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Combinatorics

Bounds on the vertex–edge domination number of a tree



Bornes sur le nombre de domination sommet–arête d'un arbre

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ABSTRACT

A vertex–edge dominating set of a graph G is a set D of vertices of G such that every edge of G is incident with a vertex of D or a vertex adjacent to a vertex of D . The vertex–edge domination number of a graph G , denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex–edge dominating set of G . We prove that for every tree T of order $n \geq 3$ with l leaves and s support vertices, we have $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$, and we characterize the trees attaining each of the bounds.

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RÉSUMÉ

Un ensemble sommet–arête dominant d'un graphe G est un ensemble D de sommets de G tel que chaque arête de G soit incidente à un sommet de D ou à un sommet adjacent à un sommet de D . Le nombre de domination sommet–arête d'un graphe G , noté $\gamma_{ve}(G)$, est le cardinal minimum d'un ensemble sommet–arête dominant de G . Nous prouvons que, pour chaque arbre T d'ordre $n \geq 3$ avec l feuilles et des sommets s de soutien, que nous avons $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$, et nous caractérisons les arbres atteignant chacune des limites.

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1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We denote by P_n the path on n vertices. Let T be a tree, and let v be a vertex of T . We say that v is adjacent to a path P_n if there is a neighbor of v , say x , such that the subtree

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resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n . By a star we mean a connected graph in which exactly one vertex has degree greater than one.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D . The domination number of G is the minimum cardinality of a dominating set of G . For a comprehensive survey of domination in graphs, see [2,3].

An edge $e \in E(G)$ is vertex-edge dominated by a vertex $v \in V(G)$ if e is incident to v , or e is adjacent to an edge incident to v . A subset $D \subseteq V(G)$ is a vertex-edge dominating set, abbreviated VEDS, of G if every edge of G is vertex-edge dominated by a vertex of D . The vertex-edge domination number of G , denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex-edge dominating set of G . A vertex-edge dominating set of G of minimum cardinality is called a $\gamma_{ve}(G)$ -set. Vertex-edge domination in graphs was introduced in [7], and further studied in [6].

Chellali and Haynes [1] established the following lower bound on the total domination number of a tree. For every tree T of order n with l leaves, we have $\gamma_t(T) \geq (n - l + 2)/2$. They also characterized the extremal trees. In [4] a lower bound on the total outer-independent domination number of a tree was given together with the characterization of the extremal trees. Lemańska [5] proved that the domination number of a tree is bounded below by $(n - l + 2)/3$.

We prove the following bounds on the vertex-edge domination number of a tree T of order $n \geq 3$ with l leaves and s support vertices, $(n - l - s + 3)/4 \leq \gamma_{ve}(T) \leq n/3$. We also characterize the trees attaining each of the bounds.

2. Results

We begin with the following straightforward observation.

Observation 1. For every connected graph G of diameter at least two, there exists a $\gamma_{ve}(G)$ -set that contains no leaf.

First we show that if T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{ve}(T)$ is bounded below by $(n - l - s + 3)/4$. For the purpose of characterizing the trees attaining this bound, we introduce a family \mathcal{T} of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_5 . If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation \mathcal{O}_1 : Attach a vertex by joining it to any support vertex of T_k .
- Operation \mathcal{O}_2 : Attach a path P_2 by joining one of its vertices to a vertex of T_k , which is not a leaf and is adjacent to a support vertex of degree two.
- Operation \mathcal{O}_3 : Attach a path P_4 by joining one of its leaves to a leaf of T_k adjacent to a weak support vertex.

We now prove that for every tree T of the family \mathcal{T} we have $\gamma_{ve}(T) = (n - l - s + 3)/4$.

Lemma 2. If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (n - l - s + 3)/4$.

Proof. We use the induction on the number k of operations performed to construct the tree T . If $T = T_1 = P_5$, then $(n - l - s + 3)/4 = (5 - 2 - 2 + 3)/4 = 1 = \gamma_{ve}(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{T} constructed by $k - 1$ operations. Let n' be the order of the tree T' , l' the number of its leaves, and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family \mathcal{T} constructed by k operations.

First assume that T is obtained from T' by operation \mathcal{O}_1 . We have $n = n' + 1$, $l = l' + 1$ and $s = s'$. It is straightforward to see that any $\gamma_{ve}(T')$ -set is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Obviously, $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') = (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$.

Now assume that T is obtained from T' by operation \mathcal{O}_2 . We have $n = n' + 2$, $l = l' + 1$ and $s = s' + 1$. It is straightforward to see that any $\gamma_{ve}(T')$ -set is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') = (n' - l' - s' + 3)/4 = (n - 2 - l + 1 - s + 1 + 3)/4 = (n - l - s + 3)/4$.

Now assume that T is obtained from T' by operation \mathcal{O}_3 . We have $n = n' + 4$, $l = l'$ and $s = s'$. We denote by x the leaf to which P_4 is attached. Let $v_1 v_2 v_3 v_4$ be the attached path. Let v_1 be joined to x . Let D' be any $\gamma_{ve}(T')$ -set. It is easy to see that $D' \cup \{v_2\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Now let us observe that there exists a $\gamma_{ve}(T)$ -set that does not contain the vertices v_4 , v_3 , and v_1 . Let D be such a set. To dominate the edge $v_3 v_4$, we have $v_2 \in D$. Observe that $D \setminus \{v_2\}$ is a VEDS of the tree T' . Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now conclude that $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. We get $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 7)/4 = (n - l - s + 3)/4$. \square

We now give a lower bound on the vertex-edge domination number of a tree together with the characterization of the extremal trees.

Theorem 3. If T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{ve}(T) \geq (n - l - s + 3)/4$ with equality if and only if $T \in \mathcal{T}$.

Proof. If $\text{diam}(T) = 1$, then $T = P_2$. We have $(n - l - s + 3)/4 = (2 - 2 - 2 + 3)/4 < 1 = \gamma_{ve}(T)$. If $\text{diam}(T) = 2$, then T is a star. We have $l = n - 1$ and $s = 1$. Consequently, $(n - l - s + 3)/4 = (n - n + 1 - 1 + 3)/4 = 3/4 < 1 = \gamma_{ve}(T)$.

Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n . Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

First assume that some support vertex of T , say x , is strong. Let y be a leaf adjacent to x . Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s$. Obviously, $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 3)/4 = (n - l - s + 3)/4$. If $\gamma_{ve}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{ve}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u . If $\text{diam}(T) \geq 5$, then let d be the parent of w . If $\text{diam}(T) \geq 6$, then let e be the parent of d . By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T .

Assume that some child of u , say x , is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$.

Now assume that among the children of u there is a support vertex other than v . Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 2 - l + 1 - s + 1 + 3)/4 = (n - l - s + 3)/4$. If $\gamma_{ve}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{ve}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_2 . Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geq 3$. First assume that some child of w , say x , is a leaf. Let $T' = T - x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') \geq (n' - l' - s' + 3)/4 = (n - 1 - l + 1 - s + 1 + 3)/4 > (n - l - s + 3)/4$.

Now assume that no child of w is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_{ve}(T)$ -set that does not contain the vertices t and v . Let D be such a set. To dominate the edge vt , we have $u \in D$. Let us observe that $D \setminus \{u\}$ is a VEDS of the tree T' . Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 3 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$.

If $d_T(w) = 1$, then $T = P_4$. We have $(n - l - s + 3)/4 = (4 - 2 - 2 + 3)/4 < 1 = \gamma_{ve}(T)$. Now assume that $d_T(w) = 2$. First assume that $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let us observe that there exists a $\gamma_{ve}(T)$ -set that does not contain the vertices t , v and w . Let D be such a set. To dominate the edge vt , we have $u \in D$. Observe that $D \setminus \{u\}$ is a VEDS of the tree T' . Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 4 - l + 1 - s + 1 + 7)/4 > (n - l - s + 3)/4$.

Now assume that $d_T(d) = 2$. First assume that some child of e is a leaf. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l$ and $s' = s - 1$. Similarly as in the previous possibility, we conclude that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 1 + 7)/4 > (n - l - s + 3)/4$.

Now assume that no child of e is a leaf. Let $T' = T - T_w$. We have $n' = n - 4$, $l' = l$ and $s' = s$. If $n' = 1$, then $T = P_5 = T_1 \in \mathcal{T}$. Assume that $n' \geq 2$. Similarly as earlier, we conclude that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1 \geq (n' - l' - s' + 3)/4 + 1 = (n - 4 - l - s + 7)/4 = (n - l - s + 3)/4$. If $\gamma_{ve}(T) = (n - l - s + 3)/4$, then obviously $\gamma_{ve}(T') = (n' - l' - s' + 3)/4$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . \square

Next we show that if T is a tree of order $n \geq 3$, then $\gamma_{ve}(T)$ is bounded above by $n/3$. For the purpose of characterizing the trees attaining this bound, we introduce a family \mathcal{F} of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_3 . If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by attaching a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_2 or P_3 .

We now prove that for every tree T of the family \mathcal{F} we have $\gamma_{ve}(T) = n/3$.

Lemma 4. *If $T \in \mathcal{F}$, then $\gamma_{ve}(T) = n/3$.*

Proof. We use the induction on the number k of operations performed to construct the tree T . If $T = T_1 = P_3$, then $\gamma_{ve}(T) = 1 = n/3$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{F} constructed by $k - 1$ operations. Let n' be the order of the tree T' . Let $T = T_{k+1}$ be a tree of the family \mathcal{F} constructed by k operations. We have $n = n' + 3$. We denote by x the vertex to which is attached P_3 . Let $v_1 v_2 v_3$ be the attached path. Let v_1 be adjacent to x . Let D' be any $\gamma_{ve}(T')$ -set. It is easy to see that $D' \cup \{v_1\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. If x is adjacent to a path P_2 , then let us observe that there exists a $\gamma_{ve}(T)$ -set that contains the vertices v_1 and x . Let D be such a set. The set D is minimal, thus $v_2, v_3 \notin D$. It is easy to observe that $D \setminus \{v_1\}$ is a VEDS of the tree T' . If x is adjacent to a path P_3 different from $v_1 v_2 v_3$, say abc , then let a and x be adjacent. Let us observe that there exists a $\gamma_{ve}(T)$ -set that contains the vertices v_1 and a . Let D be such a set. The set D is minimal, thus $v_2, v_3 \notin D$. Let us observe that $D \setminus \{v_1\}$ is a VEDS of the tree T' . We now conclude that $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$, and consequently, $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. We get $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = n'/3 + 1 = (n - 3)/3 + 1 = n/3$. \square

We now give an upper bound on the vertex-edge domination number of a tree together with the characterization of the extremal trees.

Theorem 5. *If T is a tree of order $n \geq 3$, then $\gamma_{ve}(T) \leq n/3$ with equality if and only if $T \in \mathcal{F}$.*

Proof. First assume that $\text{diam}(T) = 2$. Thus T is a star. If $T = P_3$, then $T = T_1 \in \mathcal{F}$. If T is a star different from P_3 , then we get $n/3 > 1 = \gamma_{ve}(T)$.

Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n . Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T , say x , is strong. Let y be a leaf adjacent to x . Let $T' = T - y$. It is straightforward to see that any $\gamma_{ve}(T')$ -set is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u . By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T .

Assume that some child of u , say x , is a leaf. Let $T' = T - x$. Let D' be a $\gamma_{ve}(T')$ -set that contains no leaf. It is easy to observe that D' is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

Now assume that among the children of u there is a support vertex other than v . Let $T' = T - T_v$. Let us observe that there exists a $\gamma_{ve}(T')$ -set that contains the vertex u . Let D' be such a set. It is easy to see that D' is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

Now assume that $d_T(u) = 2$. First assume that w is adjacent to a leaf, say x . Let $T' = T - x$. Let us observe that there exists a $\gamma_{ve}(T')$ -set that contains the vertex u . Let D' be such a set. It is easy to see that D' is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq n'/3 < n/3$.

Now assume that there is a child of w other than u , say x , such that the distance of w to the most distant vertex of T_x is two or three. It suffices to consider only the possibilities when T_x is a path P_2 or P_3 . Let $T' = T - T_u$. We have $n' = n - 3$. Let D' be any $\gamma_{ve}(T')$ -set. It is easy to observe that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq n'/3 + 1 = n/3$. If $\gamma_{ve}(T) = n/3$, then obviously $\gamma_{ve}(T') = n'/3$. By the inductive hypothesis we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by attaching a path P_3 by joining one of its leaves to the vertex u . Thus $T \in \mathcal{F}$.

Now assume that $d_T(w) = 2$. Let $T' = T - T_w$. We have $n' = n - 4$. If $n' = 1$, then $T = P_5$. We have $\gamma_{ve}(P_5) = 1 < 5/3$. If $n' = 2$, then $T = P_6$. The path P_6 can be obtained from two paths P_3 by joining them through leaves. Thus $T \in \mathcal{F}$. Now assume that $n' \geq 3$. Let D' be any $\gamma_{ve}(T')$ -set. Let us observe that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq n'/3 + 1 = (n - 1)/3 < n/3$. \square

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