists of  $k$  components  $G^{(1)}, G^{(2)}, \dots, G^{(k)}$ , where  $G^{(1)} = K_{1,3}$

with  $V(K_{1,3}) = \{x, y, u_1, v_1\}$ , where  $u_1$  is the central vertex, while  $G^{(i)} = K_2$  for  $i = 2, \dots, k$  (see Fig. 3). Now by adding suitable edges to  $G_{sf}$  we are able to reconstruct  $G$ .



**Fig. 3.** The spanning forest of  $G$

*Case 1.2.*  $k = 2$ . Now we start with the graph  $H_5$  (Fig. 4). In our construction of the



**Fig. 4.** The spanning forest  $H_5$

desired connected graphs we add one or more edges to  $H_5$ . Thus, let us consider the following cases regarding the number of these edges.

*Case 1.2.1. One edge* (Fig. 5). One can see that  $H_6 = G_5$  satisfies (1.1) but  $H_7$  does not.



**Fig. 5.** The graphs obtained by adding one edge to  $H_5$

*Case 1.2.2. Two edges.* For  $H_7$  we have  $\gamma_p(H_7) = 6 - 4 = |V(H_7)| - 4$ . Thus, by Fact 1.7 b) for any special supergraph  $G'$  of  $H_7$  we obtain  $\gamma_p(G') \leq |V(G')| - 4$ . Hence, we deduce that it suffices to add one edge to  $H_6$ . Since adding the edges  $u_1u_2$  or  $u_1v_2$

leads to  $H_7$ , we shall omit these edges in our construction. Now consider the graphs of Fig. 6.

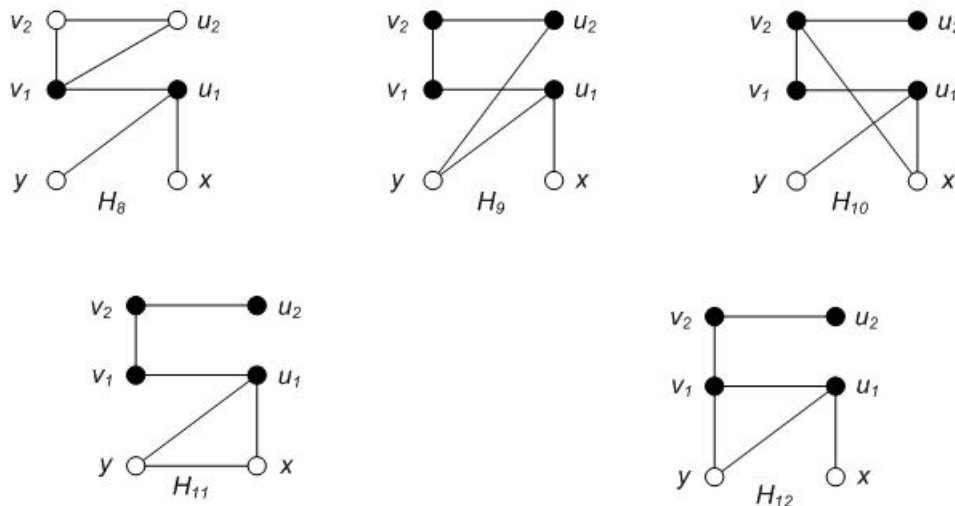


Fig. 6. Adding a new edge to  $H_6$

Certainly,  $\gamma_p(H_8) = n - 4$ ,  $\gamma_p(H_i) = n - 2$  and  $H_i = G_{i-3}$  for  $i = 9, \dots, 12$ . Using the above argument for  $H_8$  we do not take  $v_1u_2$ . Let us consider the following cases.

Case 1.2.3. Three edges. It follows from Fact 1.7 b) that it suffices to add one edge to  $H_i$  for  $i = 9, \dots, 12$ .

Case 1.2.3.1.  $H_9$ . Observe that  $H_i = G_{i-3}$ ,  $i = 13, 14, 15$ , satisfy (1.1). Moreover, the graphs depicted in Fig. 7 are the unique graphs for which (1.1) holds in this case. Indeed, the edge  $v_2y$  leads to a supergraph of  $H_8$ , and joining  $u_2$  to  $x$  we have  $H_{15}$ .

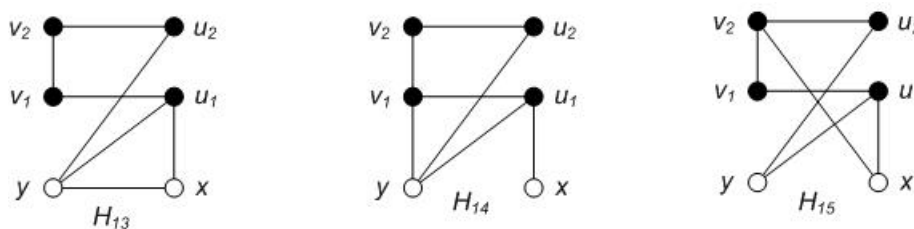


Fig. 7.  $H_9 + e$

Case 1.2.3.2.  $H_{10}$ . Then we obtain a supergraph of  $H_7$  by means of edge  $v_2y$ , a supergraph of  $H_8$  by means of  $xy$ ,  $u_2x$ , instead by adding  $u_2y$  we return to  $H_{15}$ . Therefore, it remains to research the graph of Fig. 8. It obvious that (1.1) holds for  $H_{16} = G_{13}$ .

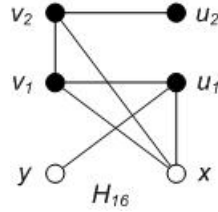


Fig. 8. The graph obtained from  $H_{10}$  by adding an edge

Case 1.2.3.3.  $H_{11}$ . Then it suffices to consider the graph of Fig. 9. Really, edges  $v_2x$ ,  $v_2y$  lead to a supergraph of  $H_8$  and  $u_2x$ ,  $u_2y$  lead to  $H_{13}$ . Observe that for  $H_{17} = G_{14}$  equality (1.1) is true.

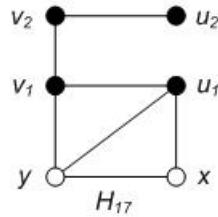


Fig. 9.  $H_{11} + e$

Case 1.2.3.4.  $H_{12}$ . Here we do not obtain any new graph satisfying (1.1). Indeed, we obtain: a supergraph of  $H_7$  (by adding  $v_1x$ ), a supergraph of  $H_8$  (by  $v_2x$ ),  $H_{13}$  (by  $u_2x$ ),  $H_{14}$  (by  $u_2y$ ) and  $H_{16}$  (by  $v_2y$ ).

Case 1.2.4. Four edges.

Case 1.2.4.1.  $H_{13}$ . Let  $G$  be a graph obtained by adding a new edge  $e$  to  $H_{13}$ . If  $e = v_1y$  then  $H_7 \subseteq G$ ; if  $e = v_2y, v_2x$ , then  $H_8 \subseteq G$  and for  $e = v_1x, u_2x$  we have the graph  $G_{15} \in \mathcal{G}$  (Fig. 10).

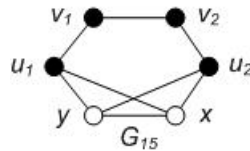


Fig. 10.  $H_{15} + e$

Case 1.2.4.2.  $H_{14}$ . Keeping the above convention we note: if  $e = xy$  then  $H_7 \subseteq G$ ; if  $e = v_2y, v_2x, u_2x$  then  $H_8 \subseteq G$ .



Case 1.2.4.3.  $H_{15}$ . If  $e = v_2y$  then  $H_7 \subseteq G$ ; if  $e = xy, v_1y, u_2x$  then  $H_8 \subseteq G$ ; if  $e = v_1x$  then  $G = G_{15}$ . It is easy to see that (1.1) is true for  $G_{15}$ .

Case 1.2.4.4.  $H_{16}$ . In this case we conclude: if  $e = xy$  then  $H_7 \subseteq G$ ; if  $e = v_2y, u_2x$  then  $H_8 \subseteq G$ ; if  $e = u_2y$  then  $G = G_{15}$ .

Case 1.2.4.5.  $H_{17}$ . Then we obtain the following results: if  $e = v_1x, v_2y, u_2y$  then  $H_7 \subseteq G$ ; if  $e = v_2x$  then  $H_8 \subseteq G$ ; if  $e = u_2x$  then we have the graph  $H_{18}$  depicted in Fig. 11. It is clear that  $H_{18} = G_{15}$ .

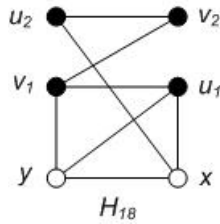


Fig. 11.  $H_{17} + e$ , where  $e = u_2x$

Case 1.2.5. Five edges.

Case 1.2.5.1.  $G_{15}$ . Then it suffices to consider the following: if  $e = v_1y$  then  $H_7 \subseteq G$ ; if  $e = v_1x$  then  $H_8 \subseteq G$ . Therefore, Case 1.2 is complete.

For case  $k \geq 3$  we only consider graphs satisfying the condition  $G[S'] = G_{sf}[S'] = K_{1,3}$  for  $S' = \{x, y, u_1, v_1\}$ . In other words,  $G$  contains the induced star  $K_{1,3}$ , where  $V(K_{1,3}) = \{x, y, u_1, v_1\}$  and  $u_1$  is the central vertex.

Case 1.3.  $k = 3$ . Then we start with the basic graph of Fig. 12. To obtain connected graphs we add two or more edges to  $H_{19}$  and investigate whether (1.1) holds for the resulting graphs. At first we find a forbidden subgraph  $H \subseteq G$  i.e. such that  $\gamma_p(H) = n - 4$ . We have already shown two forbidden special subgraphs  $H_7, H_8$ , and we now present the other one in Fig. 13. For a while we return to the general case  $k \geq 3$ . The forbidden special subgraphs  $H_7$  and  $H_{20}$  determine a means of construction of graphs  $G$  from  $G_{sf}$ .

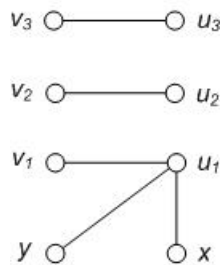


Fig. 12. The spanning forest  $G_{sf} = H_{19}$

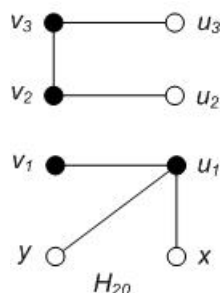


Fig. 13. The forbidden special subgraph

**Claim 1.** Let  $G$  be a connected graph satisfying (1.1) and obtained from  $G_{sf} = H_{19}$ . Then vertex  $u_i$  or  $v_i$ ,  $i = 2, \dots, k$ , can be adjacent to the vertices  $v_1, x, y$ , only.

Now we add at least two edges to  $H_{19}$ . We consider the following cases.

*Case 1.3.1. Two edges.* Then we obtain the graphs  $H_{21}$  and  $H_{22}$  for which (1.1) holds (Fig. 14).

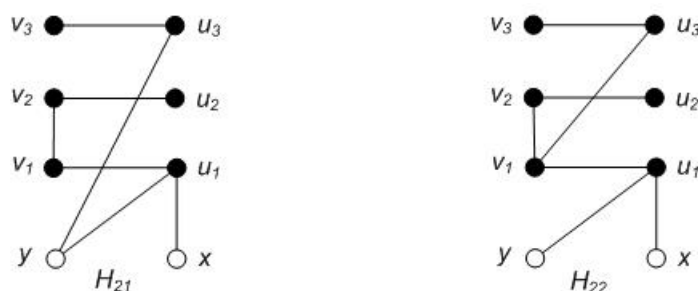


Fig. 14. Adding two edges to  $H_{19}$

*Case 1.3.2. Three edges.* At present it suffices to add one edge in  $H_{21}$ ,  $H_{22}$ . This way we obtain the graphs depicted in Figure 15.

Observe that (1.1) fails for  $H_{24}$  since  $H_{20} \subseteq H_{24}$ . Thus,  $H_{23}$  satisfies (1.1) but  $H_i$ ,  $i = 24, 25, 26$ , do not.

*Case 1.3.3. Four edges.* By adding one edge to  $H_{23}$  we obtain the unique graph for which (1.1) holds (see Fig. 16). One can verify that in the remaining options we have special supergraphs of  $H_7$ ,  $H_8$ ,  $H_{20}$ ,  $H_{25}$  or  $H_{26}$ .

*Case 1.3.4. Five edges.* Each new edge in  $H_{27}$  leads to a special supergraph of  $H_7$ ,  $H_8$ ,  $H_{20}$ ,  $H_{25}$  or  $H_{26}$ . But the following statement is obvious.

**Claim 2.** The graphs  $H_7$ ,  $H_8$ ,  $H_{20}$ ,  $H_{25}$  and  $H_{26}$  are forbidden special subgraphs for (1.1).

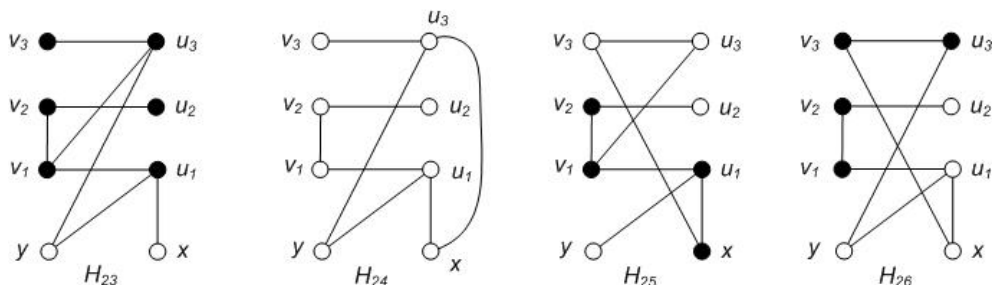


Fig. 15.  $H_{21} + e$  and  $H_{22} + e$

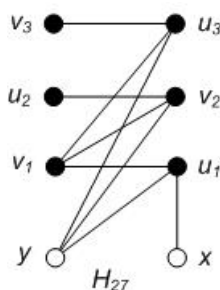


Fig. 16. The graph obtained from  $H_{23}$  by adding one edge

We now study a generalization of the case  $k = 3$ . We keep our earlier assumption regarding the induced star  $K_{1,3}$  with vertex set  $\{u_1, v_1, x, y\}$ .

Case 1.4.  $k \geq 3$ . Then we give one property of graphs satisfying (1.1).

**Claim 3.** Let  $G$  be a connected graph for which (1.1) holds and  $k \geq 3$ . If  $G$  contains the induced star  $K_{1,3}$  with  $V(K_{1,3}) = \{x, y, u_1, v_1\}$  then at least one vertex of  $K_{1,3}$  is a leaf in  $G$ .

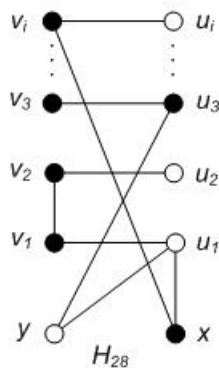
*Proof.* Consider some cases.

Case A.  $k = 3$ . It follows from our earlier investigations that  $H_{21}, H_{22}, H_{23}$  and  $H_{27}$  are the unique connected graphs satisfying (1.1) in this case. Thus, we have the desired result.

Case B.  $k \geq 4$ . Claim 1 and Fact 1.7 b) imply that a special subgraph  $G[S]$  induced by  $S = \{x, y, u_1, v_1, u_2, v_2, u_3, v_3\}$  is connected and satisfies (1.1), i.e. it must be one of the graphs  $H_{21}, H_{22}, H_{23}, H_{27}$ .

Case B.1.  $G[S] = H_{21}$ . We show that  $x$  is a leaf in  $G$ . Suppose not and let  $x$  be adjacent to  $v_i$ , where  $i \geq 4$ . Then we obtain the graph  $H_{28}$  in Fig. 17, for which (1.1) does not hold.

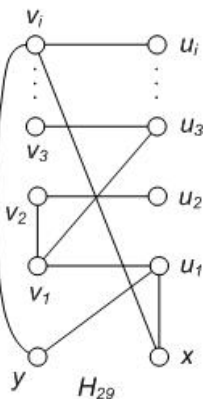




**Fig. 17.**  $x$  is adjacent to  $v_i$  for  $i \geq 4$

*Case B.2.*  $G[S] = H_{22}$ .

*Case B.2.1.* Suppose that in  $G$  vertex  $v_i$ ,  $i \geq 4$ , is adjacent to  $x$  and  $y$ . Then for graph  $H_{29}$  depicted in Fig. 18 equality (1.1) is false since  $H_{20} \subseteq H_{29}$ .



**Fig. 18.**  $v_i$ , for  $i \geq 4$ , is adjacent to  $x$  and  $y$

*Case B.2.2.* Assume that in  $G$  vertices  $v_i$  and  $u_i$ ,  $i \geq 4$  are adjacent to  $x$  and  $y$ , respectively (see Fig. 19). In this way we obtain graph  $H_{30}$  which does not satisfy (1.1) since  $H_{26} \subseteq H_{30}$ .

*Case B.2.3.* Now, in  $G$  let vertices  $v_i$  and  $u_j$ ,  $4 \leq i < j$ , be adjacent to  $x$  and  $y$ , respectively (Fig. 20). As can be seen, (1.1) fails for  $H_{31}$ , furthermore  $u_j$  is paired with  $y$ ,  $u_i$  with  $v_i$ ,  $u_3$  with  $v_3$  and  $v_1$  with  $v_2$ . It follows from the above consideration that we omit the cases:  $G[S] = H_{23}$  and  $G[S] = H_{27}$ , since  $H_{21}$ ,  $H_{22}$  are subgraphs of  $H_{23}$ ,  $H_{27}$ . In all cases we obtain special subgraphs of  $G$  for which (1.1) fails, therefore  $G$  does not satisfy (1.1), a contradiction.  $\square$

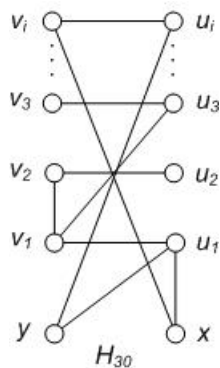


Fig. 19.  $v_i$  is adjacent to  $x$  and  $u_i$  to  $y$ , where  $i \geq 4$

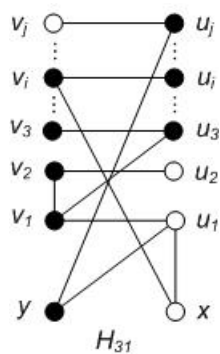


Fig. 20.  $v_i$  is adjacent to  $x$  and  $u_j$  to  $y$  for  $4 \leq i < j$

We are now in a position to construct the desired graphs for  $k \geq 3$ . Let  $G$  be a connected graph satisfying the following conditions:

- a) (1.1) holds,
- b)  $k \geq 3$ ,
- c)  $G$  contains the induced  $K_{1,3}$  with  $V(K_{1,3}) = \{x, y, u_1, v_1\}$ .

According to Claims 1–3 we can reconstruct  $G$  based on  $G_{sf}$ . By Claim 3, at least one vertex of  $K_{1,3}$ , say  $x$ , is a leaf in  $G$ . Hence, by Claim 1, a vertex  $u_i$  or  $v_i$ ,  $i = 2, \dots, k$ , can be adjacent to  $v_1, y$ , only. Observe that one vertex among  $u_i, v_i$ , for  $i = 2, \dots, k$  is a leaf. Indeed, if  $v_i y$  and  $u_i y$  ( $v_i v_1$  and  $u_i v_1$ ) are edges of  $G$  then  $H_8$  is a special subgraph of  $G$ , but if  $v_i y, u_i v_1 \in E(G)$  then  $H_{25}$  is a special subgraph of  $G$  (Fig. 21). From the above investigations we obtain the desired graph in Fig. 22. One can see that (1.1) holds for  $H_{32} = G_{16}$ . We emphasize that the numbers of edges  $yw_i$  or  $v_1 w_i, yp_j, v_1 z_m$  can be zero here.



Fig. 21. Impossible edges in  $G$

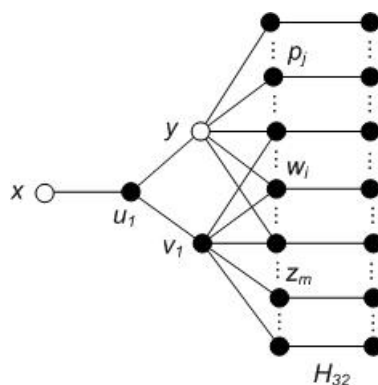


Fig. 22. The family  $G_{16}$

Note that the graphs  $H_{21}$ ,  $H_{22}$ ,  $H_{23}$  and  $H_{27}$  are particular instances of  $H_{32}$ . We next describe desired graphs  $G$  based on  $H_{32}$ . We now discard the assumption concerning the induced star  $K_{1,3}$  i.e. edges joining  $x$ ,  $y$ ,  $v_1$  are allowable. At first we add the edge  $yv_1$  to  $H_{32}$  and obtain graph  $H_{33} = G_{17}$  which satisfies (1.1) (Fig. 23). We now consider the following exhaustive cases (Fig. 24). It easy to see that (1.1) is true for  $H_{34} = G_{18}$  and  $H_{35} = G_{19}$  but is false for  $H_i$ ,  $i = 36, \dots, 39$ .

*Case 2.* Each  $\gamma_p(G)$ -set  $S$  satisfies  $\alpha(S) = 2$ .

*Case 2.1.* There exists a set  $S$  containing vertices  $u, v$  that dominate  $\{x, y\}$  such that  $u$  is paired with  $v$  in some perfect matching  $M$  of  $S$ . Without loss of generality we may assume that  $u = u_1$ ,  $v = v_1$ .

*Case 2.1.1.*  $k = 1$ . Then the unique graphs  $H_{40} = G_{20}$  and  $H_{41} = G_{21}$  satisfying (1.1) are depicted in Fig. 25.

Now for a connected graph  $G$  with  $k \geq 2$  the spanning forest  $G_{sf}(S, M, S_i) = G_{sf}$  for  $S_i = \{u, v\}$  is the sum of components  $G^{(1)}, G^{(2)}, \dots, G^{(k)}$ , where  $G^{(1)} = P_4$  and  $G^{(i)} = K_2$  for  $i = 2, \dots, k$  (Fig. 26).

*Case 2.1.2.*  $k = 2$ . Now we start with the spanning forest of Fig. 27. In our construction of the desired connected graphs we add at least one edge to the graph  $H_{42}$ . Therefore, consider the following cases.

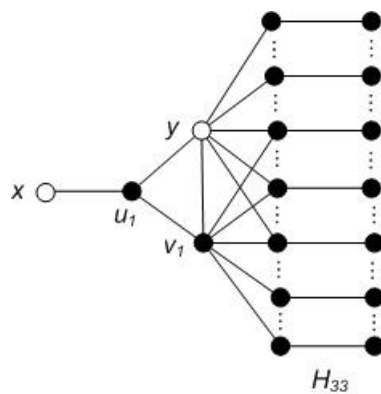


Fig. 23. The family  $G_{17}$

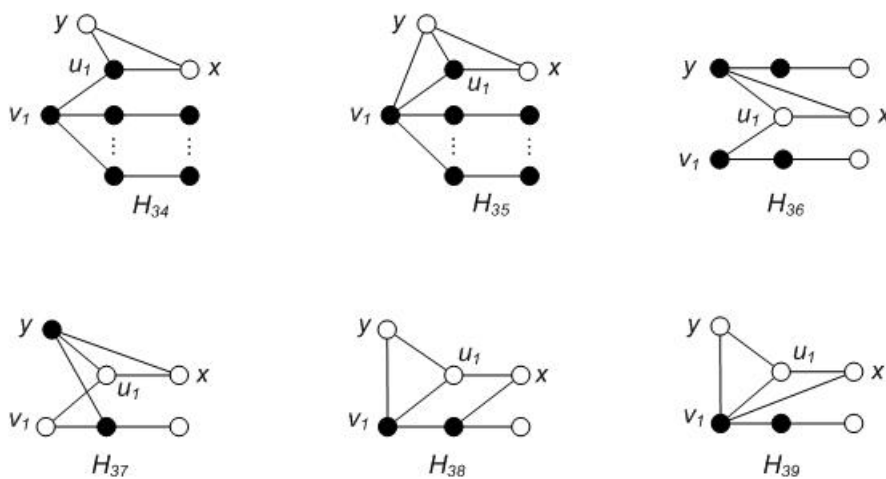


Fig. 24. The exhaustive cases



Fig. 25. The case for  $k = 1$

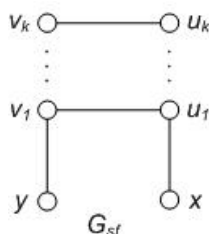


Fig. 26. The spanning forest of  $G$

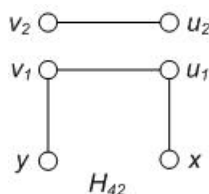


Fig. 27. The graph  $G_{sf}$  for  $k = 2$

Case 2.1.2.1. One edge (Fig. 28). Then we have  $H_{43} = G_{22}$  and  $H_{44} = G_{23}$  satisfy (1.1).



Fig. 28.  $H_{42} + e$

Case 2.1.2.2. Two edges. Now by adding one edge to  $H_{43}$  and  $H_{44}$  we obtain some graphs by exhaustion (Fig. 29). Observe (1.1) fails for  $H_{45}$ ,  $H_{46}$  and holds for  $H_{47} = G_{24}$ ,  $H_{48} = G_{25}$  and  $H_{49} = G_{26}$ . Moreover graphs  $H_i$  for  $i = 50, 51, 52$  are discussed in Case 1.

Case 2.1.2.3. Three edges. Then it suffices to add one edge to  $H_i$ ,  $i = 47, 48, 49$ . One resulting graph is the graph  $H_{53}$  depicted in Fig. 30, which does not satisfy (1.1). One can verify that the remaining graphs in this case are supergraphs of  $H_{45}$ ,  $H_{46}$  or are graphs discussed in Case 1.



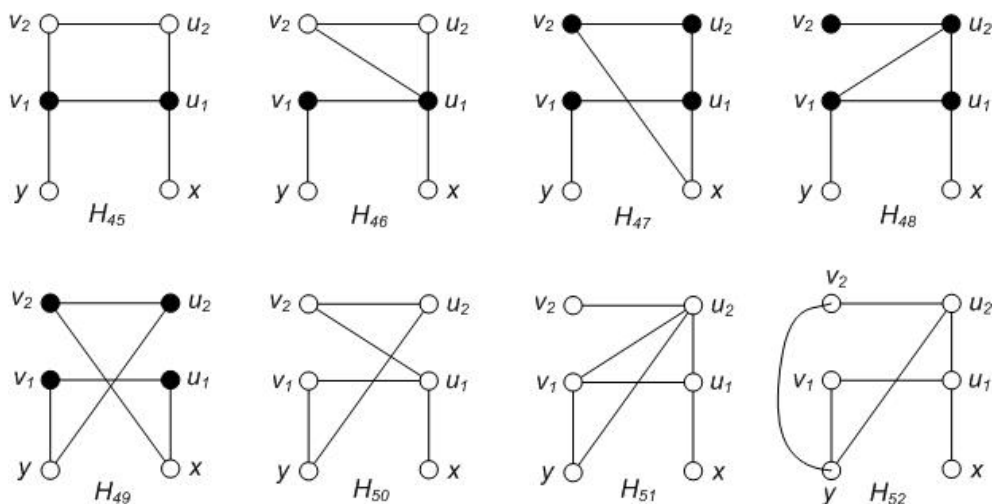


Fig. 29.  $H_{43} + e$  and  $H_{44} + e$

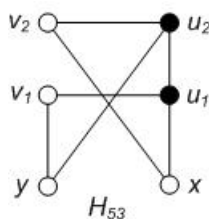


Fig. 30.  $H_i + e$  for  $i = 47, 48, 49$

Case 2.1.3.  $k \geq 3$ . At first we show some graphs for which (1.1) does not hold (Fig. 31). For  $H_i$ ,  $i = 54, \dots, 57$ , (1.1) is false; in  $H_{54}$  the vertex  $u_1$  is paired with  $u_2$  and  $v_1$  with  $u_3$ .

Now we start with the spanning forest depicted in Fig. 32.

Taking account of the forbidden special subgraphs  $H_i$ ,  $i = 54, \dots, 57$ , we can reconstruct  $G$  based on  $G_{sf}$ . By the connectedness of  $G$  it is necessary to join vertices of both the edges  $u_i v_i$ ,  $u_j v_j$  with at least one vertex among  $u_1, v_1, x, y$ . Thus we consider the following cases (without loss of generality we take the vertices  $u_i$  and  $u_j$  of the above edges). If  $u_i u_1 \in E(G)$  then we have two options:  $u_j u_1 \in E(G)$  or  $u_j x \in E(G)$ . Instead, if  $u_i x \in E(G)$  then we have the following options:  $u_j x \in E(G)$  or  $u_j u_1 \in E(G)$ . Replace  $u_1$  by  $v_1$  and  $x$  by  $y$  we obtain analogous results. This way we construct the desired graph  $G = H_{58}$  for which (1.1) holds (Fig. 33). Note that  $H_{58} = G_{27}$ . We end this case with adding new edges in  $H_{58}$ . At first, if  $u_i z \in E(G)$  and  $v_i z \in E(G)$ , where  $2 \leq i \leq k$ ,  $z = u_1, v_1, x, y$ , then we return to Case 1. Therefore,

let us consider all possible cases, which are depicted in Fig. 34. Then we obtain that (1.1) is true for  $H_{60} = G_{28}$  but is false for  $H_{59}$  and  $H_{61}$ .

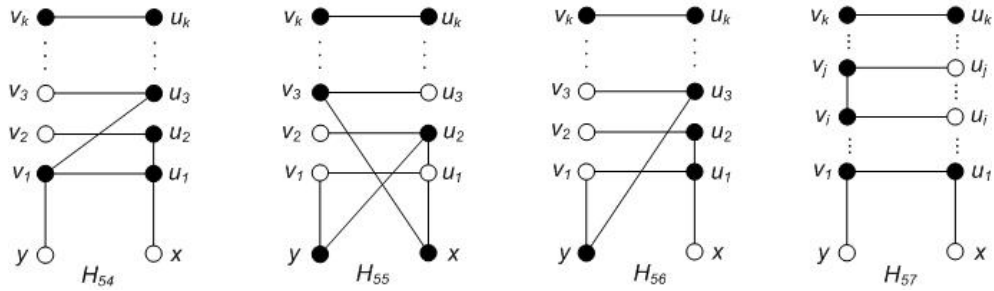


Fig. 31. The forbidden graphs

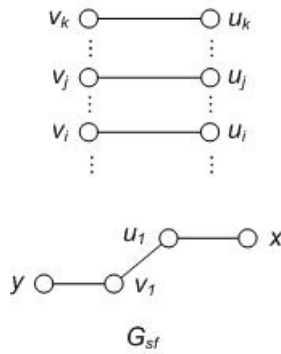


Fig. 32. The spanning forest for  $k \geq 3$ , where  $2 \leq i < j \leq k$

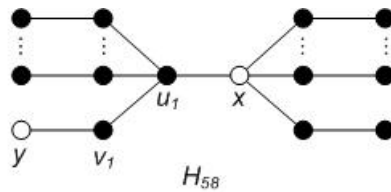


Fig. 33.  $H_{58} = G_{27}$

Case 2.2. For each  $S$  and for all vertices  $u, v \in S$  that dominate  $\{x, y\}$  the vertex  $u$  is not paired with  $v$  in any perfect matching of  $S$ . In this case the spanning forest  $G_{sf}(S, M, S_i) = G_{sf}$ , for each  $M$  and  $S_i = \{u, v\}$ , is depicted in Fig. 35.

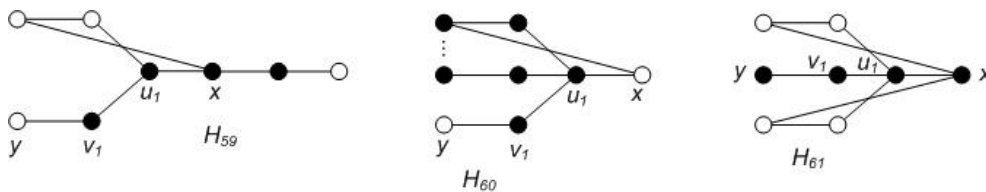


Fig. 34.  $H_{58} + e$

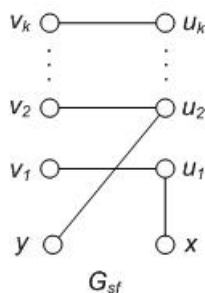


Fig. 35. The spanning forest  $G_{sf}$  of a connected graph  $G$

Now we search for connected graphs based on  $G_{sf}$  and consider the following cases.  
 Case 2.2.1.  $k = 2$ . Then by adding one edge we obtain the three options of Fig. 36:  $H_{62}$  does not satisfy (1.1) while  $H_{63} = G_5$  and  $H_{64} = G_{23}$ .

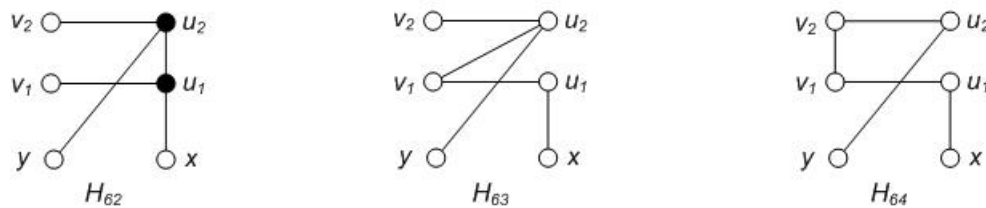


Fig. 36. The case  $k = 2$

Case 2.2.2.  $k = 3$ . Now consider the spanning forest depicted in Fig. 37. By joining the vertices  $u_1, v_1, x$  to  $u_2, v_2, y$  we could obtain  $H_i, i = 62, 63, 64$ , or their supergraphs. Hence the obtained graphs do not satisfy (1.1) or belong to Case 1 or Case 2.1. Therefore, it suffices to consider edges joining the above vertices to  $u_3$  or  $v_3$  (Fig. 38). Then  $H_i, i = 65, \dots, 69$ , do not satisfy (1.1) but  $H_{70}$  belongs to the family  $G_{16}$ .

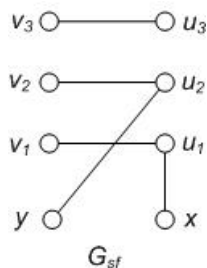


Fig. 37. The spanning forest for  $k = 3$

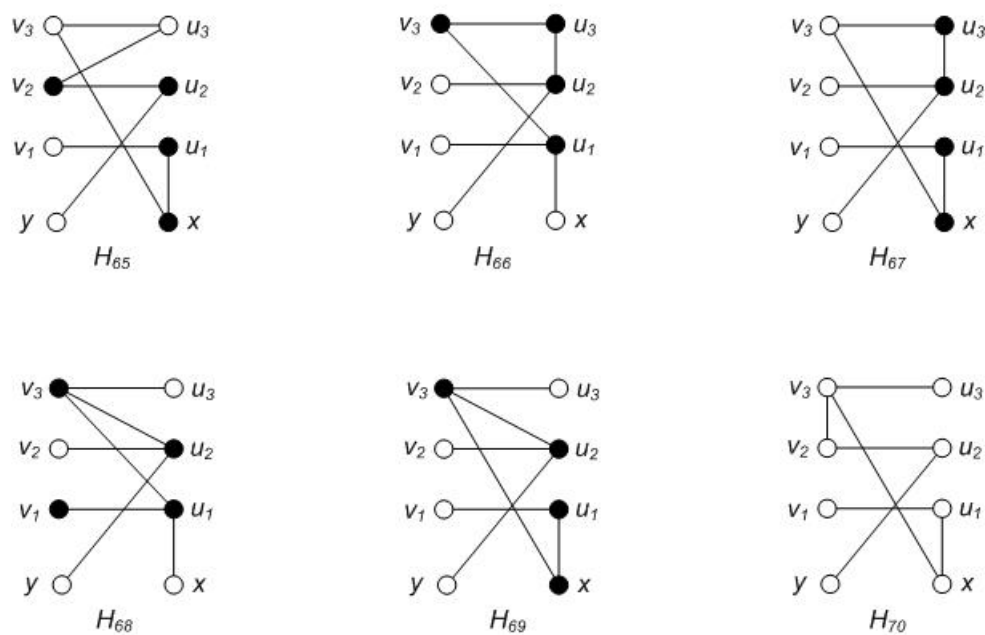


Fig. 38. The case  $k = 3$

Case 2.2.3.  $k > 3$ . Then we obtain graphs for which (1.1) fails or graphs belonging to Case 1.

Conversely, let  $G$  be any graph of the family  $\mathcal{G}$ . It follows from the former investigations that (1.1) holds for  $G$ .  $\square$

We end this paper with the following statement obtained by Theorems 1.3, 1.4, 2.1 and Corollary 1.5.

**Corollary 2.2.** *If  $G$  is a graph of order  $n \geq 4$ , then  $\gamma_p(G) = n - 2$  if and only if*

- 1) *exactly two of the components of  $G$  are isomorphic to graphs of the family  $\mathcal{F}$  given in Theorem 1.4 and every other component is  $K_2$  or*
- 2) *exactly one of the components of  $G$  is isomorphic to a graph of the family  $\mathcal{G}$  given in Theorem 2.1 and every other component is  $K_2$ .*

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