GENERALISED HERSCHEL MODEL APPLIED TO BLOOD FLOW MODELLING

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Abstract: This paper introduces a new rheological model of blood as a certain generalisation of the standard Herschel-Bulkley model (Herschel W H and Bulkley R 1926 *Kolloid-Zeitschrift* **39** (4) 291). This model is a rheological constitutive equation and belongs to the group of the so-called generalised Newtonian fluids. Experimental data (Yeleswarapu K K *et al.* 1998 *Mech. Res. Comm.* **25** (3) 257) is compared with the results obtained from the new model, to demonstrate that it allows obtaining the best agreement together with the Luo-Kuang model (Luo X and Kuang Z B 1992 *J. Biomechanics* **25** (8) 929; Easthope P L and Brooks D E 1980 *Biorheology* **17** 235). The new model may be easily implemented into commercial CFD codes, which is not that obvious for more complicated models such as differential, integral and rate type fluids (Astarita G and Marrucci G 1974 *Principles of non-Newtonian Fluid Mechanics* London, McGraw-Hill; Tesch K 2012 *Selected Topics of Blood Flows and Microclimate Modelling in Protective Suits* Gdansk, Gdansk University of Technology Press). What is more, it allows modelling of such phenomena as shear thinning, yield stress and constant viscosity values at high shear rates.

Keywords: non-Newtonian fluids, rheology

Notation

$\boldsymbol{A}_i$ – Rivlin-Ericksen tensor
$\boldsymbol{C}_t$ – left Cauchy-Green tensor
$\boldsymbol{D}$ – strain rate tensor
f – function
k – flow consistency
n – flow index
p – pressure
r – radius
γ – shear rate
$\boldsymbol{\delta}$ – Kronecker delta
λ_1 – relaxation time
λ_2 – retardation time

μ – viscosity
μ_∞ – viscosity at high shear rates
$\boldsymbol{\sigma}$ – stress tensor
$\boldsymbol{\tau}$ – viscous part of the stress tensor
τ – shear stress
τ_0 – yield stress
$\nabla\vec{U}$ – velocity gradient
$\frac{\partial\vec{U}}{\partial\vec{r}}$ – strain rate tensor

1. Introduction

Blood may be treated as a non-Newtonian fluid. This means that another constitutive equation is needed to close the system of equations describing the blood motion. The non-Newtonian nature of blood arises due to the presence of red blood cells in the plasma. The mentioned equation belongs to the class of the so-called mechanical (rheological) constitutive equations.

An ideal rheological constitutive equation for proper and complete modelling of the blood flow behaviour should take into consideration the flexibility and aggregation of red blood cells, the influence of temperature on viscosity, the yield stress and thixotropy. It is hardly possible to satisfy all of these conditions. The more attributes are satisfied, the better the model is. One has to keep in mind that such a model is also more complicated then.

2. Models

Typically, we can divide rheological constitutive equations into categories of Newtonian-, generalised Newtonian-, differential-, integral- and rate type fluids. For any case the stress tensor $\boldsymbol{\sigma}$ is decomposed into a reversible and irreversible (viscous) part, $\boldsymbol{\tau}$. If the density is constant we have:

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau} \tag{1}$$

For Newtonian fluids we have a linear relationship between the viscous part of the stress tensor $\boldsymbol{\tau}$ and the strain rate tensor $\boldsymbol{D}$:

$$\boldsymbol{\tau} = 2\mu\boldsymbol{D} \tag{2}$$

where the dynamic viscosity μ is a factor of proportionality. The definition (2) is a generalisation of the following one-dimensional expression taken from an experiment:

$$\tau = \mu\gamma \tag{3}$$

All the fluids that do not fulfil Equations (2) or (3) are known to be non-Newtonian.

2.1. Newtonian fluids

For Newtonian fluids the relationship between shear stress and shear rate is linear which also means that viscosity is constant:

$$\tau = \mu\gamma \tag{4a}$$

$$\mu = \text{const.} \tag{4b}$$

The above constitutive equation will not allow making a correct description of the blood behaviour such as shear thinning and yield stress and many others. The only advantage of this model is that it keeps viscosity constant at high shear rates.

2.2. Generalised Newtonian fluids

Generalised Newtonian fluids satisfy the following rheological equation:

$$\tau = \mu(\gamma)\gamma \tag{5}$$

where the viscosity depends on the shear rate γ. Despite the name these fluids are non-Newtonian. The Newtonian rheological Equation (4) may be always obtained as a certain simplification of the selected generalised Newtonian fluid. The most popular generalised Newtonian fluids, easily applied to blood flow modelling, are the Bingham, Ostwald-de Waele, Herschel-Bulkley, Casson and Luo-Kuang models.

The Bingham model [1] expresses the shear rate and dynamic viscosity in the following manner:

$$\tau = \tau_0 + k\gamma \tag{6a}$$

$$\mu = \frac{\tau_0}{|\gamma|} + k \tag{6b}$$

We have an additional term responsible for the yield stress τ_0 in comparison with the Newtonian model (4). If $\tau < \tau_0$ the Bingham fluid behaves as a solid, otherwise it behaves as a fluid. Except for the yield stress modelling ability it is not the best model for a blood flow description. This is simply because it cannot mimic the shear thinning. If $\tau_0 = 0$, we have the Newtonian fluid (4).

The Ostwald-de Waele [2, 3] or so-called power-law model is given by:

$$\tau = k\gamma^n \tag{7a}$$

$$\mu = k|\gamma|^{n-1} \tag{7b}$$

where k is a flow consistency index and n is a flow index. It is probably the simplest model allowing the shear thinning phenomenon. This is because of the dimensionless flow index n present in Equation (7). However, it suffers from the lack of yield stress. A disadvantage of this model is the problem of correct prediction of viscosity at low and high stresses. For $n = 1$ we obtain the Newtonian fluid (4).

The Herschel-Bulkley model [4] combines the two previous models, *i.e.* the Bingham and Herschel-Bulkley models. This results in:

$$\tau = \tau_0 + k\gamma^n \tag{8a}$$

$$\mu = \frac{\tau_0}{|\gamma|} + k|\gamma|^{n-1} \tag{8b}$$

This also means that it is now possible to model the shear thinning behaviour and the yield stress. One has to keep in mind that the Herschel-Bulkley model inherits the disadvantages of the Ostwald-de Waele model, *i.e.* it cannot correctly predict the blood behaviour at high and low shear stresses. For $\tau_0 = 0$ and $n = 1$ we can obtain the Newtonian fluid (4). It is also possible to obtain both the Ostwald-de Waele and Bingham models.

The Casson model [5] follows the definitions:

$$\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{k\gamma} \tag{9a}$$

$$\sqrt{\mu} = \sqrt{\frac{\tau_0}{|\gamma|}} + \sqrt{k} \tag{9b}$$

It cannot be derived from the three above models. The advantages, however, are exactly the same as previously. It allows shear thinning and yield stress modelling. What is even more important this model is able to keep constant viscosity at high shear rates and is commonly used for blood flow modelling. For $\tau_0 = 0$ we have the Newtonian fluid (4). There is also the generalised Casson model. The only difference is that we do not have the power $\frac{1}{2}$ ($\sqrt{\ }$) but an optional power m instead. This makes it even more flexible in the sense of the shear thinning level calibration.

The Luo-Kuang model [6] has been the best discussed blood model so far. It is defined by means of the following equations:

$$\tau = \tau_0 + k\sqrt{\gamma} + \mu_\infty \gamma \tag{10a}$$

$$\mu = \frac{\tau_0}{|\gamma|} + \frac{k}{\sqrt{|\gamma|}} + \mu_\infty \tag{10b}$$

where μ_∞ stands for constant viscosity at high shear rates. It cannot be derived from any of the previously discussed models. It allows modelling of the shear thinning behaviour and the yield stress as well as the correct prediction of viscosity at high stresses. It allows modelling of the best blood behaviour [7] at least within the frame of generalised Newtonian fluids. For $\tau_0 = 0$ and $k = 0$ we have the Newtonian fluid (4).

The generalised Herschel model, introduced here, is given by the following definition:

$$\tau = \tau_0 + k\gamma^n + \mu_\infty \gamma \tag{11a}$$

$$\mu = \frac{\tau_0}{|\gamma|} + k|\gamma|^{n-1} + \mu_\infty \tag{11b}$$

A new rheological parameter, n, is introduced here. It allows better flexibility in comparison with the Luo-Kuang model (10). There are three components of viscosity in Equations (10) and (11) that may be easily distinguished. The first component, $\tau_0|\gamma|^{-1}$, is responsible for the yield stress. The second component, $k|\gamma|^{n-1}$, is responsible for the non-Newtonian behaviour and the last component, μ_∞, allows constant viscosity at high shear rates. The combination of these components makes these two models the best among the models discussed in this paper. Similarly as for the Luo-Kuang model, for $\tau_0 = 0$ and $k = 0$ we have the Newtonian fluid (4). Obviously for $n = \frac{1}{2}$ we obtain the Luo-Kuang model (10).

Figure 1 shows relationships between all the generalised Newtonian models discussed in this paper except for the Szulman model. This model is a certain

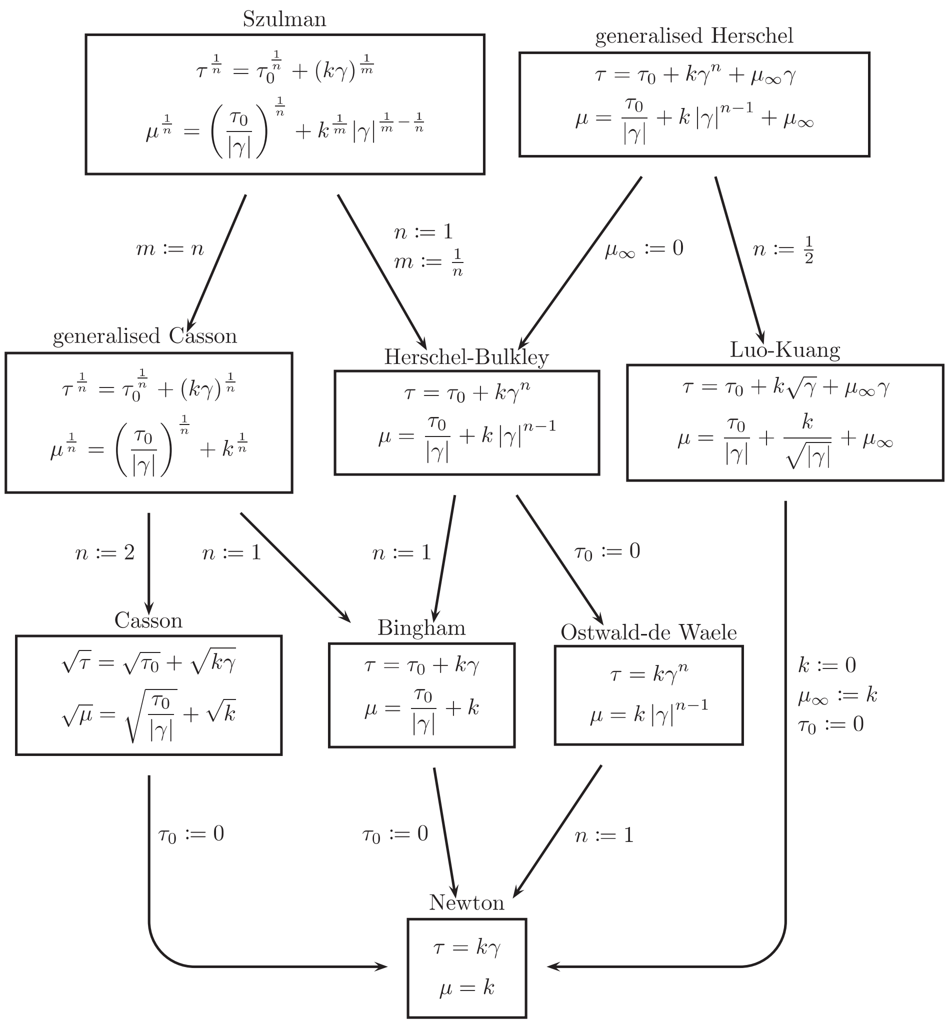

Figure 1. Various models dependency

generalisation of the generalised Casson and Herschel-Bulkley models. There is no direct connection between the Szulman and Generalised Herschel models. The level of complexity of these two is more or less the same. Theoretically, the Szulman model allows for more than the Casson model. However, one has to be careful when fitting the values of the constants as it is quite easy to end up with values that do not allow constant viscosity at high shear rates.

2.3. Differential type fluids

The viscous part of the stress tensor is expressed explicitly as a function of other tensors and their derivatives, both being of a kinematic nature. These may be Rivlin-Ericksen [8] tensors $\boldsymbol{A}_i$:

$$\boldsymbol{\tau} = f(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots) \tag{12}$$

defined by means of the following recurrent equation:

$$\boldsymbol{A}_{i+1} = \frac{\mathrm{d}\boldsymbol{A}_i}{\mathrm{d}t} + \boldsymbol{A}_i \cdot \frac{\partial \vec{U}}{\partial \vec{r}} + \nabla \vec{U} \cdot \boldsymbol{A}_i, \quad i = 1,2,\ldots \tag{13}$$

2.4. Integral type fluids

The viscous part of the stress tensor is expressed explicitly as a function of one or more integrals of other tensors of a kinematic nature:

$$\boldsymbol{\tau} = \int_{-\infty}^{t} f(t-\tau)\left(\boldsymbol{\delta} - \boldsymbol{C}_t(\tau)\right) \mathrm{d}\tau \tag{14}$$

2.5. Rate type fluids

Equations describing the rate type fluids are not explicit for the stress tensors. This simply means that the constitutive equation involves both the viscous part of the stress tensor and its derivatives:

$$\dot{\boldsymbol{\tau}} = f\left(\boldsymbol{\tau}, \boldsymbol{D}, \dot{\boldsymbol{D}}\right) \tag{15}$$

The most popular variants of rate type fluids are Maxwell and Oldroyd equations. The former follows the definition:

$$\boldsymbol{\tau} + \lambda_1 \dot{\boldsymbol{\tau}} = 2\mu \boldsymbol{D} \tag{16}$$

It generalises the Newtonian hypothesis by means of an additional term containing the time derivative of the stress tensor. This term is responsible for the so-called fluid memory. λ_1 stands for the relaxation time. The latter adds another term containing the time derivative of the strain rate tensor:

$$\boldsymbol{\tau} + \lambda_1 \dot{\boldsymbol{\tau}} = 2\mu\left(\boldsymbol{D} + \lambda_2 \dot{\boldsymbol{D}}\right) \tag{17}$$

and λ_2 is the retardation time. It can be considered as the time needed to strain relaxation when the stress is removed.

3. Comparison with experimental data

This paragraph shows a comparison between the experimental data and the predictions of the generalised Newtonian models discussed earlier. More complex models (integral, differential and rate type fluids) are not discussed here. This is simply because they cannot be easily implemented into commercial CFD codes.

The experimental velocity profiles [9] were measured by means of Doppler velocimetry inside a straight plexiglass tube (0.25 inch in diameter and 6 feet long). The investigated fluid was composed of porcine blood and 10% sodium citrate. The viscosity of the fluid was measured by means of Couette and capillary viscometers. All experiments were performed at room temperature within six hours of the blood collection.

3.1. Viscosity

Figure 2 shows a comparison of various models *vs.* experimental data for blood [9]. The values of constants such as τ_0, k and n were obtained by means of the least squares method. It is obvious that the Newtonian fluid model is the worst. The best agreement is achieved for the Luo-Kuang and generalised Herschel models. Models such as Ostwald-de Waele and Hereschel-Bulkley are not able to predict viscosity at high shear rates whereas they fit perfectly for middle values of shear rates. Surprisingly enough, the Casson model does not fit the experimental data well except for constant values (overestimated) at high shear rates.

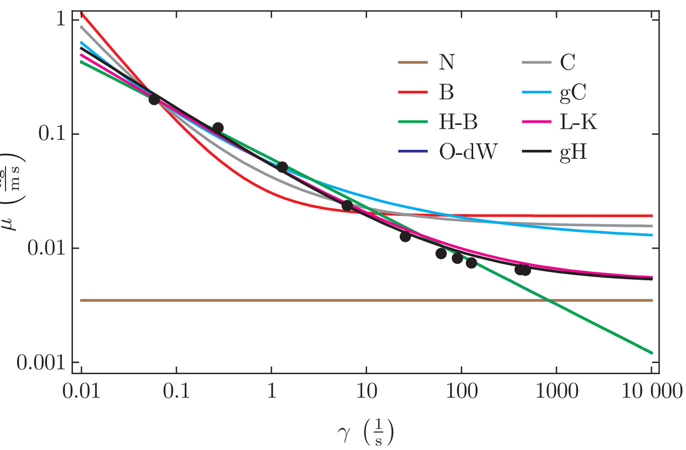

Figure 2. Various models *vs.* experimental data: N – Newtonian model, B – Bingham, H-B – Herschel-Bulkley, O-dW – Ostwald-de Waele, C – Casson, gC – generalised Casson, L-K – Luo Kuang, gH – generalised Herschel

3.2. Velocity

Figures 3–9 show a comparison of various models and the experimental data for three different volumetric flow rates. The velocity profiles may be solved either

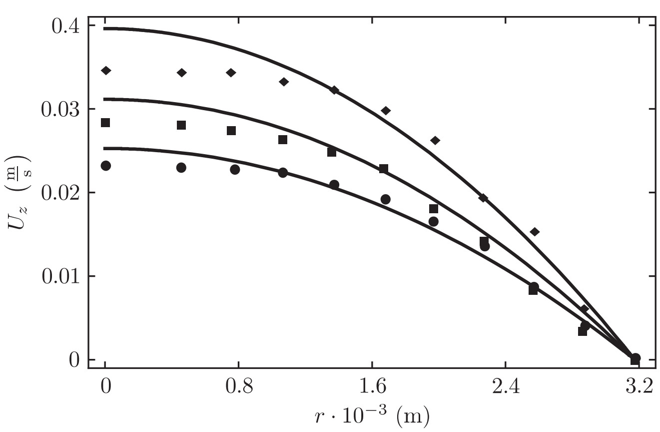

Figure 3. Newton model

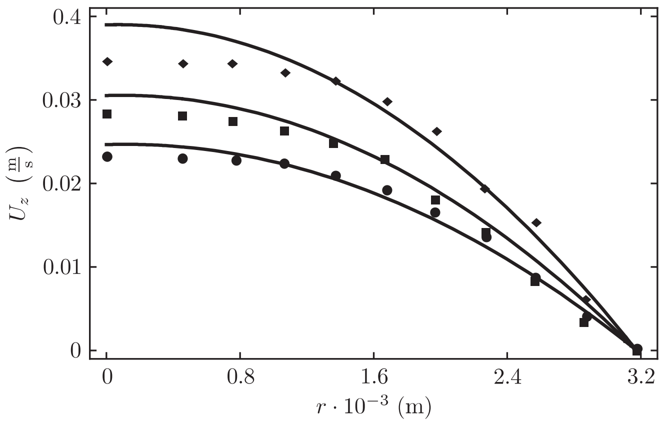

Figure 4. Bingham model

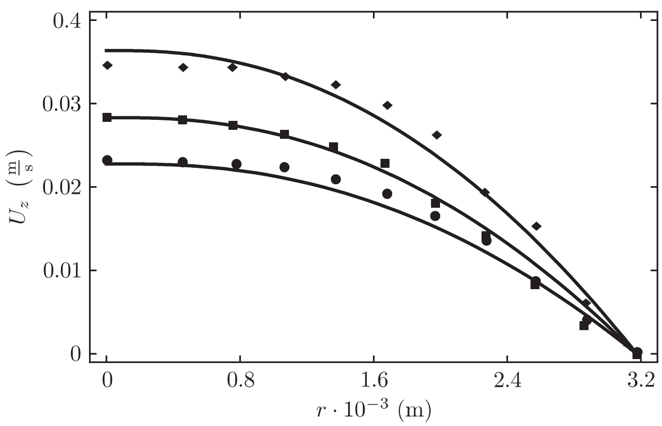

Figure 5. Casson model

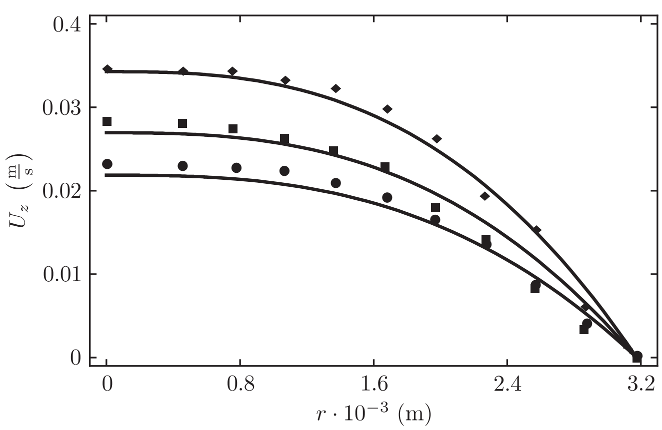

Figure 6. Ostwald-de Waele model

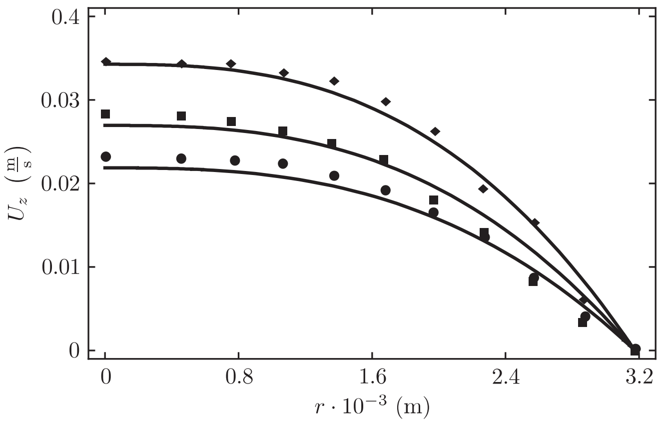

Figure 7. Herschel-Bulkley model

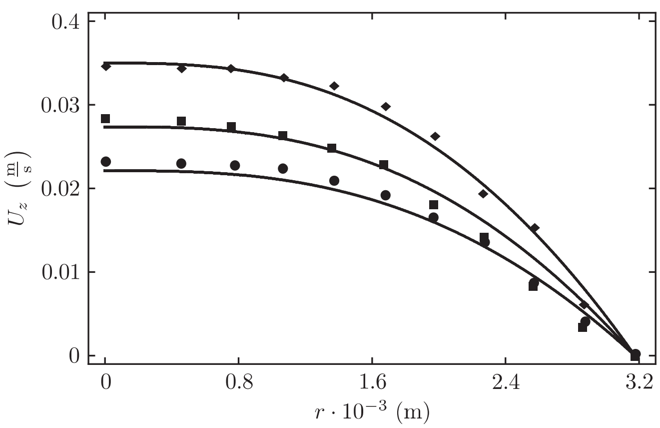

Figure 8. Generalised Herschel model

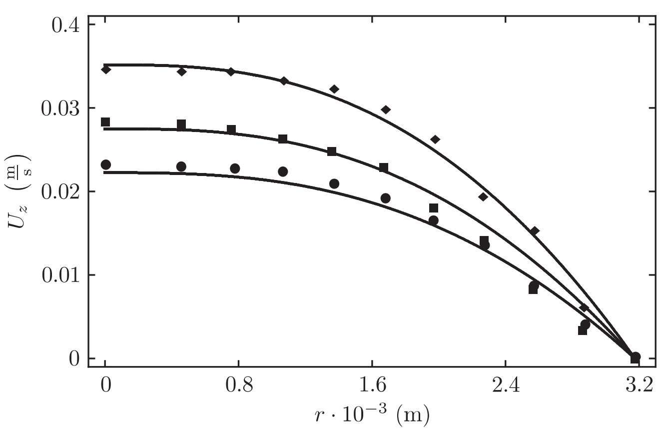

Figure 9. Luo-Kuang model

numerically or analytically [10]. Again, it is not surprising that the Newtonian and Bingham models do not predict the velocity profiles well. As for the other models, the prediction is quite accurate. The best agreement was obtained for the Luo-Kuang and generalised Hereschel models.

4. Conclusions

The generalised Newtonian fluids are the simplest and easiest to implement into commercial CFD codes. The Luo-Kuang and generalised Herschel models allow the best experimental fitting of data. This has been shown in this paper. More advanced models such as those of Oldroyd and Maxwell have a potential for even better approximation of blood features but they cannot be directly implemented into commercial CFD codes. Generally speaking, there is no universal constitutive equation that would be able to model all the features of blood. This means that there is still need for further research in this field.

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