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Existence of solutions with exponential growth for nonlinear differential-functional parabolic equations

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Abstract. We consider the Cauchy problem for nonlinear parabolic equations with functional dependence. We prove Schauder-type existence results for unbounded solutions. We also prove existence of maximal solutions for a wide class of differential functional equations.

1. Introduction. We study the Cauchy problem for parabolic equations with general functional dependence (see [7, 8]), and with Hölder continuous leading terms. Under natural assumptions we prove Schauder-type existence results in classes of continuous functions such that $|u(t,x)| \leq C\exp(\psi(t)|x|^2)$.

Existence results for parabolic equations with functionals of the unknown function and its spatial first-order derivatives were considered in [14] as fixed points of suitable integral operators. Our research into existence and uniqueness of such solutions for various parabolic differential-functional problems has gradually developed by the use of iterative methods or the Banach contraction principle (see [3, 5, 10, 11]). In all these works estimates of the fundamental solutions based on [6, 9, 12, 18] are applied. The quasilinearization method is described e.g. in [1, 4]. It is worth noting that differential inequalities play an important role in proving existence and uniqueness theorems. Such methods for parabolic equations can be found in [15, 16], and for differential-functional equations in [13, 18]. The article [18] focuses on the existence of maximal solutions of parabolic problems, treated as abstract ordinary differential equations, thus it is strongly related to the general theory of ODEs (see [17]). Monotone iterative techniques were developed in [2], based on weak and strong maximum principles and comparison inequalities (see [15]).

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If we consider the Schauder theory for ODEs such as $dz/dt = F(t,z)$, then the right-hand side is assumed to be sublinear: $|F(t,z)| \le$ const + const $\cdot |z|$. When we are looking for bounded solutions of parabolic problems with right-hand sides depending on $t, x, u(t,x), \frac{\partial u}{\partial x}(t,x)$, sublinearity means that the right-hand side is estimated by const + const $\cdot\, |u(t,x)|$ + const $\cdot \left|\frac{\partial u}{\partial x}(t,x)\right|$. If we consider unbounded solutions with the growth restriction $|u(t,x)| \le C \exp(\psi(t)|x|^2)$, then this estimate changes to

$$\text{const} \cdot \exp(\psi(t)|x|^2) + \text{const} \cdot |u(t,x)| + \text{const} \cdot \left|\frac{\partial u}{\partial x}(t,x)\right|.$$

It has a similar form for the functional dependence $u(s,x), \frac{\partial u}{\partial x}(s,x)$, where $s \le t$. Any functional dependence with deviating arguments such as $u(t, x \pm \tau)$ or $\frac{\partial u}{\partial x}(t, x \pm \tau)$ demands a deflator which behaves like $\exp(-K|x|)$, with a sufficiently large constant K. Accordingly, the right-hand side is majorized by

$$\text{const} \cdot \exp(\psi(t)|x|^2) + \text{const} \cdot |u(t,x)| + \text{const} \cdot \exp(-K|x|) \sup_{|y-x| \le \tau} |u(t,y)|.$$

In this paper we express these sublinearity conditions in terms of the Hale operator: see Assumption $[a]$ and Theorem 4.1.

The paper is organized as follows. In Section 2 we list basic properties of the fundamental solution and deduce a priori estimates of unbounded solutions. In Sections 3 and 4 we formulate theorems on the existence of solutions in some classes of continuous functions satisfying the growth condition $|u(t,x)| \le C \exp(K|x|^2)$. The proof is based on the Schauder fixed point theory in some metric spaces contained in the set of continuous extensions of the initial data. In Section 5 we prove the existence of maximal solutions satisfying a similar growth condition. Maximal solutions for ODEs $dz/dt = F(t,z)$, $z(0) = z_0$, are obtained by means of strong differential inequalities as decreasing limits of solutions to approximate Cauchy problems $dz_k/dt = F(t, z_k) + \varepsilon_k$, $z(0) = z_0 + \varepsilon_k$, where (ε_k) is any sequence decreasing to zero. In the case of parabolic problems in unbounded domains there is no satisfactory result in the form of a strong differential inequality, thus any statement on maximal solutions must be somehow tempered: maximal solutions are restricted to a specific subclass of functions and the whole theory demands relatively strong assumptions, especially quasimonotonicity and strengthened sublinearity.

1.1. Motivation. Consider the heat transfer equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ whose natural bounded initial distributions result in bounded heat dynamics for $t > 0$. In the case of nuclear plants there is an obvious interest in security measures which exceed 1000%. The heat equation is well posed in the class of bounded functions, unbounded solutions are not physical. However, it turns out that the equation has 'exotic' solutions outside the classes $|u(t,x)| \leq Ce^{Kx^2}$ (see Tikhonov's counterexample [6, 9]). Thus the safety measure for the heat equation can be specified by this growth restriction. It is known that solutions of the heat equation with unbounded initial data may blow up in finite time, e.g. the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad u(0,x) = e^{x^2},$$

blows up at $t = 1/4$. The heat equation with sources $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t,x)$ must have the properties:

(i) any nonnegative initial distribution $u(0,x) \geq 0$ implies $u(t,x) \geq 0$ for $t > 0$,

(ii) the initial growth condition $|u(0,x)| \leq Ce^{Kx^2}$ implies $|u(t,x)| \leq C_1 e^{K_1 x^2}$.

If the heat flow (convection) depends on the density u of the heat (temperature) and the source $f(t,x,u)$ satisfies the Lipschitz condition in u and the growth restriction $|f(t,x,0)| \leq Ce^{Kx^2}$, then the above postulates (i)–(ii) are satisfied. The situation changes when the source f depends on the temperature at perturbed points $x \pm \Delta x$. For example, consider the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u(t,x+1), \qquad u(0,x) = e^{x^2}.$$

Although the Lipschitz condition is satisfied, there is no solution to this problem in the class of functions

$$u(t,x) \geq 0 \quad \text{and} \quad |u(t,x)| \leq Ce^{Kx^2}.$$

Assuming that such a solution existed it would immediately explode at any time $t > 0$.

Let us go back to the case without uniqueness. For example, consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sqrt{u(t,x)}, \qquad u(0,x) = 0.$$

The nonuniqueness property is inherited from ODEs, the problem has at least two solutions $u = 0$ and $u = t^2/4$. Our theorems on extremal solutions show that 0 is the minimal solution and $t^2/4$ is the maximal solution in the classes $|u(t,x)| \leq Ce^{Kx^2}$. The same result is valid for the Cauchy problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sqrt{u(t,x+1)}, \qquad u(0,x) = 0.$$

Due to the above counterexample, it is not obvious whether the Cauchy problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sqrt{u(t,x+1)}, \qquad u(0,x) = e^{x^2},$$

has multiple solutions. Schauder-type theorems imply existence. The upper and lower solution method surprisingly yields uniqueness in the classes $|u(t,x)| \le Ce^{Kx^2}$ because the maximal solution is equal to the minimal solution. This means that extremal solutions provide additional information missed by fixed point theory.

Finally, we comment on the main difficulties in the paper:

(a) Different asymptotic behaviour of u and $\frac{\partial u}{\partial x}$ at $t = 0^+$ and different properties of the potentials $\int_{\mathbb{R}^n} \Gamma\varphi\, dy$ and $\int_0^t \int_{\mathbb{R}^n} \Gamma f\, dy\, ds$. Remedy: we construct metric spaces which consist of functions u satisfying the inequalities
$$\Big|u - \int_{\mathbb{R}^n} \Gamma\varphi\, dy\Big| \le \gamma(t)\exp(\psi(t)|x|^2),$$
$$\Big|\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\int_{\mathbb{R}^n} \Gamma\varphi\, dy\Big| \le \gamma_1(t)\exp(\psi(t)|x|^2).$$

(b) No comparison theorem in the classes $|u(t,x)| \le Ce^{K|x|^2}$. Remedy: we build a sequence of upper solutions u_n such that $|u_n| \le C_n e^{K_n(t)|x|^2}$, where the functions $K_n(t)$ strictly decrease. Since the functions u_n and u_{n+1} have different asymptotics at $|x| \to \infty$, it is possible to get the inequality $u_{n+1} \le u_n$ by Nagumo's method.

2. Preliminaries. We recall basic properties of fundamental solutions and their applications to the existence and uniqueness theory for differential equations (see [5]).

Let $E = (0,a]\times\mathbb{R}^n$, $E_0 = [-\tau_0, 0]\times\mathbb{R}^n$, $\widetilde{E} = E_0 \cup E$, $B = [-\tau_0, 0]\times[-\tau,\tau]$, where $a > 0$, $\tau_0, \tau_1, \ldots, \tau_n \in \mathbb{R}_+ = [0,\infty)$, and
$$\tau = (\tau_1,\ldots,\tau_n), \qquad [-\tau,\tau] = [-\tau_1,\tau_1]\times\cdots\times[-\tau_n,\tau_n].$$
If $u : E_0\cup E \to \mathbb{R}$ and $(t,x) \in E$, then the Hale-type functional $u_{(t,x)} : B \to \mathbb{R}$ is defined by
$$u_{(t,x)}(s,y) = u(t+s, x+y) \quad \text{ for } (s,y) \in B.$$
Let $C(X)$ be the set of all real continuous functions defined on a metric space X, and $L^1[0,a]$ the space of integrable functions on $[0,a]$. Denote by $\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}$ the operators of partial derivatives with respect to $t, x_1, \ldots, x_n$. Let $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and let $\partial_{x_j x_l}$ $(j,l = 1,\ldots,n)$ denote the second order derivatives with respect to the spatial variables x.

Suppose that $f : E \times C(B) \to \mathbb{R}$ and $\varphi : E_0 \to \mathbb{R}$ are given functions. We consider the Cauchy problem
$$\mathcal{P}u(t,x) = f(t,x,u_{(t,x)}), \tag{2.1}$$
$$u(t,x) = \varphi(t,x) \quad \text{ on } E_0, \tag{2.2}$$

where the differential operator $\mathcal{P}$ is defined by

$$\mathcal{P}u(t,x) = \partial_t u(t,x) - \sum_{j,l=1}^{n} a_{jl}(t,x)\partial_{x_j x_l} u(t,x).$$

The Cauchy problem (2.1)–(2.2) can be transformed into the following integral equation:

$$u(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(0,y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(t,x;s,y) f(s,y,u_{(s,y)})\,dy\,ds, \tag{2.3}$$

where $\Gamma(t,x;s,y)$ is the fundamental solution of the above parabolic problem (2.1)–(2.2).

Definition 2.1. A continuous and nondecreasing function $\chi : [-\tau_0, a] \to [0,\infty)$ is said to be of *class* $\mathcal{C}^+$ if $\chi(t) = \chi(0)$ on $[-\tau_0, 0]$.

Definition 2.2. Let $u \in C(\widetilde{E})$.

(i) A function u is called a C^0 *solution* of problem (2.1)–(2.2) if u coincides with φ on E_0 and satisfies (2.3) on E. Such C^0 solutions are known as 'mild solutions'.
(ii) A function u is called a $C^{0,1}$ *solution* of problem (2.1)–(2.2) if u is a C^0 solution whose derivatives $\partial_{x_j} u$ $(j = 1,\ldots,n)$ are continuous on E.

Define a function $\widetilde{\varphi} : \widetilde{E} \to \mathbb{R}$ by

$$\widetilde{\varphi}(t,x) = \begin{cases} \varphi(t,x) & \text{for } (t,x) \in E_0, \\ \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varphi(0,y)\,dy & \text{for } (t,x) \in E. \end{cases}$$

By $|\cdot|$ we denote absolute values of real numbers as well as Euclidean norms of vectors in $\mathbb{R}^n$. Furthermore, if $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, then $|y| = (|y_1|,\ldots,|y_n|)$. In particular, $|\,|y| + \tau|^2 = (|y_1| + \tau_1)^2 + \cdots + (|y_n| + \tau_n)^2$. The supremum norm in $C(B)$ will be denoted by $\|\cdot\|$. If $x, y \in \mathbb{R}^n$, then we denote by $\langle x, y\rangle$ the inner product of the vectors x, y.

The following assumption will be needed throughout the paper:

Assumption $[a]$. Suppose that

(1) the operator $\mathcal{P}$ is *uniformly parabolic*, i.e. there are $C', c' > 0$ such that

$$C'|\xi|^2 \ge \sum_{j,l=1}^{n} a_{jl}(t,x)\xi_j\xi_l \ge c'|\xi|^2 \quad \text{for all } (t,x) \in E,\ \xi \in \mathbb{R}^n,$$

(2) the coefficients a_{jl} are continuous on E for $j,l = 1,\dots,n$ and satisfy the Hölder condition

$$|a_{jl}(t,x) - a_{jl}(\bar{t},\bar{x})| \le c''(|t-\bar{t}|^{\alpha/2} + |x-\bar{x}|^{\alpha}) \qquad (j,l = 1,\dots,n),$$

where $c'' > 0$, $\alpha \in (0,1]$.

We recall a priori estimates for the fundamental solution (see [6] and [9]).

LEMMA 2.3. *If Assumption* [a] *holds, then there are positive constants* k_0, c_0, c_1, c_α, $c_{1+\alpha}$ *such that*

$$|\Gamma(t,x;s,y)| \le c_0(t-s)^{-n/2}\exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big),$$

$$\|\partial_x \Gamma(t,x;s,y)\| \le c_1(t-s)^{-(n+1)/2}\exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big),$$

$$|\Gamma(t,x;s,y) - \Gamma(\bar{t},\bar{x};s,y)| \le c_\alpha(t-s)^{-(n+\alpha)/2}\exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big)[|t-\bar{t}|^{\alpha/2} + |x-\bar{x}|^{\alpha}],$$

$$\|\partial_x \Gamma(t,x,s,y) - \partial_x \Gamma(\bar{t},\bar{x},s,y)\| \le c_{1+\alpha}(t-s)^{-(n+1+\alpha)/2}\exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big)[|t-\bar{t}|^{\alpha/2} + |x-\bar{x}|^{\alpha}],$$

for all $0 \le s < t < \bar{t} \le a$ *and* $x, \bar{x}, y \in \mathbb{R}^n$ *and* $\alpha \in (0,1]$.

From Lemma 2.3 we deduce that

$$\int_{\mathbb{R}^n} |\Gamma(t,x;s,y)|\,dy \le \widetilde{c}_0, \quad \text{where} \quad \widetilde{c}_0 = c_0\Big(\frac{4\pi}{k_0}\Big)^{n/2}.$$

Since $\int_{\mathbb{R}^n} \Gamma(t,x;s,y)\,dy = 1$, we have the obvious inequality $\widetilde{c}_0 \ge 1$. We introduce the auxiliary function $S_0 = 1/\sqrt{t}$ for $t > 0$ and the convolution operator $*$ as follows:

$$(g_1 * g_2)(t) := \int_0^t g_1(t-s)g_2(s)\,ds \qquad \text{for } t > 0,$$

where $g_1, g_2 \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

We now formulate the main assumptions on the right side of the equation (2.1) and the initial function φ.

ASSUMPTION $[f,\varphi]$. Suppose that

(1) the function $f(\cdot,x,w) : [0,a] \to \mathbb{R}$ is measurable for $(x,w) \in \mathbb{R}^n \times C(B)$ and $f(t,\cdot) : \mathbb{R}^n \times C(B) \to \mathbb{R}$ is continuous for all $t \in [0,a]$,

(2) there is a positive function $\psi \in \mathcal{C}^+$ such that

$$\frac{k_0\psi(s)}{k_0 - 4\psi(s)(t-s)} \le \psi(t) \quad \text{for } 0 \le s \le t \le a, \tag{2.4}$$

where k_0 is the constant of Lemma 2.3,

(3) $\varphi \in C(E_0)$ and $|\varphi(t,x)| \le K_\varphi \exp(\psi(0)|x|^2)$ on E_0, where $K_\varphi > 0$,

(4) there are functions $m, \lambda, \widetilde{\lambda} \in L^1[0,a]$ such that

$$\begin{aligned}|f(t,x,w)| \le{}& m(t)\exp(\psi(t)|x|^2) \\ &+ \lambda(t)\|w(\cdot,0)\| + \widetilde{\lambda}(t)\|w\|\exp(-2\psi(t)\langle |x|,\tau\rangle)\end{aligned} \tag{2.5}$$

on $E \times C(B)$.

Assumption $[W+]$. Suppose that

(1) the function f is *quasimonotone nondecreasing*, i.e.

$$w \le \bar{w} \text{ and } w(0,0) = \bar{w}(0,0) \;\Rightarrow\; f(t,x,w) \le f(t,x,\bar{w}),$$

(2) there is an upper function $u^{(0)}$ of the form $u^{(0)}(t,x) = C(t)e^{\psi(t)|x|^2}$ on $E_0 \cup E$, where $C, \psi \in \mathcal{C}^+$, which means that $u^{(0)} \ge \varphi$ on E_0 and

$$\mathcal{P}u^{(0)}(t,x) \ge f(t,x,u^{(0)}_{(t,x)}) \quad \text{on } E,$$

(3) there is $\psi_0 \in \mathcal{C}^+$ satisfying (2.4) on $[0,a]$ with $\psi_0 > \psi$ and there are functions $m, \lambda, \widetilde{\lambda} \in L^1[0,a]$ such that

$$\begin{aligned}|f(t,x,w)| \le{}& m(t)\exp(\psi_0(t)|x|^2) + \lambda(t)\|w(\cdot,0)\| \\ &+ \widetilde{\lambda}(t)\|w\|\exp(-2\psi_0(t)\langle |x|,\tau\rangle)\end{aligned}$$

on $E \times C(B)$.

Remark 2.4. The most important example of a function $\psi \in \mathcal{C}^+$ satisfying (2.4) is

$$\psi(t) = \begin{cases} k_0C/(k_0 - 4Ct) & \text{for } 0 \le t < 1/(4C), \\ C & \text{for } t \le 0, \end{cases}$$

where $C \ge 0$ (see [11]).

Remark 2.5. Assumption $[W+](1)$ generalizes Ważewski's condition $(W+)$, called quasi-monotonicity, meaning that the right-hand side is nondecreasing in all arguments of the unknown functions but (t,x); for instance the function

$$f(t,x,u_{(t,x)}) = u(t,x+\tau/2) - \sqrt{|u(t,x)|} + \int_{-\tau}^{\tau} u(t,y)\,dy$$

satisfies the latter condition. This example does not satisfy condition (3), because the coefficients of $u(t,x+\tau/2)$ and $u(t,y)$ do not tend to zero as

$|x| \to \infty$. The following modification satisfies this condition:

$$f(t,x,u_{(t,x)}) = u(t,x+\tau/2)e^{-K|x+\tau/2|} - \sqrt{|u(t,x)|} + \int_{-\tau}^{\tau} u(t,y)e^{-K|y|}\,dy,$$

provided that K is sufficiently large.

REMARK 2.6. It follows from Assumption $[W+](3)$ that for all $\widetilde{\psi} \in \mathcal{C}^+$ satisfying (2.4) on $[0,a]$ such that $\psi \le \widetilde{\psi} \le \psi_0$ we have

$$\forall_{u\in C(\widetilde{E})}\Big(\sup_{(t,x)\in\widetilde{E}} |u(t,x)|e^{-\widetilde{\psi}(t)|x|^2} < \infty \\ \Rightarrow\ \exists_{\widetilde{m}\in L^1[0,a]}\,\forall_{t\in[0,a]}\ |f(t,x,u_{(t,x)})|e^{-\widetilde{\psi}(t)|x|^2} \le \widetilde{m}(t)\Big).$$

3. Existence of C^0 solutions. In this section we prove the existence of weak solutions of problem (2.1)–(2.2) in a class of continuous functions satisfying the growth condition $|u(t,x)| \le C\exp(K|x|^2)$.

Suppose that $\gamma,\psi \in \mathcal{C}^+$ are given functions. Define the set of admissible functions

$$\mathcal{X} = \{u \in C(\widetilde{E}) : \forall_{(t,x)\in\widetilde{E}}\ |(u-\widetilde{\varphi})(t,x)| \le \gamma(t)\exp(\psi(t)|x|^2)\} \tag{3.1}$$

and the following metric $d_{\mathcal{X}} : \mathcal{X}\times\mathcal{X} \to \mathbb{R}_+$:

$$d_{\mathcal{X}}(u,\bar{u}) = \sup_{\substack{(t,x)\in E\\ \gamma(t)\ne 0}} \frac{|u(t,x)-\bar{u}(t,x)|}{\gamma(t)\exp(\psi(t)|x|^2)} \tag{3.2}$$

for $u,\bar{u} \in \mathcal{X}$. We formulate a technical lemma on functions $u \in \mathcal{X}$ and their compositions with the generalized Hale operator.

LEMMA 3.1. *Let $\gamma,\psi \in \mathcal{C}^+$. If $u \in \mathcal{X}$ and $(s,y) \in E$, then $d_{\mathcal{X}}(u,\widetilde{\varphi}) \le 1$ and*

$$\|u_{(s,y)} - \widetilde{\varphi}_{(s,y)}\| \le \gamma(s)\exp(\psi(s)|\,|y|+\tau|^2).$$

Proof. It follows from (3.2) that $d_{\mathcal{X}}(u,\widetilde{\varphi}) \le 1$ for $u \in \mathcal{X}$. Moreover, taking into account the monotonicity of ψ and γ, we have

$$\begin{aligned}\|u_{(s,y)} - \widetilde{\varphi}_{(s,y)}\| &= \sup_{(s',y')\in B} |(u-\widetilde{\varphi})(s'+s,y'+y)| \\ &\le d_{\mathcal{X}}(u,\widetilde{\varphi}) \sup_{\substack{(s',y')\in B\\ \gamma(s'+s)\ne 0}} [\gamma(s'+s)\exp(\psi(s'+s)|y'+y|^2)] \\ &\le \gamma(s)\exp(\psi(s)|\,|y|+\tau|^2). \ \blacksquare\end{aligned}$$

LEMMA 3.2 ([11]). *If $0 \le B < A$, then*

$$\int_{\mathbb{R}^n} \exp(-A|x-y|^2 + B|y|^2)\,dy = \Big(\frac{\pi}{A-B}\Big)^{n/2} \exp\Big(\frac{AB}{A-B}|x|^2\Big).$$

LEMMA 3.3. *If Assumption* $[f,\varphi]$ *is satisfied, then there is* $\widetilde{m}\in L^1[0,a]$ *such that*

$$|f(t,x,u_{(t,x)})|\le \widetilde{m}(t)\exp(\psi(t)|x|^2)\quad \text{for } u\in\mathcal{X} \text{ and } t\in[0,a].$$

Proof. Applying the estimates of the Green function Γ, Assumption $[f,\varphi](1)$–(3) and Lemma 3.2, we get

$$\begin{aligned}|\widetilde{\varphi}(t,x)| &\le \widetilde{K}_\varphi \exp(\psi(t)|x|^2),\\ \|\widetilde{\varphi}_{(t,x)}\| &\le \widetilde{K}_\varphi \exp(\psi(t)|\,|x|+\tau|^2),\end{aligned}\tag{3.3}$$

where $\widetilde{K}_\varphi=\widetilde{c}_0K_\varphi[\psi(a)/\psi(0)]^{n/2}$. Then in view of Assumption $[f,\varphi](4)$ we have

$$\begin{aligned}|f(t,x,u_{(t,x)})|&\\ &\le m(t)\exp(\psi(t)|x|^2)+\lambda(t)\sup_{0\le t'\le t}|u(t',x)-\widetilde{\varphi}(t',x)|\\ &\quad+\lambda(t)\sup_{0\le t'\le t}|\widetilde{\varphi}(t',x)|+\widetilde{\lambda}(t)\|u_{(t,x)}-\widetilde{\varphi}_{(t,x)}\|\exp(-2\psi(t)\langle|x|,\tau\rangle)\\ &\quad+\widetilde{\lambda}(t)\|\widetilde{\varphi}_{(t,x)}\|\exp(-2\psi(t)\langle|x|,\tau\rangle).\end{aligned}$$

Finally Assumption $[f,\varphi](1)$–(3), Lemma 3.1 and the estimates (3.3) imply

$$|f(t,x,u_{(t,x)})|\le \widetilde{m}(t)\exp(\psi(t)|x|^2),$$

where

$$\widetilde{m}(t)=m(t)+(\gamma(t)+\widetilde{K}_\varphi)\big(\lambda(t)+\widetilde{\lambda}(t)\exp(\psi(t)|\tau|^2)\big).\ \blacksquare\tag{3.4}$$

We now formulate sufficient conditions for existence of unbounded weak solutions of (2.1)–(2.2) which satisfy natural growth restrictions. Let $S_0^\alpha(t)=(1/\sqrt{t}\,)^\alpha$.

THEOREM 3.4. *Suppose that Assumption* $[f,\varphi]$ *is fulfilled and the convolutions* $S_0^\alpha * m$, $S_0^\alpha*\lambda$ *and* $S_0^\alpha*\widetilde{\lambda}$ *exist. Then there exists a* C^0 *solution* u *to problem* (2.1)–(2.2) *satisfying the growth condition*

$$|u(t,x)|\le C\exp(\psi(t)|x|^2)\quad \text{for } (t,x)\in\widetilde{E}.$$

The proof of the main existence theorem will be preceded by a useful lemma. First, we define an integral operator on $\mathcal{X}$ by

$$\mathcal{T}u(t,x)=\widetilde{\varphi}(t,x)+\int_0^t\int_{\mathbb{R}^n}\Gamma(t,x;s,y)f(s,y,u_{(s,y)})\,dy\,ds\tag{3.5}$$

for $(t,x)\in E$, and $\mathcal{T}u(t,x)=\varphi(t,x)$ on E_0. Then problem (2.1)–(2.2) in $\mathcal{X}$ is equivalent to the fixed-point equation $u=\mathcal{T}u$.

Lemma 3.5. *If Assumption* $[f,\varphi]$ *is satisfied and* $\mathcal{X}$ *is defined by* (3.1), *then* $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ *provided that*

$$\gamma(t)=\widetilde{c_0}\int_0^t\Big[\frac{\psi(t)}{\psi(s)}\Big]^{n/2} \times\big\{m(s)+(\gamma(s)+\widetilde{K}_\varphi)\big(\lambda(s)+\widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2)\big)\big\}\,ds. \tag{3.6}$$

Proof. The proof of the implication $u\in\mathcal{X}\Rightarrow\mathcal{T}u\in\mathcal{X}$ will be carried out in two steps.

Step I. First, we show that

$$|\mathcal{T}u(t,x)-\widetilde{\varphi}(t,x)|\le\gamma(t)\exp(\psi(t)|x|^2)\quad\text{ on } E.$$

Suppose that $u\in\mathcal{X}$ and $(t,x)\in E$. The operator $\mathcal{T}$, by its definition (3.5), satisfies

$$|(\mathcal{T}u-\widetilde{\varphi})(t,x)|\le\int_0^t\int_{\mathbb{R}^n}|\Gamma(t,x;s,y)f(s,y,u_{(s,y)})|\,dy\,ds.$$

Applying Lemmas 2.3, 3.1 and 3.3 we get

$$|(\mathcal{T}u-\widetilde{\varphi})(t,x)| \le\int_0^t\int_{\mathbb{R}^n}c_0(t-s)^{-n/2}\exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big)\widetilde{m}(s)\exp(\psi(s)|y|^2)\,dy\,ds.$$

By Lemma 3.2 with $A=\frac{k_0}{4(t-s)}$, $B=\psi(s)$ and the inequality (2.4) we have

$$|(\mathcal{T}u-\widetilde{\varphi})(t,x)|\le\widetilde{c_0}[\psi(t)]^{n/2}\exp(\psi(t)|x|^2)\int_0^t\frac{\widetilde{m}(s)}{[\psi(s)]^{n/2}}\,ds,$$

where the function $\widetilde{m}(t)$ is given by (3.4). Then

$$|(\mathcal{T}u-\widetilde{\varphi})(t,x)|\le\gamma(t)\exp(\psi(t)|x|^2), \tag{3.7}$$

where γ satisfies the integral equation (3.6).

Step II. We prove that $\mathcal{T}u$ is continuous. Take $t\in(0,a]$ and $x,\bar{x}\in\mathbb{R}^n$. From Lemmas 2.3 and 3.3 we get

$$\begin{aligned}|\mathcal{T}u(t,\bar{x})-\mathcal{T}u(t,x)| &\le|\widetilde{\varphi}(t,\bar{x})-\widetilde{\varphi}(t,x)|+\int_0^t\int_{\mathbb{R}^n}|\Gamma(t,\bar{x};s,y)-\Gamma(t,x;s,y)|\,|f(s,y,u_{(s,y)})|\,dy\,ds\\ &\le\widetilde{K}_{\varphi,\alpha}|\bar{x}-x|^\alpha\exp(\psi(t)|x|^2)\Big\{t^{-\alpha/2}+\frac{1}{K_\varphi}(S_0^\alpha*\widetilde{m})(t)\Big\},\end{aligned}$$

where $\widetilde{K}_{\varphi,\alpha} = \widetilde{c}_\alpha K_\varphi[\psi(a)/\psi(0)]^{n/2}$ and $\widetilde{m}$ is given by (3.4). Hence the set $\mathcal{T}(\mathcal{X})$ is equicontinuous in x on all compact subsets of E. We show its continuity in t. Take arbitrary t and $\bar{t}$ such that $0 < t < \bar{t} \le a$. Then

$$\begin{aligned}
|\mathcal{T}u(t,x) - \mathcal{T}u(\bar{t},x)| &\le |\widetilde{\varphi}(t,x) - \widetilde{\varphi}(\bar{t},x)| \\
&\quad + \int_0^t \int_{\mathbb{R}^n} |\Gamma(t,x;s,y) - \Gamma(\bar{t},x;s,y)|\,|f(s,y,u_{(s,y)})|\,dy\,ds \\
&\quad + \int_t^{\bar{t}} \int_{\mathbb{R}^n} |\Gamma(\bar{t},x;s,y)|\,|f(s,y,u_{(s,y)})|\,dy\,ds \\
&\le |\widetilde{\varphi}(t,x) - \widetilde{\varphi}(\bar{t},x)| + \widetilde{K}_{\varphi,\alpha}|\bar{t}-t|^{\alpha/2}\exp(\psi(t)|x|^2)\int_0^t (t-s)^{-\alpha/2}\widetilde{m}(s)\,ds \\
&\quad + \widetilde{K}_\varphi \exp(\psi(\bar{t})|x|^2)\int_t^{\bar{t}} \widetilde{m}(s)\,ds.
\end{aligned}$$

Since $\gamma(0) = 0$, continuity of $\mathcal{T}u$ at $t = 0$ follows from (3.7). Thus $\mathcal{T}$ maps $\mathcal{X}$ into itself, and the proof is complete. ∎

Proof of Theorem 3.4. Observe that $\mathcal{X}$ is the unit ball centered at $\widetilde{\varphi}$ in $\mathcal{X}$ with respect to $d_{\mathcal{X}}$. Thus, this set is bounded, convex and closed with respect to $d_{\mathcal{X}}$. Lemma 3.5 shows that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$. Moreover, $\mathcal{T}$ is compact, because $\mathcal{X}$ is a closed set and $\mathcal{T}(\mathcal{X})$ is a family of uniformly bounded and equicontinuous functions on all compact subsets of E. The equicontinuity of $\mathcal{T}(\mathcal{X})$ follows from the estimates in Step II of the proof of Lemma 3.5, which are independent of the choice of $u \in \mathcal{X}$.

Now, we show that $\mathcal{T}(\mathcal{X})$ is uniformly bounded on compact sets. This follows from the estimate

$$\begin{aligned}
|\mathcal{T}u(t,x)| &\le |(\mathcal{T}u - \widetilde{\varphi})(t,x)| + |\widetilde{\varphi}(t,x)| \\
&\le \gamma(a)\exp(\psi(t)|x|^2) + |\widetilde{\varphi}(t,x)|.
\end{aligned}$$

The continuity of $f = f(t,x,w)$ in w implies that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is continuous. By the Schauder fixed point theorem there exists $u \in \mathcal{X}$ such that $u = \mathcal{T}u$, which is a C^0 solution of problem (2.1)–(2.2). ∎

We now illustrate the above existence theory.

Example 3.6. Let $n = 1$. Consider the Cauchy problem

$$\partial_t u(t,x) - \partial_{xx}u(t,x) = \sqrt{u(t,x+1)}, \qquad u(0,x) = e^{x^2}.$$

We are looking for nonnegative solutions. It is clear that Theorem 3.4 is applicable to this problem. It is known (from ODE theory) that the same equation with another initial condition $u(0,x) = 0$ has at least two solutions $u \equiv 0$ and $u(t,x) = t^2/4$.

4. Existence of $C^{0,1}$ solutions. Fix $\psi, \gamma_1 \in \mathcal{C}^+$. Define the set of admissible functions

$$\mathcal{X}' = \big\{u \in \mathcal{X} : \partial_x u \in C(\widetilde{E}, \mathbb{R}^n);\\ \forall_{(t,x)\in\widetilde{E}}\ \|\partial_x(u - \widetilde{\varphi})(t,x)\| \le \gamma_1(t)\exp(\psi(t)|x|^2)\big\}$$

and the function $d_{\mathcal{X}'} : \mathcal{X}' \times \mathcal{X}' \to \mathbb{R}_+$ by

$$d_{\mathcal{X}'}(u,\bar{u}) = \sup_{\substack{(t,x)\in E\\ \gamma_1(t)\neq 0}} \frac{\|\partial_x u(t,x) - \partial_x \bar{u}(t,x)\|}{\gamma_1(t)\exp(\psi(t)|x|^2)} \tag{4.1}$$

for $u, \bar{u} \in \mathcal{X}'$. It is easily seen that $d_{\mathcal{X}'}(u, \widetilde{\varphi}) \le 1$ for $u \in \mathcal{X}'$.

THEOREM 4.1. *Suppose that Assumption* $[f, \varphi]$ *is satisfied and*

(1) $\partial_x \varphi \in C(E_0, \mathbb{R}^n)$ *and* $\|\partial_x\varphi(t,x)\| \le K_{\varphi'}\exp(\psi(t)|x|^2)$ *with* $K_{\varphi'} > 0$,
(2) *there are functions* $m, \lambda, \widetilde{\lambda}, \lambda_1, \widetilde{\lambda}_1 \in L^1[0,a]$ *such that*

$$|f(t,x,w)| \le m(t)\exp(\psi(t)|x|^2) + \lambda(t)\|w(\cdot,0)\|\\ + \widetilde{\lambda}(t)\|w\|\exp(-2\psi(t)\langle|x|,\tau\rangle) + \lambda_1(t)\|\partial_x w(\cdot,0)\|\\ + \widetilde{\lambda}_1(t)\|\partial_x w\|\exp(-2\psi(t)\langle|x|,\tau\rangle)$$

on $E \times C(B)$,
(3) *the functions* γ *and* γ_1 *satisfy the system of integral equations*

$$\begin{cases} \gamma(t) = \displaystyle\int_0^t \widetilde{c}_0 \left[\frac{\psi(t)}{\psi(s)}\right]^{n/2} \\ \qquad \times \big\{m(s) + (\gamma(s) + \widetilde{K}_\varphi)\big(\lambda(s) + \widetilde{\lambda}(s)\exp(\psi(s)|\tau|^2)\big) \\ \qquad\quad + (\gamma_1(s) + K)\big(\lambda_1(s) + \widetilde{\lambda}_1(s)\exp(\psi(s)|\tau|^2)\big)\big\}\, ds, \\ \gamma_1(t) = \displaystyle\int_0^t \widetilde{c}_1 (t-s)^{-1/2}\left[\frac{\psi(t)}{\psi(s)}\right]^{n/2} \\ \qquad \times \big\{m(s) + (\gamma(s) + \widetilde{K}_\varphi)\big(\lambda(s) + \widetilde{\lambda}(s), \exp(\psi(s)|\tau|^2)\big) \\ \qquad\quad + (\gamma_1(s) + K)\big(\lambda_1(s) + \widetilde{\lambda}_1(s)\exp(\psi(s)|\tau|^2)\big)\big\}\, ds, \end{cases} \tag{4.2}$$

where the constants $\widetilde{K}_\varphi$ *and* $K = K_{\varphi,\psi,c_0,c_1}$ *are positive,*
(4) *the convolutions* $S_0^{1+\alpha} * m$, $S_0^{1+\alpha} * \lambda$, $S_0^{1+\alpha} * \widetilde{\lambda}$, $S_0^{1+\alpha} * \lambda_1$ *and* $S_0^{1+\alpha} * \widetilde{\lambda}_1$ *exist, where* $S_0^{1+\alpha}(t) = (1/\sqrt{t}\,)^{1+\alpha}$.

Then there exists a $C^{0,1}$ *solution* u *to problem* (2.1)–(2.2) *satisfying the growth condition*

$$|u(t,x)| \le C\exp(\psi(t)|x|^2) \quad \textit{for } (t,x) \in \widetilde{E}.$$

Proof. We introduce the metric $\max\{d_{\mathcal{X}}, d_{\mathcal{X}'}\}$, where $d_{\mathcal{X}}$ and $d_{\mathcal{X}'}$ are defined by (3.2) and (4.1). We see at once that the inequalities (3.3) are satisfied and in a similar way we can show that

$$\|\partial_x \widetilde{\varphi}(t,x)\| \le K \exp(\psi(t)|x|^2),$$
$$\|\partial_x \widetilde{\varphi}_{(t,x)}\| \le K \exp(\psi(t)||x|+\tau|^2),$$

where $K = K_{\varphi,\psi,c_0,c_1}$. Taking arbitrary $u, \bar{u} \in \mathcal{X}'$, similarly to Theorem 3.4, we obtain

$$\begin{aligned}
|f(t,x,u_{(t,x)})| &\le m(t)\exp(\psi(t)|x|^2) + \lambda(t) \sup_{0\le t'\le t} |(u-\widetilde{\varphi})(t',x)| \\
&\quad + \lambda(t) \sup_{0\le t'\le t} |\widetilde{\varphi}(t',x)| + \lambda_1(t)\|\partial_x(u-\widetilde{\varphi})(t,x)\| + \lambda_1(t)\|\partial_x\widetilde{\varphi}(t,x)\| \\
&\quad + \widetilde{\lambda}(t)\exp(-2\psi(t)\langle |x|,\tau\rangle)\big[\|(u-\widetilde{\varphi})_{(t,x)}\| + \|\widetilde{\varphi}_{(t,x)}\|\big] \\
&\quad + \widetilde{\lambda}_1(t)\exp(-2\psi(t)\langle |x|,\tau\rangle)\big[\|\partial_x(u-\widetilde{\varphi})_{(t,x)}\| + \|\partial_x\widetilde{\varphi}_{(t,x)}\|\big] \\
&\le \exp(\psi(t)|x|^2)\big\{\widetilde{m}(t) + (\gamma_1(t)+K)(\lambda_1(t) + \widetilde{\lambda}_1(t)\ \exp(\psi(t)|\tau|^2))\big\},
\end{aligned}$$

where $\widetilde{m}$ is defined by (3.4). Applying Lemmas 2.3, 3.2 and the above inequalities, we get

$$\begin{aligned}
&|(\mathcal{T}u-\widetilde{\varphi})(t,x)| \\
&\qquad \le \int_0^t \int_{\mathbb{R}^n} c_0(t-s)^{-n/2}\exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big)\widetilde{m}_1(s)\exp(\psi(s)|y|^2)\,dy\,ds \\
&\qquad \le \widetilde{c}_0\exp(\psi(t)|x|^2)\int_0^t \Big[\frac{\psi(t)}{\psi(s)}\Big]^{n/2}\widetilde{m}_1(s)\,ds
\end{aligned}$$

and

$$\begin{aligned}
|\partial_{x_j}(\mathcal{T}u-\widetilde{\varphi})(t,x)| &\le \int_0^t \int_{\mathbb{R}^n} |\partial_{x_j}\Gamma(t,x,s,y)|\,|f(s,y,u_{(s,y)})|\,dy\,ds \\
&\le \int_0^t \int_{\mathbb{R}^n} c_1(t-s)^{-(n+1)/2}\ \exp\Big(-\frac{k_0|x-y|^2}{4(t-s)}\Big)\widetilde{m}_1(s)\exp(\psi(s)|y|^2)\,dy\,ds \\
&\le \widetilde{c}_1\exp(\psi(t)|x|^2)\int_0^t \Big[\frac{\psi(t)}{\psi(s)}\Big]^{n/2}\widetilde{m}_1(s)(t-s)^{-1/2}\,ds,
\end{aligned}$$

where $\widetilde{m}$ is given by (3.4) and

$$\widetilde{m}_1(t) = \widetilde{m}(t) + (\gamma_1(s)+K)\big(\lambda_1(s) + \widetilde{\lambda}_1(s)\exp(\psi(s)|\tau|^2)\big).$$

Then

$$|(\mathcal{T}u-\widetilde{\varphi})(t,x)| \le \gamma(t)\exp(\psi(t)|x|^2),$$
$$\|\partial_x(\mathcal{T}u-\widetilde{\varphi})(t,x)\| \le \gamma_1(t)\exp(\psi(t)|x|^2),$$

where γ and γ_1 satisfy (4.2). Hence $\mathcal{T}u \in \mathcal{X}'$, which means that $\mathcal{T}$ maps $\mathcal{X}'$ into itself. Observe that $\mathcal{T}(\mathcal{X}')$ and $\partial_x\mathcal{T}(\mathcal{X}') = \{\partial_x\mathcal{T}u : u \in \mathcal{X}'\}$ consist of uniformly bounded functions on compact subsets of E. The analysis similar to that in Step II of the proof of Lemma 3.5 shows that $\mathcal{T}(\mathcal{X}')$ is a set of equicontinuous functions. What is left is to show that $\partial_x\mathcal{T}(\mathcal{X}')$ is a set of equicontinuous functions. If $u \in \partial_x\mathcal{T}(\mathcal{X}')$, then

$$\|\partial_x\mathcal{T}u(\bar{t},x) - \partial_x\mathcal{T}u(t,x)\| \le \|\partial_x\widetilde{\varphi}(\bar{t},x) - \partial_x\widetilde{\varphi}(t,x)\|$$
$$+ \int_0^t \widetilde{c}_{1+\alpha}(t-s)^{-1+\alpha/2}|\bar{t}-t|^{\alpha/2}\widetilde{m}_1(s)\,ds + \int_t^{\bar{t}} \widetilde{c}_{1+\alpha}(t-s)^{-1+\alpha/2}\widetilde{m}_1(s)\,ds$$

for all $0 < t < \bar{t} \le a$. Note that the right side does not depend on u. The equicontinuity of $\partial_x\mathcal{T}(\mathcal{X}')$ in x can be shown in a similar way. Thus $\mathcal{T}(\mathcal{X}')$ is a relatively compact set, so the operator $\mathcal{T}$ is compact. Furthermore, since f is a continuous function in w, the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is continuous with respect to the metric $\max\{d_{\mathcal{X}}, d_{\mathcal{X}'}\}$. By the Schauder fixed point theorem, $\mathcal{T}$ has a fixed point. ■

Remark 4.2. In Theorem 4.1 we can replace the system (4.2) by a single equation

$$\gamma^*(t) = \int_0^t [\widetilde{c}_0 + \widetilde{c}_1(t-s)^{-1/2}][M_0(s) + M_1(s)\gamma^*(s)]\,ds,$$

where $M_0, M_1 \in L^1[0,a]$ depend on $m, \lambda, \widetilde{\lambda}, \lambda_1, \widetilde{\lambda}_1$. Its solution γ^* estimates $\gamma + \gamma_1$.

Example 4.3. Let $n = 1$. Consider the integro-differential equation

$$\partial_t u(t,x) - \partial_{xx}u(t,x) = \Big(\int_{x-1}^{x+1} \sin^2(\partial_x u(t,y))\,dy\Big)^{1/4}.$$

The right-hand side $f : E \times C(B) \to \mathbb{R}$ can be specified as follows:

$$f(t,x,w) = \Big(\int_{-1}^{1} \sin^2(\partial_x w(0,y))\,dy\Big)^{1/4} \quad \text{for } (t,x) \in E,\ w \in C(B)$$

(see Preliminaries). The assumptions of Theorem 4.1 are satisfied with $\tau = 1$, $m(t) = 2^{1/4}$ and $\lambda(t) = \widetilde{\lambda}(t) = \lambda_1(t) = \widetilde{\lambda}_1(t) = 0$ for all t.

Remark 4.4. If the functional dependence applies only to the unknown function, and its derivative appears in the classical way, a result on existence of unbounded solutions can be formulated as follows: there are solutions in the class of functions satisfying $|(u - \widetilde{\varphi})(t,x)| \le \gamma(t)\exp(\psi(t)|x|^2)$ and

$\|\partial_x(u-\widetilde{\varphi})(t,x)\| \le \gamma_1(t)\exp(\psi(t)|x|^2)$ if we assume that

$$\begin{aligned}|f(t,x,w)| \le m(t) + \lambda(t)\|w(\cdot,0)\| + \widetilde{\lambda}(t)\|w\|\exp(-2\psi(t)\langle|x|,\tau\rangle)\\ + \lambda_1(t)\sup_{\substack{-\tau_0\le s\le 0\\ 0<s+t}}\sqrt{s+t}\,\|\partial_x w(s,0)\|\\ + \widetilde{\lambda}_1(t)\sup_{\substack{(s,y)\in B\\ 0<s+t}}\sqrt{s+t}\,\|\partial_x w(s,y)\|\exp(-2\psi(t)\langle|x|,\tau\rangle).\end{aligned}$$

The function $\psi \in \mathcal{C}^+$ satisfies (2.4), while γ, γ_1 are chosen so that the respective closed and convex sets of functions are transformed by the operator $\mathcal{T}$ into themselves. In this case we must assume the existence of the following convolutions: $S_0^{1+\alpha} * m$, $S_0^{1+\alpha} * \lambda$, $S_0^{1+\alpha} * \widetilde{\lambda}$ and $S_0^{1+\alpha} * (S_0 \cdot \lambda_1)$, $S_0^{1+\alpha} * (S_0 \cdot \widetilde{\lambda}_1)$.

5. Maximal solutions. Before the general existence theorem, whose main idea goes back to [13] and [18], we provide an example which illustrates our technique. It is well known from ODE theory that the Cauchy problem $dz/dt = \sqrt{z}$ with initial condition $z(0) = 0$ has the minimal solution $z(t) = 0$ for $t \ge 0$ and the maximal solution $z = t^2/4$ for $t \ge 0$. The maximal solution can be approximated by any decreasing sequence of functions $z_k : \mathbb{R}_+ \to \mathbb{R}$ which satisfy for $k \ge 1$ the Cauchy problems

$$\frac{dz_k}{dt} = \sqrt{z_k} + \varepsilon_k, \qquad z_k(0) = \varepsilon_k, \qquad \lim_{k\to\infty} \varepsilon_k = 0.$$

Since the function $g(x) = \sqrt{x}$ satisfies a local Lipschitz condition for $x > 0$, these iterations for sufficiently small $t \ge 0$ can be replaced by

$$\frac{dz_k}{dt} = \sqrt{z_{k-1}}, \qquad z_k(0) = \varepsilon_k, \qquad k \ge 1,$$

starting from any constant $z_0 \equiv C > 0$. This concept will be exploited in the following example.

EXAMPLE 5.1. Let $n = 1$. Consider the Cauchy problem

$$\partial_t u(t,x) - \partial_{xx} u(t,x) = \sqrt{u_+(t,x+1)}, \qquad u(0,x) = 0, \tag{5.1}$$

where $u_+ = \max\{0,u\}$. We prove that equation (5.1) has the same minimal solution $u(t,x) = 0$ and the same maximal solution $u(t,x) = t^2/4$ in the classes $|u(t,x)| \le C\exp(Kx^2)$. It is clear that zero is the minimal solution, because the right-hand side is nonnegative. Fix $C > 0$ and $K > 0$ in the growth restriction. Then we can find an upper solution u_0 of the form $u_0(t,x) = C(t)\exp(K(t)x^2)$, where $C(0) = C$ and $K(0) = K$. That u_0 is an upper solution means that

$$\partial_t u_0(t,x) - \partial_{xx} u_0(t,x) \ge \sqrt{u_0(t,x+1)}.$$

This inequality can be satisfied for sufficiently small $t \ge 0$ when $K(t)$ satisfies the Riccati equation $K'(t) - 4K^2(t) = 0$ and $C(t)$ satisfies the ODE

$$C'(t) - 2K(t)C(t) = \sqrt{C(t)}\, e^{K^2(t)}.$$

Define an approximate sequence u_k which satisfies for $k \geq 1$ the Cauchy problems

$$\partial_t u_k(t,x) - \partial_{xx} u_k(t,x) = \sqrt{u_{k-1}(t,x+1)}, \qquad u_k(0,x) = \varepsilon_k,$$

where the sequence ε_k increases to zero and $\varepsilon_0 = C$. It is easy to check that $u_0 \geq u_1 \geq u_2 \geq \cdots$. Thus the decreasing sequence of nonnegative functions u_k has a limit $u \geq 0$. We show that $u(t,x) = t^2/4$. One can prove by induction on k that these functions are estimated by $C_k(t)\exp(K_k(t)x^2)$, where

$$K_k'(t) = 4K_k^2(t), \qquad K_k(0) = K(3/4)^k,$$
$$C_k'(t) = 2K_k(t)C_k(t) + \sqrt{C_{k-1}}\exp\left(\frac{K_{k-1}(t)K_k(t)}{2(K_k(t) - K_{k-1}(t)/2)}\right),\ C_k(0) = \varepsilon_k,$$

for $k \geq 1$. By continuous dependence the sequence $K_k(t)$ tends to zero, hence $C_k(t)$ tends to the maximal solution of the Cauchy problem

$$\frac{dz}{dt} = 0 + \sqrt{z}\, e^0, \qquad z(0) = 0.$$

Hence $u(t,x) \leq z(t) = t^2/4$. On the other hand, this function satisfies equation (5.1), thus it is the maximal solution of (5.1).

Consider again problem (2.1)–(2.2). We still need Assumptions $[a]$ and $[f,\varphi](1)$–(3) concerning the coefficients of the differential operator $\mathcal{P}$ and the right-hand side of (2.1). Instead of condition (4) of Assumption $[f,\varphi]$ we impose Assumption $[W+]$.

We formulate our main result on maximal solutions

Theorem 5.2. *Suppose that Assumptions* $[a]$, $[f,\varphi]$ *and* $[W+]$ *are satisfied. Then there exists a maximal* C^0 *solution* u *of* (2.1)–(2.2) *such that*

$$|u(t,x)| \leq C\exp(\psi(t)|x|^2) \quad \textit{for } (t,x) \in \widetilde{E}.$$

Proof. Assume $u^{(0)}(t,x) \geq \varphi(t,x) + \varepsilon_0 e^{\psi_0(t)|x|^2}$ on E_0 and

$$\mathcal{P}u^{(0)}(t,x) \geq f(t,x,u^{(0)}_{(t,x)}) + \varepsilon_0 e^{\psi_0(t)|x|^2} \quad \text{on } E,$$

where $\varepsilon_0 = \psi_0(0) - \psi(0)$. Choose a sequence $\varepsilon_0 > \varepsilon_1 > \cdots$, decreasing to zero. Take functions $\psi_k \in \mathcal{C}^+$, satisfying (2.4) and $\psi_0 > \psi_1 > \cdots > \psi$, $\psi_k(0) - \psi(0) = \varepsilon_k$. Define a sequence of approximate solutions

$$\mathcal{P}u^{(k)}(t,x) = f(t,x,u^{(k)}_{(t,x)}) + \varepsilon_k e^{\psi_k(t)|x|^2} \quad \text{on } E,$$
$$u^{(k)}(t,x) = \varphi(t,x) + \varepsilon_k e^{\psi_k(t)|x|^2} \quad \text{on } E_0.$$

By Theorem 3.4 there exist solutions $u^{(k)}$ of these Cauchy problems such that $|u^{(k)}(t,x)| \le C\exp(\psi_k(t)|x|^2)$. We claim that the sequence is bounded and monotone, converging to a solution u^* of (2.1)–(2.2). Moreover, if u is a solution of (2.1)–(2.2) such that $|u(t,x)| \le C\exp(\psi(t)|x|^2)$, then $u \le u^{(k)}$ for each k, thus $u \le u^*$ (u^* is maximal).

Since the proof of monotonicity is based on the same idea as $u \le u^{(k)}$, we prove the second assertion. Denote $\omega^{(k)} = u^{(k)} - u$. We will prove that $\omega^{(k)} > 0$ on E. Observe that $\omega^{(k)}(t,x) = \varepsilon_k e^{\psi_k(t)|x|^2} > 0$ on E_0 and

$$\mathcal{P}\omega^{(k)}(t,x) = f(t,x,u^{(k)}_{(t,x)}) - f(t,x,u_{(t,x)}) + \varepsilon_k e^{\psi_k(t)|x|^2} \quad \text{on } E.$$

Since, near $t = 0$, we have $\omega^{(k)}(t,x) \to \infty$ as $|x| \to \infty$, the function $\omega^{(k)}(t,\cdot)$ attains its minimum for each t near 0. Suppose that there is a Nagumo point, that is, $(t,x) \in E$ with the smallest t such that $\omega^{(k)}(t,x) \le 0$. Because $\omega^{(k)}(s,y) > 0$ for all $(s,y) \in \widetilde{E}$, $s < t$, we have $\omega^{(k)}(t,x) = 0$, $\partial_t\omega^{(k)}(t,x) \le 0$, $\partial_x\omega^{(k)}(t,x) = 0$ and the quadratic form $\sum_{j,l=1}^n a_{jl}(t,x)\partial_{x_jx_l}\omega^{(k)}(t,x)$ is nonnegative definite, thus $0 \ge \mathcal{P}\omega^{(k)}(t,x)$ provided that these partial derivatives exist. On the other hand, we have $u^{(k)}(t,x) = u(t,x) + \omega^{(k)}(t,x) = u(t,x)$ and $u^{(k)}(s,y) = u(s,y) + \omega^{(k)}(s,y) < u(s,y)$ for (s,y), $s < t$. Thus by quasimonotonicity we get

$$f(t,x,u^{(k)}_{(t,x)}) - f(t,x,u_{(t,x)}) \ge 0.$$

This leads to the contradiction that

$$0 \ge \mathcal{P}\omega^{(k)}(t,x) \ge f(t,x,u^{(k)}_{(t,x)}) - f(t,x,u_{(t,x)}) + \varepsilon_k e^{\psi_k(t)|x|^2} \ge \varepsilon_k e^{\psi_k(t)|x|^2} > 0.$$

If at least one of the derivatives $\partial_t\omega^k(t,x)$ or $\partial_{x_j}\omega^{(k)}(t,x)$ or $\partial_{x_jx_l}\omega^{(k)}(t,x)$ does not exist, then we arrive at a contradiction in another way. We still have $\omega^{(k)}(t,x) = 0$. We use the inequality $\Gamma(t,x;s,y) \ge 0$ and the integral equation

$$\omega^{(k)}(t,x) = 0 = \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varepsilon_k e^{\psi_k(0)|y|^2}\,dy$$
$$+ \int_0^t \int_{\mathbb{R}^n} \Gamma(t,x;s,y)\{f(s,y,u^{(k)}_{(s,y)}) - f(s,y,u_{(s,y)}) + \varepsilon_k e^{\psi_k(s)|y|^2}\}\,dy\,ds.$$

By quasimonotonicity we get

$$0 \ge \int_{\mathbb{R}^n} \Gamma(t,x;0,y)\varepsilon_k e^{\psi_k(0)|y|^2}\,dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(t,x;s,y)\varepsilon_k e^{\psi_k(s)|y|^2}\,dy\,ds,$$

which is a contradiction since $\Gamma(t,x;s,y) \ge 0$ and the above integrals are greater than zero. ∎

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