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DERIVATION OF THE SCALING LAWS USED IN GEOTECHNICAL CENTRIFUGE MODELLING-APPLICATION OF DIMENSIONAL ANALYSIS AND BUCKINGHAM Π THEOREM

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Abstract

Geotechnical centrifuge modelling has been a world-wide used technology in physical tests. In this papers a derivation of scaling laws by dimensional analysis for the centrifugal modelling is presented. Basic principles of centrifuge modelling are described. Scaling laws for slow events like consolidation and fast events like dynamic loads are shown. The differences in scale factors for both processes are noticed. The aim of this paper is to introduce geotechnical centrifuge technology to a wider Polish audience.

Introduction

Geotechnical centrifuge modelling is a technique of testing *1/n* scaled models subjected to gravitational field increased by a factor n. Similitude laws are a group of rules which links behaviour of a model to the prototype in the field. A set of scaling laws for static tests, dynamic tests, water flow and consolidation can be derived by dimensional analysis using assumption that stress level in prototype and centrifuge is the same (SCHOFIELD 1980, TAYLOR 1995, JOSEPH et al. 1988). They can be also derived by dimension analysis of equilibrium equation in continuum mechanics. However, only the results of this process were presented (CORTÉ 1989, FUGLSANG, OVESEN 1988). In this paper a full derivation using this method was shown. Consequently, the phenomena of centrifuge modelling based on scaling laws enables to maintain

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Fig. 1. Geotechnical beam centrifuge in Davis, University of California
Source: KUTTER (1998).

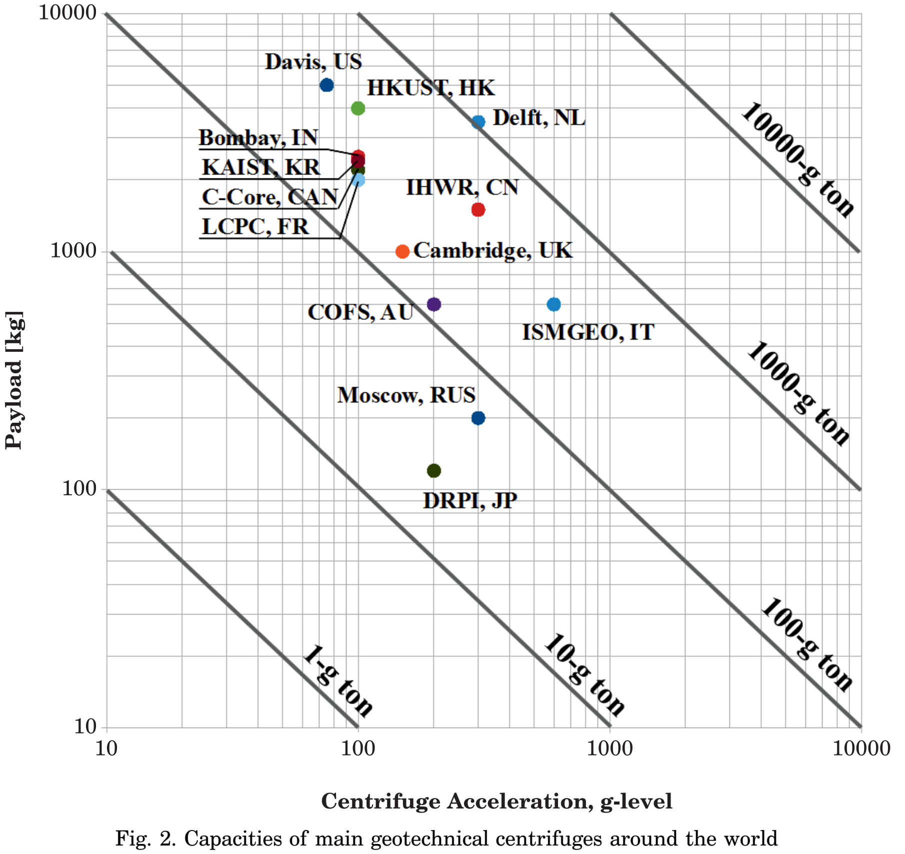

Fig. 2. Capacities of main geotechnical centrifuges around the world
Source: modified after NG et al. (2001) and KIM et al. (2013)

the same stress and strain levels in a model and the prototype. This is the main advantage of centrifuge tests. Catalogue of scaling laws and similitude questions in geotechnical centrifuge modelling was presented by TC2 committee of ISSMGE (GARNIER et al. 2007).

In figure 1 a view of typical centrifuge is shown. There are two types of geotechnical centrifuges. The first one is a drum centrifuge. Drum centrifuges enable to maintain high acceleration levels (up to 500 g), but require small models. Also, higher distortions in modelling occur for drum centrifuges with small radius than in a beam centrifuge. The second group is beam centrifuges. The radius of beam can reach up to 9 meters. They have lower acceleration levels, but they can carry a huge payload. For example, the Hong Kong centrifuge can carry 4.0 tons payload and its acceleration is up to 150 g (NG et al. 2001). Beam centrifuges enable to conduct more complicated tests like earthquake modelling and in-flight tests (MADABHUSHI, SCHOFIELD 1993). The capacities of centrifuges around the world are show in figure 2.

Scaling laws for dynamic events

A geotechnical model conducted under *ng* gravitational field, where *g* is acceleration due to earth gravity, and a prototype under one gravity have to be linked by some laws. Some requirements have to be fulfilled to achieve a strict similitude. These are geometric similarity, kinematic similarity and dynamic similarity (LANGHAAR 1951). Geometric similarity refers to the model and prototype with homologous physical dimensions. Kinematic similarity defines a model and prototype with homologous particle flow. Dynamic similarity means that net forces acting on model and prototype are homologous. It is often impossible to fulfil all these criteria during a model tests. Then partial similitude occurs and scale effects must be taken into account.

The relationship between model and prototype for centrifuge tests is generally derived through dimensional analysis. The dimensional analysis involves application of the Buckingham Π theorem. The Statement is: If there are n variables in a problem and these variables contain k primary dimensions the equation relating all the variables will have $(n\text{-}k)$ dimensionless Π groups. Scaling relation may be resolved by equating Π terms in model and prototype.

Scaling laws for centrifuge tests may be derived form momentum conservation equation (CORTÉ 1989) using Buckingham theorem. Momentum conservation law is given by equation:

$$\mathrm{div}(\tilde{\sigma}) + \tilde{\rho} \cdot \left(\tilde{g} - \frac{\partial^2 \tilde{u}}{\partial t^2}\right) = 0 \tag{1}$$

Equation may be also rewritten using variables:

$$f(\sigma, \rho, g, u, t, x) = 0 \tag{2}$$

where:
σ – stress tensor [kg/(ms^2)],
ρ – density vector [kg/m^3],
g – gravitational acceleration vector [m/s^2],
u – displacement vector [m],
t – time [s],
x – position vector [m]

In this problem occurs 6 independent variables, so $n=6$. These variables contain 3 primary dimensions: length (l), mass (m) and time (t), making $k=3$. We can also use another variables, for example time (t), density (ρ) and position (x). By invoking the Buckingham theorem it can be shown that there are 3 non-dimensional Π terms:

$$n - k = 6 - 3 = 3 \tag{3}$$

Let us define the non-dimensional terms by grouping the variables into n-k groups. Each group contains 3 repeating variables and one non-repeating. This makes:

$$\Pi_1 = \sigma \cdot t^{p_1} \cdot x^{p_2} \cdot \rho^{p_3} \tag{4}$$

$$\Pi_2 = u \cdot t^{p_4} \cdot x^{p_5} \cdot \rho^{p_6} \tag{5}$$

$$\Pi_3 = g \cdot t^{p_7} \cdot x^{p_8} \cdot \rho^{p_9} \tag{6}$$

All variables may be expressed in terms of its dimensions, as shown in table 1.

Table 1
Variable's dimension

Variable	Dimension
σ	kg/(ms^2)
u	m
g	m/s^2
ρ	kg/m^3
t	s
x	m

By substituting these dimensions into eq. (4), eq. (5) and eq. (6) , we have:

$$\Pi_1 = \left[\frac{\text{kg}}{\text{m} \cdot \text{s}^2}\right] \cdot [\text{s}]^{p_1} \cdot [\text{m}]^{p_2} \cdot \left[\frac{\text{kg}}{\text{m}^3}\right]^{p_3} \tag{7}$$

$$\Pi_2 = [\text{m}] \cdot [\text{s}]^{p_4} \cdot [\text{m}]^{p_5} \cdot \left[\frac{\text{kg}}{\text{m}^3}\right]^{p_6} \tag{8}$$

$$\Pi_3 = \left[\frac{\text{m}}{\text{s}^2}\right] \cdot [\text{s}]^{p_7} \cdot [\text{m}]^{p_8} \cdot \left[\frac{\text{kg}}{\text{m}^3}\right]^{p_9} \tag{7}$$

The Π numbers are dimensionless ($\Pi_1 = \Pi_2 = \Pi_3 = 1$). Thus, by solving the eq. (7), (8) and (9) we have:

$$p_1 = 2, p_2 = -2, p_3 = -1 \tag{10}$$

$$p_4 = 0, p_5 = -1, p_6 = -0 \tag{11}$$

$$p_7 = 2, p_8 = -1, p_9 = -0 \tag{12}$$

Non-dimensional Π numbers become then:

$$\Pi_1 = \frac{\sigma \cdot t^2}{x^2 \cdot \rho} \tag{13}$$

$$\Pi_2 = \frac{u}{x} \tag{14}$$

$$\Pi_3 = \frac{g \cdot t^2}{x} \tag{15}$$

Variable $[g]$ may be expressed in dimensions of $[x]$ and $[t]$. Then Π_1 can be written as:

$$\Pi_1 = \frac{\sigma}{x \cdot g \cdot \rho} \tag{16}$$

Hence, momentum conservation equation in dimensionless form can be written as:

$$f\left(\frac{\sigma}{x \cdot g \cdot \rho}, \frac{u}{x}, \frac{g \cdot t^2}{x}\right) = 0 \tag{17}$$

Let us define a scale factors, which are described below:

$$\sigma^* = \frac{\sigma_m}{\sigma_p} \tag{18}$$

$$\rho^* = \frac{\rho_m}{\rho_p} \tag{19}$$

$$x^* = \frac{x_m}{x_p} \tag{20}$$

$$g^* = \frac{g_m}{g_p} \tag{21}$$

$$t^* = \frac{t_m}{t_p} \tag{22}$$

$$u^* = \frac{u_m}{u_p} \tag{23}$$

where suffix m indicates model and p prototype. Scale factors are non-dimensional numbers, so it may be written that:

$$f\left(\frac{\sigma^*}{x^* \cdot g^* \cdot \rho^*}, \frac{u^*}{x^*}, \frac{g \cdot t^{*2}}{x^*}\right) = 0 \tag{24}$$

$$\sigma^* = x^* \cdot g^* \cdot \rho^* \tag{25}$$

$$g^* \cdot t^{*2} = x^* \tag{26}$$

$$u^* = x^* \tag{27}$$

Equations from (25) to (27) are fundamental rules for derivation of scaling laws in physical modelling.

Understanding of centrifuge phenomenon can be explained by the comparison with traditional 1 g test ($g^* = 1$). Let us consider a $1/n$ scale model ($x^* = 1/n$). By using the same soil ($\rho^* = 1$) in a model and prototype, the scale factor for stresses following by eq. (25) will be described as:

$$\sigma^* = \frac{1}{n} \cdot 1 \cdot 1 = \frac{1}{n} \tag{28}$$

Consequently, deformation scale factor is given as:

$$u^* = \frac{1}{n} \tag{29}$$

Further, strain scale factor may be written as:

$$\varepsilon^* = \frac{\Delta u^*}{\Delta x^*} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1 \tag{30}$$

Hence, the constitutive relations governed by equations (HEINBOCKEL 2001):

$$\sigma_{ij} = \frac{E}{1 + \upsilon} \left(\varepsilon_{ij} + \frac{\upsilon}{1 - 2\upsilon} \varepsilon_{kk} \delta_{ij} \right) \tag{31}$$

will be correct only for modified mechanical characteristics of material.

$$E_m = \frac{1}{n} E_p \tag{32}$$

where:
E_m – elastic modulus for model [Pa],
E_p – elastic modulus for prototype [Pa],
n – scale factor [–].

The choice of elastic moduli in soils described by eq. (32) may be problematic to solve. If small-scale models are tested, stress-strain relation is incorrect for full scale construction. The proper stiffness of the soil in the scale model can be achieved by using geotechnical centrifuge.

Now consider a $1/n$ scale model testing in a centrifuge ($g^* = n$). If the model is subjected to acceleration ng and the same materials for model and prototype are used ($\rho^* = 1$), the stress scale factor will be given as:

$$\sigma^* = \frac{1}{n} \cdot n \cdot 1 = 1 \tag{33}$$

Further, the deformation and strain scale factors will be described as:

$$u^* = \frac{1}{n} \tag{34}$$

$$\varepsilon^* = \frac{\Delta u^*}{\Delta x^*} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1 \tag{35}$$

From above equations it can be seen, that stress and strain have a scaling factor of 1. This is one of the main advantages of centrifuge modelling. If the scale factors of g^* and x^* for centrifuge test are inserted into eq. (26), hence time scale factor for dynamic events becomes:

$$t^* = \frac{1}{n} \tag{36}$$

The list of scaling laws in geotechnical centrifuge modelling including dynamic events is presented in table 2. Presented scaling laws, not derived above in this paper, may be obtained from basic physical laws (KONKOL 2013).

Table 2

Scaling laws for centrifuge tests

Type of test	Parameter	Units	Notation	Scaling law model/prototype
Common	length	m	L^*	$1/n$
	area	m^2	A^*	$1/n^2$
	volume	m^3	V^*	$1/n^3$
	density	kg/m^3	ρ^*	1
	mass	kg	m^*	$1/n^3$
	gravitational acceleration	m/s^2	g^*	n
	unit weight	N/m^3	γ^*	n
	stress	N/m^2	σ^*	1
	strain	–	ε^*	1
	force (static)	N	F^*	$1/n^2$
	displacement	m	u^*	$1/n$
Dynamic	bending Moment energy	Nm J	M^* E^*	$1/n^3$ $1/n^3$
	time	s	t^*	$1/n$
	velocity	m/s	v^*	1
	acceleration	m/s^2	a^*	n
	frequency	s^{-1}	f^*	n

Scaling laws for consolidation

Derivation of scaling laws for dynamic events do not include water flow in centrifuge models. Diffusion events like consolidation are slow. Consolidation process is governed by the diffusion equation given as (WIŁUN 2010):

$$\frac{\partial u}{\partial t} = C_v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{37}$$

where:
u – excess pore pressure [Pa],
t – time [s],
C_v – coefficient of consolidation [m^2/s].

Coefficient of consolidation is linked to time by equation (WIŁUN 2010):

$$T_v = C_v \cdot \frac{t}{h^2} \tag{38}$$

where:
T_v – time factor [–],
h – drainage path [m].

Time factor T_v is non-dimensional term. Hence,

$$T_v{}^* = \frac{T_{v,w}}{T_{v,p}} = 1 \tag{39}$$

By expanding eq. (39) we will have:

$$T_v{}^* = \frac{T_{v,w}}{T_{v,p}} = \frac{C_{v,m} \cdot \dfrac{t_m}{h_m^2}}{C_{v,p} \cdot \dfrac{t_p}{h_p^2}} = 1 \tag{40}$$

When the same soil is used in the model and the prototype, coefficients of consolidation for the model and the prototype will also be the same ($C_{v,m} = C_{v,p}$). Consequently, in this case the following time factor t^* will be satisfied:

$$t^* = \frac{t_m}{t_p} = \left(\frac{h_m}{h_p}\right)^2 = \left(\frac{1}{n}\right) = \frac{1}{n^2} \tag{41}$$

Here an interesting feature in consolidation process appears. The time factor of $1/n^2$ enables modelling of settlements in short period of time. For example, let us consider a 3 m deep soft soil layer. It consolidates 15 months to reach 90% degree of consolidation. In geotechnical centrifuge, a 30 mm thick layer tested at 100g will reach the same degree of consolidation in a period of 1,1 h.

Now, let us analyse seepage velocity. Water flow velocity v_{fl} is governed by the Darcy's law and it can be written in two ways (THUSYANTHAN, MADABHUSHI 2003):

$$v_{ij} = k \cdot i \tag{42}$$

$$v_{ij} = -\frac{K}{\mu} \cdot \text{grad}(P) \tag{43}$$

where:
k – coefficient of permeability [m/s],
i – hydraulic gradient [–],
K – hydraulic conductivity (intrinsic permeability) [m²],
μ – dynamic viscosity [Pa · s],
grad(P) – pore pressure gradient [Pa/m].

Coefficient of permeability is linked to unit weight γ, hydraulic conductivity K and dynamic viscosity μ by equation (MUSKAT 1937):

$$k = K \cdot \frac{\gamma}{\mu} \tag{44}$$

By using scaling factors given as:

$$v_{fl}^* = \frac{v_{fl,m}}{v_{fl,p}} \tag{45}$$

$$k^* = \frac{k_m}{k_p} \tag{46}$$

$$i^* = \frac{i_m}{i_p} \tag{47}$$

$$K^* = \frac{K_m}{K_p} \tag{48}$$

$$\mu^* = \frac{\mu_m}{\mu_p} \tag{49}$$

$$\text{grad}(P^*) = \frac{\text{grad}(P_m)}{\text{grad}(P_p)} \tag{50}$$

The Darcy law can be written in dimensionless form as:

$$v_{fl}^* = k^* \cdot i^* \tag{51}$$

$$v_{fl}^* = -\frac{K^*}{\mu^*} \cdot \text{grad}(P^*) \tag{52}$$

If the same soil ($K^* = 1$) and the same fluid ($\mu^* = 1$) are used in the model and the prototype, then the coefficient of permeability can be written as:

$$k^* = K^* \cdot \frac{\gamma^*}{\mu^*} = 1 \cdot \frac{n}{1} = n \tag{53}$$

Thus, because of non-dimensional value of hydraulic gradient ($i^*=1$), water flow velocity is given as:

$$v_{fl}^* = k^* \cdot i^* = n \cdot 1 = n \tag{54}$$

Notice that the time factor will be the same as for consolidation:

$$t^* = \frac{L^*}{v_{fl}^*} = \frac{\frac{1}{n}}{n} = \frac{1}{n^2} \tag{55}$$

Further, pressure gradient can be derived by eq. (52). By substituting $K^* = 1$, $\mu^*=1$ and $v_{fl} = n$ to eq. (52), we have

$$n = -\frac{1}{1} \cdot \mathrm{grad}(P^*) \tag{56}$$

$$\mathrm{grad}(P^*) = n \tag{57}$$

Now let us compare a scale factors for dynamic and consolidation events. In geotechnical modelling diffusion events often occur with dynamic events. As can be seen in table 3, excess pore water pressure in the model dissipate n time faster than in the prototype. For proper matching time, scaling factors for diffusion and dynamic events should be the same. This can be achieved in two different ways. Firstly, we may reduce the permeability of soil by decreasing soil grains in the model. This solution is not desirable because of strain-stress relation. Another approach is to increase viscosity of the fluid used in the model by a factor n. That is a better clue. In centrifuge modelling the fluids like silicone oil or methyl cellulose successfully are used (STEWART et al. 1998). In this case, viscosity scale factor μ^* is given as:

$$\mu^* = \frac{\mu_m}{\mu_p} = n \tag{58}$$

Because the permeability of soil and hydraulic gradient remains the same, the permeability factor will be described as:

$$k^* = K^* \cdot \frac{\gamma^*}{\mu^*} = 1 \cdot \frac{n}{n} = 1 \tag{59}$$

Consequently, the water flow velocity factor will be given as:

$$v_{fl}^* = k^* \cdot i^* = 1 \cdot 1 = 1 \tag{60}$$

Hence, the time factor may be written as:

$$t^* = \frac{L^*}{v_{fl}^*} = \frac{\frac{1}{n}}{1} = \frac{1}{n} \tag{61}$$

Notice that the pressure gradient factor remain unchanged:

$$v_{fl}^* = \frac{K^*}{\mu^*} \cdot \mathrm{grad}(P^*) = \frac{1}{n} \cdot \mathrm{grad}(P^*) = 1 \tag{62}$$

$$\text{grad}(P^*) = n \tag{63}$$

By increasing the viscosity of the model's fluid we gain another profit. Reynolds number, which is given as:

$$\text{Re}^* = \frac{v_{fl}^* \cdot d_e^* \cdot \rho^*}{\mu^*} \tag{64}$$

where:
v_{fl}^* – seepage velocity factor [–],
d_e – effective diameter of soil factor [–],
ρ^* – density of soil factor [–],
μ^* – dynamic viscosity factor [–]

When water in centrifuge model is used we have:

$$\text{Re}^* = \frac{n \cdot 1 \cdot 1}{1} = n \tag{65}$$

But, when using alternative fluid such as silicone oil is used, we have:

$$\text{Re}^* = \frac{1 \cdot 1 \cdot 1}{n} = \frac{1}{n} \tag{66}$$

Consequently, laminar flow in prototype will be always laminar in the model, but turbulent flow in prototype may also be laminar in the model.

Table 3
Different scaling laws for dynamics and consolidation

Parameter	Notation	Scaling law model/prototype	
		dynamics	consolidation
Seepage velocity	v_{fl}^*	1	n
Time	t^*	$1/n$	$1/n^2$

Conclusions

The derivation of scaling laws used in geotechnical centrifuge was show. The application of Buckingham Π theorem in dimensional analysis was presented. The dimensionless form of momentum conservation equation was obtained and the application of this equation in the derivation of scaling laws in physical modelling was featured. The differences in seepage velocity and time scale factors for consolidation and dynamic events were noticed. Currently used solutions of this problem were cited. The advantages of geotechnical centrifuge technology were described and finally, the basic principles of centrifuge modelling were featured.

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