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APPROXIMATIVE SEQUENCES AND ALMOST HOMOCLINIC SOLUTIONS FOR A CLASS OF SECOND ORDER PERTURBED HAMILTONIAN SYSTEMS

MAREK IZYDOREK and JOANNA JANCZEWSKA

Faculty of Applied Physics and Mathematics, Gdańsk University of Technology
G. Narutowicza 11/12, 80-233 Gdańsk, Poland
E-mail: izydorek@mif.pg.gda.pl, janczewska@mif.pg.gda.pl

Abstract. In this work we will consider a class of second order perturbed Hamiltonian systems of the form $\ddot{q} + V_q(t,q) = f(t)$, where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$, with a superquadratic growth condition on a time periodic potential $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a small aperiodic forcing term $f : \mathbb{R} \to \mathbb{R}^n$. To get an almost homoclinic solution we approximate the original system by time periodic ones with larger and larger time periods. These approximative systems admit periodic solutions, and an almost homoclinic solution for the original system is obtained from them by passing to the limit in $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ when the periods go to infinity. Our aim is to show the existence of two different approximative sequences of periodic solutions: one of mountain pass type and the second of local minima.

1. Introduction. In this work we will consider a class of second order perturbed Hamiltonian systems

$$\ddot{q} + V_q(t,q) = f(t) \tag{1}$$

where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$, and a potential $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a forcing term $f : \mathbb{R} \to \mathbb{R}^n$ satisfy the following hypotheses:

(H1) $V(t,q) = -K(t,q) + W(t,q)$, where $K, W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-maps, T-periodic with respect to t, $T > 0$,

(H2) there are constants $b_1, b_2 > 0$ such that for all $(t,q) \in \mathbb{R} \times \mathbb{R}^n$,

$$b_1|q|^2 \leq K(t,q) \leq b_2|q|^2,$$

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(H3) for all $(t,q) \in \mathbb{R} \times \mathbb{R}^n$, $K(t,q) \leq (q, K_q(t,q)) \leq 2K(t,q)$,
(H4) $W_q(t,q) = o(|q|)$ as $|q| \to 0$ uniformly with respect to t,
(H5) there is a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \setminus \{0\}$,

$$0 < \mu W(t,q) \leq (q, W_q(t,q)),$$

(H6) $f : \mathbb{R} \to \mathbb{R}^n$ is a continuous bounded function.

Here and subsequently, $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the standard inner product and $|\cdot| : \mathbb{R}^n \to [0,\infty)$ is the Euclidean norm.

In the literature, (H2) is called a pinching type condition, and (H5) is a superquadratic growth condition due to A. Ambrosetti and P. H. Rabinowitz. (1) is also called the Lagrangian or Newtonian system.

Let us remark that if the forcing term f is trivial and the conditions (H1)–(H5) are fulfilled then $0 \in \mathbb{R}^n$ is a stationary point of (1). Therefore it is natural to ask for the existence of homoclinic (to 0) solution of (1), i.e. a solution $Q : \mathbb{R} \to \mathbb{R}^n$ such that $(Q(t), \dot{Q}(t)) \to (0,0)$ as $t \to \pm\infty$. If f is nontrivial then $0 \in \mathbb{R}^n$ is no longer a stationary point of (1), and hence (1) does not possess homoclinics (to 0) in the classical sense. Nevertheless we can still study the existence of a solution $q : \mathbb{R} \to \mathbb{R}^n$ such that $q(t) \to 0$ as $t \to \pm\infty$. Moreover, under suitable assumptions on V, $\dot{q}(t) \to 0$ as $t \to \pm\infty$.

In many papers concerning systems of second order ODE's, solutions vanishing at $\pm\infty$ (both with the extra property that $\dot{q}(t) \to 0$ as $t \to \pm\infty$, and without it) are called homoclinic, too. See for example: [IJ, LJ, S, TL, TX, ZY]. As it might be confusing to people working in the theory of dynamical systems, in [J1] we introduced and since then we have consistently used the notion of an almost homoclinic solution (to 0) of (1).

Definition 1.1. A solution $q : \mathbb{R} \to \mathbb{R}^n$ of (1) is said to be *almost homoclinic* if $q(t) \to 0$ as $t \to \pm\infty$.

For each $k \in \mathbb{N}$, let $E_k = W^{1,2}_{2kT}(\mathbb{R}, \mathbb{R}^n)$, the Sobolev space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|q\|_{E_k} = \left(\int_{-kT}^{kT} (|q(t)|^2 + |\dot{q}(t)|^2) dt \right)^{1/2}.$$

Let $L^\infty_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote the space of $2kT$-periodic essentially bounded measurable functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|q\|_{L^\infty_{2kT}} = \operatorname{ess\,sup}\big\{|q(t)| : t \in [-kT, kT]\big\}.$$

Proposition 1.2 (see [IJ, Proposition 1.1]). *There is a positive constant C such that for each $k \in N$ and $q \in E_k$,*

$$\|q\|_{L^\infty_{2kT}} \leq C\|q\|_{E_k}. \tag{2}$$

If $T \geq \frac{1}{2}$ then the inequality (2) holds with $C = \sqrt{2}$ (see [IJ, Fact 2.8]). Set

$$\begin{aligned} M &= \sup\big\{W(t,q) : t \in [0,T],\ |q| = 1\big\}, \\ m &= \inf\big\{W(t,q) : t \in [0,T],\ |q| = 1\big\}, \\ \bar{b}_1 &= \min\{1, 2b_1\}, \\ \bar{b}_2 &= \max\{1, 2b_2\}. \end{aligned}$$

We will also assume that the forcing term is sufficiently small in $L^2(\mathbb{R}, \mathbb{R}^n)$. Precisely,

(H7) $2M < \bar{b}_1$ and $\|f\|_{L^2(\mathbb{R},\mathbb{R}^n)} \leq \frac{\beta}{2C}$, where $0 < \beta < \bar{b}_1 - 2M$ and C is the constant from the inequality (2).

In [IJ] we proved the following theorem.

THEOREM 1.3 (see [IJ, Theorem 1.2]). *Under the conditions* (H1)–(H7)*, the system* (1) *has an almost homoclinic solution* $q_0 \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ *such that* $\dot{q}_0(t) \to 0$ *as* $t \to \pm\infty$.

If the forcing term f is trivial and the conditions (H1)–(H5) are fulfilled then (1) possesses a nontrivial homoclinic solution.

To prove this theorem we applied the approximative method stated in a general setting and proved in [J2].

THEOREM 1.4 (see [J2, Theorem 1.2]). *Let* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *and* $f : \mathbb{R} \to \mathbb{R}^n$ *satisfy the following hypotheses*:

(C1) V *is* C^1*-smooth with respect to all variables and* T*-periodic with respect to* t, $T > 0$,
(C2) f *is nontrivial, bounded, continuous and square integrable.*

Assume that for each $k \in \mathbb{N}$, *the Newtonian system*

$$\ddot{q} + V_q(t, q) = f_k(t), \tag{3}$$

where $f_k : \mathbb{R} \to \mathbb{R}^n$ *is the* $2kT$*-periodic extension of* f *restricted to the interval* $[-kT, kT)$ *over* $\mathbb{R}$, *has a solution* $q_k \in E_k$. *If* $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ *is a bounded sequence in* $\mathbb{R}$ *then there exist a subsequence* $\{q_{k_j}\}_{j\in\mathbb{N}}$ *and a function* $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ *such that*

$$q_{k_j} \to q \ \text{as}\ j \to \infty,$$

in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ *and* q *is an almost homoclinic solution of the Newtonian system* (1).

By a solution $q_k \in E_k$ we mean a $2kT$-periodic solution of (3), i.e. $q_k(kT) - q_k(-kT) = 0 = \dot{q}_k(kT) - \dot{q}_k(-kT)$ and $q_k \in C^2((-kT, kT), \mathbb{R}^n)$.

Such a sequence $\{q_k\}_{k\in\mathbb{N}}$ as in Theorem 1.4 is called *an approximative sequence* for (1).

Here and subsequently, $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ denotes the space of C^2 functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the topology of uniformly convergence of functions and all derivatives up to the order 2 on every compact subsets of $\mathbb{R}$.

Let $I_k : E_k \to \mathbb{R}$ be defined by

$$I_k(q) = \int_{-kT}^{kT} \Big(\frac{1}{2} |\dot{q}(t)|^2 - V(t, q(t)) + (f_k(t), q(t)) \Big)\, dt. \tag{4}$$

The functional I_k is C^1 and its critical points are $2kT$-periodic solutions of (3). In [IJ] by the use of the Mountain Pass Theorem (see for example [MW, Theorem 4.10]) we received an approximative sequence $\{q_k\}_{k\in\mathbb{N}}$ for (1). In consequence, we got an almost homoclinic solution q_0 of (1) as the limit in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ of a certain subsequence $\{q_{k_j}\}_{j\in\mathbb{N}}$. For a fixed $k \in \mathbb{N}$, q_k is a critical point of mountain pass type for the Lagrangian functional I_k.

According to MathSciNet up till now the paper [IJ] has been cited over 50 times by many authors. In some papers (see for example: [LJ, TL, TX, ZY]) Theorem 1.3 was extended to a broader class of potentials. However, it seems that no one has asked for the multiplicity of almost homoclinic solutions.

The aim of this paper is to show the existence of another approximative sequence $\{p_k\}_{k\in\mathbb{N}}$ for (1). For each $k \in \mathbb{N}$, p_k is a point of minimum of I_k on a ball in E_k.

Theorem 1.5. *Under the assumptions* (H1)–(H7), *the system* (1) *possesses two approximative sequences*: *one of mountain pass type and the second of local minima.*

Using Theorem 1.4 we get an almost homoclinic solution p_0 of (1) as the limit in $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ of a subsequence $\{p_{k_j}\}_{j\in\mathbb{N}} \subset \{p_k\}_{k\in\mathbb{N}}$. Unfortunately, we are not able to answer for the question whether p_0 is different from q_0.

2. An approximative sequence of local minima. To prove the existence of an approximative sequence $\{p_k\}_{k\in\mathbb{N}}$ (of local minima) for (1) we will need the following result.

Lemma 2.1. *If V and f satisfy* (H1)–(H7) *then for every $k \in \mathbb{N}$ the functional I_k satisfies the Palais-Smale condition, i.e. every sequence $\{u_m\}_{m\in\mathbb{N}} \subset E_k$ such that $\{I_k(u_m)\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and $I_k'(u_m) \to 0$ in E_k^*, as $m \to \infty$, contains a convergent subsequence.*

The proof of this lemma can be found in [IJ] (see [IJ, the proof of Lemma 2.4]).

Applying (H5) we see that for each $q \neq 0$ and $t \in [0,T]$ the function

$$(0,+\infty) \ni \zeta \mapsto W(t,\zeta^{-1}q)\zeta^{\mu}$$

is nonincreasing. On account of the above remark, we have

$$W(t,q) \le W\Big(t,\frac{q}{|q|}\Big)|q|^{\mu}, \quad \text{if } 0<|q|\le 1,\ t\in[0,T] \tag{5}$$

and

$$W(t,q) \ge W\Big(t,\frac{q}{|q|}\Big)|q|^{\mu}, \quad \text{if } |q|\ge 1,\ t\in[0,T]. \tag{6}$$

Let

$$\varrho = \frac{1}{C}\,.$$

Fix $k \in \mathbb{N}$. Assume that $q \in E_k$ and $\|q\|_{E_k} \le \varrho$. From (2) it follows that $\|q\|_{L^\infty_{2kT}} \le 1$. Applying (H2), (5) and (H7) we get

$$\begin{aligned} I_k(q) &\ge \frac{1}{2}\,\bar b_1\|q\|^2_{E_k} - M\|q\|^2_{E_k} - \|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}} \\ &\ge \frac{1}{2}\,\bar b_1\|q\|^2_{E_k} - M\|q\|^2_{E_k} - \frac{\beta}{2C}\,\|q\|_{E_k} \\ &= \frac{1}{2}\,(\bar b_1-\beta-2M)\|q\|^2_{E_k} + \frac{\beta}{2}\,\|q\|^2_{E_k} - \frac{\beta}{2C}\,\|q\|_{E_k}. \end{aligned} \tag{7}$$

Set

$$\alpha = \frac{\bar b_1-\beta-2M}{2C^2}\,.$$

The condition (H7) implies $\alpha > 0$. Define

$$d_k = \inf_{\|q\|_{E_k}\le\varrho} I_k(q).$$

Since $I_k(0)=0$ we have $d_k \le 0$. By Ekeland's variational principle (see [MW, Theorem 4.1]), there is a minimizing sequence $\{u_m\}_{m\in\mathbb{N}} \subset \{q \in E_k : \|q\|_{E_k} \le \varrho\}$ such that

$I_k(u_m) \to d_k$ and $I'_k(u_m) \to 0$ as $m \to \infty$. From Lemma 2.1 we conclude that d_k is a critical value of I_k. Consequently, there exists $p_k \in E_k$ such that $\|p_k\|_{E_k} \le \varrho$, $I_k(p_k) = d_k$ and $I'_k(p_k) = 0$. Thus p_k is a $2kT$-periodic solution of (3), and Theorem 1.4 now implies $p_k \to p_0$ in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, where p_0 is an almost homoclinic solution of (1).

For the unperturbed system ($f \equiv 0$) it suffices to take $p_k \equiv 0$ for each $k \in \mathbb{N}$, therefore $p_0 \equiv 0$.

3. An open problem. At the beginning we briefly sketch the proof of existence of an approximative sequence $\{q_k\}_{k\in\mathbb{N}}$, of mountain pass type, for (1).

Using (H2) and (6) we obtain

$$I_k(q) \le \frac{1}{2}\,\bar{b}_2 \|q\|^2_{E_k} - m \int_{-kT}^{kT} |q(t)|^\mu \, dt + \|f_k\|_{L^2_{2kT}} \|q\|_{E_k} + 2kTm \tag{8}$$

for each $k \in \mathbb{N}$. We conclude from (8) that there exists $e_1 \in E_1$ such that $\|e_1\|_{E_1} > \varrho$ and $I_1(e_1) < 0$. Define

$$e_k(t) = \begin{cases} e_1(t) & \text{for } |t| \le T, \\ 0 & \text{for } T < |t| \le kT \end{cases}$$

for $k > 0$. Then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \varrho$ and $I_k(e_k) = I_1(e_1) < 0$. As (7), (8) and Lemma 2.1 hold, by the Mountain Pass Theorem we see that I_k possesses a critical value $c_k \ge \alpha$ given by

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} I_k(g(s)),$$

where

$$\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0,\ g(1) = e_k\}.$$

Hence there is $q_k \in E_k$ such that $I_k(q_k) = c_k$ and $I'_k(q_k) = 0$. Moreover, the sequence $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$ (see the proof of [IJ, Lemma 2.6]).

Since $c_k \ge \alpha > 0 \ge d_k$, we have $p_k \ne q_k$ for each $k \in \mathbb{N}$. Thus Theorem 1.5 is proved.

Theorem 1.4 implies $\{q_k\}_{k\in\mathbb{N}}$ goes to q_0 along a subsequence in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. In [IJ] it was proved that for the unperturbed system $q_0 \ne 0 \equiv p_0$.

The question still unanswered is whether $q_0 \ne p_0$ in the case $f \ne 0$.

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