



On the partition dimension of trees

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ABSTRACT

Given an ordered partition $\Pi = \{P_1, P_2, \dots, P_t\}$ of the vertex set V of a connected graph $G = (V, E)$, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), \dots, d(v, P_t))$, where $d(v, P_i)$ represents the distance between the vertex v and the set P_i . A partition Π of V is a *resolving partition* of G if different vertices of G have different partition representations, i.e., for every pair of vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G . In this paper we obtain several tight bounds on the partition dimension of trees.

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1. Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [17]. After these papers were published several authors developed diverse theoretical works about this topic [3,2,4–10,14,19]. Slater described the usefulness of these ideas into long range aids to navigation [17]. Also, these concepts have some applications in chemistry for representing chemical compounds [12,13] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [15]. Other applications of this concept to navigation of robots in networks and other areas appear in [5,11,14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [16], locating domination [10], resolving domination [1] and resolving partitions [4,7,8,19].

Given a graph $G = (V, E)$ and an ordered set of vertices $S = \{v_1, v_2, \dots, v_k\}$ of G , the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), \dots, d(v, v_k))$, where $d(v, v_i)$ denotes the distance between the vertices v and v_i , $1 \leq i \leq k$. We say that S is a *resolving set* of G if different vertices of G have different metric representations, i.e., for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension*¹ of G is the minimum cardinality of any resolving set of G , and it is denoted by $\dim(G)$. The metric dimension of graphs is studied in [3,2,4–6,18].

Given an ordered partition $\Pi = \{P_1, P_2, \dots, P_t\}$ of the vertices of G , the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), \dots, d(v, P_t))$, where $d(v, P_i)$, with $1 \leq i \leq t$, represents the distance between the vertex v and the set P_i , i.e., $d(v, P_i) = \min_{u \in P_i} \{d(v, u)\}$. We say that Π is a *resolving partition* of G if different vertices of G have different partition representations, i.e., for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$.

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¹ Also called the locating number.

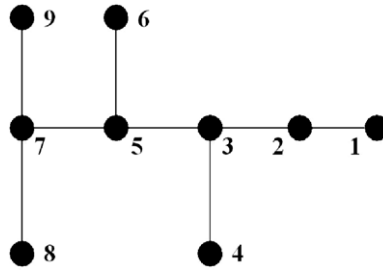


Fig. 1. In this tree the vertex 3 is an exterior major vertex of terminal degree two: 1 and 4 are terminal vertices of 3.

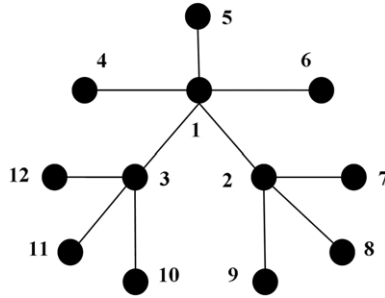


Fig. 2. $\Pi = \{\{1, 4, 9, 12\}, \{3, 5, 8, 11\}, \{2, 6, 7, 10\}\}$ is a resolving partition.

$r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G and it is denoted by $pd(G)$. The partition dimension of graphs is studied in [4,7,8,18].

2. The partition dimension of trees

It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph G we have

$$pd(G) \leq \dim(G) + 1. \tag{1}$$

We know that the partition dimension of any path is two. That is, for any path graph P , it follows $pd(P) = \dim(P) + 1 = 2$. A formula for the dimension of trees that are not paths has been established in [5,9,17]. In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a tree T will be called a *major vertex* of T . Any leaf u of T is said to be a *terminal vertex* of a major vertex v of T if $d(u, v) < d(u, w)$ for every other major vertex w of T . The *terminal degree* of a major vertex v is the number of terminal vertices of v . A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree.

Let $n_1(T)$ denote the number of leaves of T , and let $ex(T)$ denote the number of exterior major vertices of T . We can now state the formula for the dimension of a tree [5,9,17]: if T is a tree that is not a path, then

$$\dim(T) = n_1(T) - ex(T). \tag{2}$$

As a consequence, if T is a tree that is not a path, then

$$pd(T) \leq n_1(T) - ex(T) + 1. \tag{3}$$

The above bound is tight, it is achieved for the graph in Fig. 1 where $\Pi = \{\{8\}, \{4, 9\}, \{1, 2, 3, 5, 6, 7\}\}$ is a resolving partition and $pd(T) = 3$. However, there are graphs for which the following bound gives better result than bound (3), for instance, the graph in Fig. 2.

Let $S = \{s_1, s_2, \dots, s_\kappa\}$ be the set of exterior major vertices of $T = (V, E)$ with terminal degree greater than one; let $\{s_{i1}, s_{i2}, \dots, s_{il_i}\}$ be the set of terminal vertices of s_i and let $\tau = \max_{1 \leq i \leq \kappa} \{l_i\}$. With the above notation we have the following result.

Theorem 1. For any tree T which is not a path,

$$pd(T) \leq \kappa + \tau - 1.$$

Proof. For a terminal vertex s_{ij} of a major vertex $s_i \in S$ we denote by S_{ij} the set of vertices of T , different from s_i , belonging to the $s_i - s_{ij}$ path. If $l_i < \tau - 1$, we assume $S_{ij} = \emptyset$ for every $j \in \{l_i + 1, \dots, \tau - 1\}$. Now for every $j \in \{2, \dots, \tau - 1\}$, let

$B_j = \cup_{i=1}^{\kappa} S_{ij}$ and, for every $i \in \{1, \dots, \kappa\}$, let $A_i = S_{i1}$. Let us show that $\Pi = \{A, A_1, A_2, \dots, A_\kappa, B_2, \dots, B_{\tau-1}\}$ is a resolving partition of T , where $A = V - ((\cup_{i=1}^{\kappa} A_i) \cup (\cup_{j=2}^{\tau-1} B_j))$. We consider two different vertices $x, y \in V$. Note that if x and y belong to different sets of Π , we have $r(x|\Pi) \neq r(y|\Pi)$.

Case 1: $x, y \in S_{ij}$. If $j = \tau$, then we have that $x, y \in A$ and it follows that $d(x, A_i) \neq d(y, A_i)$. Otherwise, we obtain that $d(x, A) = d(x, s_i) \neq d(y, s_i) = d(y, A)$.

Case 2: $x \in S_{ij}$ and $y \in S_{kl}$, $i \neq k$. If $j = 1$ or $l = 1$, then x and y belong to different sets of Π . So we suppose $j \neq 1$ and $l \neq 1$. Hence, if $d(x, A_i) = d(y, A_i)$, then

$$\begin{aligned} d(x, A_k) &= d(x, s_i) + d(s_i, s_k) + 1 \\ &= d(x, A_i) + d(s_i, s_k) \\ &= d(y, A_i) + d(s_i, s_k) \\ &= d(y, s_k) + 2d(s_k, s_i) + 1 \\ &= d(y, A_k) + 2d(s_k, s_i) \\ &> d(y, A_k). \end{aligned}$$

Case 3: $x \in S_{i\tau}$ and $y \in A - \cup_{l=1}^{\kappa} S_{i\tau}$. If $d(x, A_i) = d(y, A_i)$, then $d(x, s_i) = d(y, s_i)$. Since $y \notin S_{i\tau}$, $l \in \{1, \dots, \kappa\}$, there exists $A_j \in \Pi$, $j \neq i$, such that s_i does not belong to the $y - s_j$ path. Now let Y be the set of vertices belonging to the $y - s_j$ path, and let $v \in Y$ such that $d(s_i, v) = \min_{u \in Y} \{d(s_i, u)\}$. Hence,

$$\begin{aligned} d(x, A_j) &= d(x, s_i) + d(s_i, v) + d(v, s_j) + 1 \\ &= d(y, s_i) + d(s_i, v) + d(v, s_j) + 1 \\ &= d(y, v) + 2d(v, s_i) + d(v, s_j) + 1 \\ &= d(y, A_j) + 2d(v, s_i) \\ &> d(y, A_j). \end{aligned}$$

Case 4: $x, y \in A' = A - \cup_{l=1}^{\kappa} S_{i\tau}$. If for some exterior major vertex $s_i \in S$, the vertex x belongs to the $y - s_i$ path or the vertex y belongs to the $x - s_i$ path, then $d(x, A_i) \neq d(y, A_i)$. Otherwise, there exist at least two exterior major vertices s_i, s_j such that the $x - y$ path and the $s_i - s_j$ path share more than one vertex (if not, then $x, y \notin A'$). Let W be the set of vertices belonging to the $s_i - s_j$ path. Let $u, v \in W$ such that $d(x, u) = \min_{z \in W} \{d(x, z)\}$ and $d(y, v) = \min_{z \in W} \{d(y, z)\}$. We suppose, without loss of generality, that $d(s_i, u) > d(v, s_i)$. Hence, if $d(x, v) = d(y, v)$, then $d(x, u) \neq d(y, u)$, and if $d(x, u) = d(y, u)$, then $d(x, v) \neq d(y, v)$. We have

$$\begin{aligned} d(x, A_j) &= d(x, u) + d(u, s_j) + 1 \\ &\neq d(y, u) + d(u, s_j) + 1 \\ &= d(y, A_j) \end{aligned}$$

or

$$\begin{aligned} d(x, A_i) &= d(x, v) + d(v, s_i) + 1 \\ &\neq d(y, v) + d(v, s_i) + 1 \\ &= d(y, A_i). \end{aligned}$$

Therefore, for different vertices $x, y \in V$, we have $r(x|\Pi) \neq r(y|\Pi)$. \square

One example where $pd(T) = \kappa + \tau - 1$ is the tree in Fig. 1.

Any vertex adjacent to a leaf of a tree T is called a *support* vertex. In the following result ξ denotes the number of support vertices of T and θ denotes the maximum number of leaves adjacent to a support vertex of T .

Corollary 2. For any tree T of order $n \geq 2$, $pd(T) \leq \xi + \theta - 1$.

Proof. If T is a path, then $\xi = 2$ and $\theta = 1$, so the result follows. Now we suppose T is not a path. Let v be an exterior major vertex of terminal degree τ . Let x be the number of leaves adjacent to v and let $y = \tau - x$. Since $\kappa + y \leq \xi$ and $x \leq \theta$, we deduce $\kappa + \tau \leq \xi + \theta$. \square

The above bound is achieved, for instance, for the graph of order six composed of two support vertices a and b , where a is adjacent to b and four leaves; two of them are adjacent to a and the other two leaves are adjacent to b . One example of a graph for which Theorem 1 gives a better result than Corollary 2 is the graph in Fig. 1.

Since the number of leaves, $n_1(T)$, of a tree T is bounded below by $\xi + \theta - 1$, Corollary 2 leads to the following bound.

Remark 3. For any tree T of order $n \geq 2$, $pd(T) \leq n_1(T)$.

Now we are going to characterize all the trees for which $pd(T) = n_1(T)$. It was shown in [7] that $pd(G) = 2$ if and only if the graph G is a path. So by the above remark we obtain the following result.

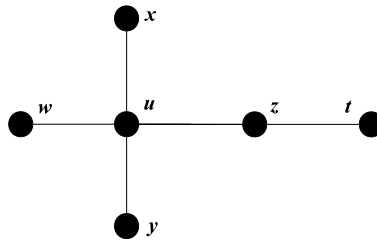


Fig. 3. A comet graph where $3 = \theta = pd(T) < n_1(T)$.

Remark 4. Let T be a tree of order $n \geq 4$. If $n_1(T) = 3$, then $pd(T) = 3$.

Theorem 5. Let T be a tree with $n_1(T) \geq 4$. Then $pd(T) = n_1(T)$ if and only if T is the star graph.

Proof. If $T = S_n$ is a star graph, it is clear that $pd(T) = n_1(T)$. Now, let $T = (V, E) \neq S_n$, such that $pd(T) = n_1(T) \geq 4$. Note that by (3) we have $ex(T) = 1$. Let $t = n_1(T)$ and let $\Omega = \{u_1, u_2, \dots, u_t\}$ be the set of leaves of T . Let $u \in V$ be the unique exterior major vertex of T . Let us suppose, without loss of generality, u_t is a leaf of T such that $d(u_t, u) = \max_{u_i \in \Omega} \{d(u_i, u)\}$.

For the leaves $u_1, u_2, u_t \in \Omega$ let the paths $P = uu_{t_1}u_{t_2}, \dots, u_{t_r}u_t$, $Q = uu_{11}u_{12}, \dots, u_{1r_1}u_1$ and $R = uu_{21}u_{22}, \dots, u_{2r_2}u_2$. Now, let us form the partition $\Pi = \{A_1, A_2, \dots, A_{t-2}, A\}$, such that $A_1 = \{u_{11}, u_{12}, \dots, u_{1r_1}, u_1, u_{t_2}, u_{t_3}, \dots, u_{t_r}, u_t\}$, $A_2 = \{u_{21}, u_{22}, \dots, u_{2r_2}, u_2, u_{t_1}\}$, $A_i = \{u_i\}$, $i \in \{3, \dots, t-2\}$ and $A = V - \cup_{i=1}^{t-2} A_i$. Let us consider two different vertices $x, y \in V$. Hence, we have the following cases.

Case 1: $x, y \in A_1$. Let us suppose $x \in P$ and $y \in Q$. If $d(x, A_2) = d(y, A_2)$, then we have

$$\begin{aligned} d(x, A) &= d(x, u_{t_1}) + 1 \\ &= d(x, A_2) + 1 \\ &= d(y, A_2) + 1 \\ &= d(y, A) + 2 \\ &> d(y, A). \end{aligned}$$

Now, if $x, y \in P$ or $x, y \in Q$, then $d(x, A) \neq d(y, A)$.

Case 2: $x, y \in A_2$. If $x = u_{t_1}$ or $y = u_{t_1}$, then let us suppose for instance, $x = u_{t_1}$, so we have $d(x, A_1) = 1 < 2 \leq d(y, A_1)$. On the contrary, if $x, y \in R$, then $d(x, A) \neq d(y, A)$.

Case 3: $x, y \in A$. If $d(x, A_1) = d(y, A_1)$, then $t \geq 5$ and there exists a leaf u_i , $i \neq 1, 2, t-1, t$, such that $d(x, A_i) = d(y, u_i) \neq d(y, u_i) = d(y, A_i)$.

Therefore, for different vertices $x, y \in V$ we have $r(x|\Pi) \neq r(y|\Pi)$ and Π is a resolving partition in T , a contradiction. \square

Let T be the comet graph shown in Fig. 3. A resolving partition for T is $\Pi = \{A_1, A_2, A_3\}$, where $A_1 = \{x, t\}$, $A_2 = \{y, z\}$ and $A_3 = \{u, w\}$. In this case, $\theta = pd(T) = 3 < 4 = n_1(T)$.

Remark 6. For any tree T of order $n \geq 2$, $pd(T) \geq \theta$.

Proof. Since different leaves adjacent to the same support vertex must belong to different sets of a resolving partition, the result follows. \square

Other examples where $pd(T) = \theta$ are the star graphs and the graph in Fig. 2.

Theorem 7. Let T be a tree which is not a path. If every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then

$$pd(T) \leq \max\{\kappa, \tau + 1\}.$$

Proof. We suppose $T = (V, E)$ is not a path. Let $S = \{s_1, s_2, \dots, s_\kappa\}$ be the set of exterior major vertices of T with terminal degree greater than one and let $B_i = \{s_i\}$, $i = 1, \dots, \kappa$. If $\kappa < \tau + 1$, then for $i \in \{\kappa + 1, \dots, \tau + 1\}$ we assume $B_i = \emptyset$. Let l_i be the terminal degree of s_i , $i \in \{1, \dots, \kappa\}$. If $l_i < i$, then we denote by $\{s_{i1}, \dots, s_{il_i}\}$ the set of terminal vertices of s_i . On the contrary, if $l_i \geq i$, then the set of terminal vertices of s_i is denoted by $\{s_{i1}, \dots, s_{ii-1}, s_{ii+1}, \dots, s_{il_i+1}\}$. Also, for a terminal vertex s_{ij} of a major vertex s_i we denote by S_{ij} the set of vertices of T , different from s_i , belonging to the $s_i - s_{ij}$ path. Moreover, we assume $S_{ij} = \emptyset$ for the following three cases: (1) $i = j$, (2) $i \leq l_i < \tau$ and $j \in \{l_i + 2, \dots, \tau + 1\}$, and (3) $i > l_i$ and $j \in \{l_i + 1, \dots, \tau + 1\}$. Now, let $t = \max\{\kappa, \tau + 1\}$ and let $\Pi = \{A_1, A_2, \dots, A_t\}$ be composed of the sets $A_i = B_i \cup (\cup_{j=1}^{\kappa} S_{ij})$, $i = 1, \dots, t$. Since every vertex belonging to the path between two exterior major vertices of terminal degree greater than one is an exterior major vertex of terminal degree greater than one, then Π is a partition of V .

Let us show that Π is a resolving partition. Let $x, y \in V$ be different vertices of T . If $x, y \in A_i$, we have the following three cases.

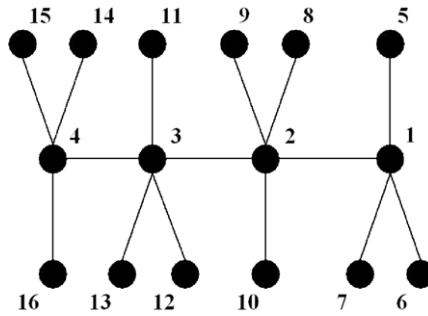


Fig. 4. $\Pi = \{\{1, 8, 11, 14\}, \{2, 5, 12, 15\}, \{3, 6, 9, 16\}, \{4, 7, 10, 13\}\}$ is a resolving partition.

Case 1: $x, y \in S_{ji}$. In this case $d(x, A_j) = d(x, s_j) \neq d(y, s_j) = d(y, A_j)$.

Case 2: $x \in S_{ji}$ and $y \in S_{ki}$, $j \neq k$. If $d(x, A_k) = d(y, A_k)$ we have $d(y, A_j) > d(y, s_k) = d(y, A_k) = d(x, A_k) > d(x, s_j) = d(x, A_j)$.

Case 3: $x = s_i$ and $y \in S_{ji}$. As s_i has at least two terminal vertices, there exists a terminal vertex s_{il} of s_i , $l \neq j$, such that $d(x, A_l) = d(x, s_{il}) = 1$. Hence, $d(y, A_l) > d(y, s_j) \geq 1 = d(x, A_l)$. Therefore, for different vertices $x, y \in V$, we have $r(x|\Pi) \neq r(y|\Pi)$. \square

The above bound is achieved, for instance, for the graph in Fig. 4.

3. On the partition dimension of generalized trees

A *cut vertex* in a graph is a vertex whose removal increases the number of components of the graph and an *extreme vertex* is a vertex such that its closed neighborhood forms a complete graph. Also, a *block* is a maximal biconnected subgraph of the graph. Now, let \mathfrak{F} be the family of sequences of connected graphs G_1, G_2, \dots, G_k , $k \geq 2$, such that G_1 is a complete graph K_{n_1} , $n_1 \geq 2$, and G_i , $i \geq 2$, is obtained recursively from G_{i-1} by adding a complete graph K_{n_i} , $n_i \geq 2$, and identifying a vertex of G_{i-1} with a vertex in K_{n_i} .

From this point we will say that a connected graph G is a *generalized tree* if and only if there exists a sequence $\{G_1, G_2, \dots, G_k\} \in \mathfrak{F}$ such that $G_k = G$ for some $k \geq 2$. Notice that in these generalized trees every vertex is either a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every G_i is isomorphic to K_2 , then G_k is a tree, thus justifying the terminology used. In this section we will be centered in the study of partition dimension of generalized trees.

Let $G = (V, E)$ be a generalized tree and let R_1, R_2, \dots, R_k be the blocks of G . A cut vertex $v \in V$ is a *support cut vertex* if there is at least one block R_i of G , in which v is the unique cut vertex belonging to the block R_i . An extreme vertex is an *exterior extreme vertex* if it is adjacent to only one cut vertex. Let $S = \{s_1, s_2, \dots, s_\zeta\}$ be the set of support cut vertices of G and let $\{s_{i1}, s_{i2}, \dots, s_{il_i}\}$ be the set of exterior extreme vertices adjacent to $s_i \in S$. Also, let $Q = \{Q_1, Q_2, \dots, Q_\vartheta\}$ be the set of blocks of G which contain more than one cut vertex and more than one extreme vertex and let $\{q_{i1}, q_{i2}, \dots, q_{it_i}\}$ be the set of extreme vertices belonging to $Q_i \in Q$. Now, let $\phi = \max_{1 \leq i \leq \zeta, 1 \leq j \leq \vartheta} \{l_i, t_j\}$. With the above notation we have the following result.

Theorem 8. For any generalized tree G ,

$$pd(G) \leq \begin{cases} \zeta + \vartheta + \phi - 1, & \text{if } \phi \geq 3; \\ \zeta + \vartheta + 1, & \text{if } \phi \leq 2. \end{cases}$$

Proof. For each support cut vertex $s_i \in S$, let $A_i = \{s_{i1}\}$ and for each block $Q_j \in Q$, let $B_j = \{q_{j1}\}$. Let us suppose $\phi \geq 3$. For every $j \in \{2, \dots, l_i\}$ we take $M_{ij} = \{s_{ij}\}$ and, if $l_i < \phi - 1$, then for every $j \in \{l_{i+1}, \dots, \phi - 1\}$ we consider $M_{ij} = \emptyset$. Analogously, for every $j \in \{2, \dots, t_i\}$ we take $N_{ij} = \{q_{ij}\}$ and, if $t_i < \phi - 1$, then for every $j \in \{t_{i+1}, \dots, \phi - 1\}$ we consider $N_{ij} = \emptyset$. Now, let $C_j = \bigcup_{i=1}^{\max\{\zeta, \vartheta\}} (M_{ij} \cup N_{ij})$, with $j \in \{2, \dots, \phi - 1\}$.

Let us prove that $\Pi = \{A, A_1, A_2, \dots, A_\zeta, B_1, B_2, \dots, B_\vartheta, C_2, C_3, \dots, C_{\phi-1}\}$ is a resolving partition of G , where $A = V - \bigcup_{i=1}^\zeta A_i - \bigcup_{i=1}^\vartheta B_i - \bigcup_{i=2}^{\phi-1} C_i$. To begin with, let x, y be two different vertices of G . We have the following cases.

Case 1: x is a cut vertex or y is a cut vertex. Let us suppose, for instance, x is a cut vertex. So there exists an extreme vertex s_{i1} such that x belongs to a shortest $y - s_{i1}$ path or y belongs to a shortest $x - s_{i1}$ path. Hence, we have $d(x, A_i) = d(x, s_{i1}) \neq d(y, s_{i1}) = d(y, A_i)$.

Case 2: x, y are extreme vertices. If x, y belong to the same block of G , then x, y belong to different sets of Π . On the contrary, if x, y belong to different blocks in G , then let us suppose that there exists an extreme vertex c such that $d(x, c) \leq 1$ or $d(y, c) \leq 1$. We can suppose $c \in A_i$, for some $i \in \{1, \dots, \zeta\}$, or $c \in B_j$, for some $j \in \{1, \dots, \vartheta\}$. Without the loss of generality, we suppose that $d(x, c) \leq 1$. Since x and y belong to different blocks of G , we have $d(y, c) > 1$. So we obtain either $d(x, A_i) = d(x, c) \leq 1 < d(y, c) = d(y, A_i)$ or $d(x, B_j) = d(x, c) \leq 1 < d(y, c) = d(y, B_j)$.

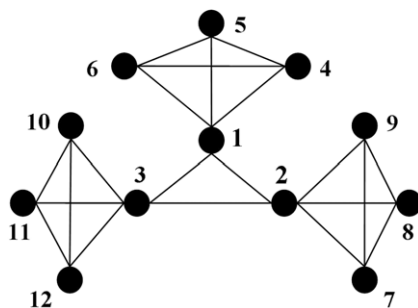


Fig. 5. $\Pi = \{\{4\}, \{7\}, \{10\}, \{5, 8, 11\}, \{1, 2, 3, 6, 9, 12\}\}$ is a resolving partition for the generalized tree.

Now, if there exists no such a vertex c , then there exist two blocks $H, K \notin Q$ with $x \in H$ and $y \in K$, which contain more than one cut vertex and only one extreme vertex. So $x, y \in A$. Let $u \in H$ be a cut vertex such that $d(y, u) = \max_{v \in H} d(y, v)$. Hence, there exists an extreme vertex s_{i1} such that u belongs to a shortest $x - s_{i1}$ path and $d(y, s_{i1}) = d(y, u) + d(u, s_{i1})$. As x, y belong to different blocks and $d(y, u) = \max_{v \in H} d(y, v)$ we have $d(y, u) \geq 2$. Thus,

$$\begin{aligned} d(y, A_i) &= d(y, s_{i1}) \\ &= d(y, u) + d(u, s_{i1}) \\ &\geq 2 + d(u, s_{i1}) \\ &> 1 + d(u, s_{i1}) \\ &= d(x, u) + d(u, s_{i1}) \\ &= d(x, A_i). \end{aligned}$$

Hence, we conclude that if $\phi \geq 3$, then for every $x, y \in V$, $r(x|\Pi) \neq r(y|\Pi)$. Therefore, Π is a resolving partition.

On the other hand, if $\phi \leq 2$, then $\Pi' = \{A, A_1, A_2, \dots, A_\zeta, B_1, B_2, \dots, B_\vartheta\}$ is a partition of V . Proceeding as above we obtain that Π' is a resolving partition. \square

The above bound is achieved, for instance, for the graph in Fig. 5, where $\zeta = 3$, $\vartheta = 0$ and $\phi = 3$. Also, notice that for the particular case of trees we have $\zeta = \xi$, $\phi = \theta$ and $\vartheta = 0$. So the above result leads to Corollary 2.

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