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MODULE STRUCTURE IN CONLEY THEORY WITH SOME APPLICATIONS

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Abstract. A multiplicative structure in the cohomological version of Conley index is described following a joint paper by the author with K. Gęba and W. Uss. In the case of equivariant flows we apply a normalization procedure known from equivariant degree theory and we propose a new continuation invariant. The theory is applied then to obtain a mountain pass type theorem. Another illustrative application is a result on multiple bifurcations for some elliptic PDE.

1. Introduction. In this paper, we consider a module structure of cohomology Conley index of local flows determined by equivariant smooth vector fields in $\mathbb{R}^n$. By using this structure, in [7] a continuation invariant called *a relative cup-length* has been described. Motivated also by [3], we present here a normalization technique known from equivariant degree theory. This allows to define a version of the relative cup-length as an element of the Euler ring of a group G (comp. [6]). Let us observe that the module structure can be used also in the infinite-dimensional version of the Conley index (see [11]), since it is preserved after suspension. Some application to PDE is also briefly presented in the last section.

The paper is organized as follows. Section 2 contains some standard notation for compact Lie group actions. In Section 3 we recall necessary notions from (equivariant) Conley index theory and in Section 4 a normalization process is described. In Section 5 we describe an abstract notion of a relative cup-length of an index pair with respect to an isolating domain. Some simple applications are given in the last two sections. A mountain pass type theorem is proved in Section 6. Elliptic PDE with Dirichlet boundary condition and with $\mathbb{Z}_2$-symmetry is considered in Section 7. We prove a multiple bifurcation theorem.

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2. Preliminaries on group actions. We start from some notation about group actions (see [6] for more details). Let G be a group. If $H \subset G$ is a subgroup, we denote by G/H the set of left cosets gH. Two subgroups H and K of G are *conjugate* if there exists $g \in G$ such that $K = g^{-1}Hg$. The conjugacy class of H is denoted by (H). There is a natural partial order in the set $\Phi(G)$ of conjugacy classes:

$$(K) \leqslant (H) \text{ if there exist } \bar{K} \in (K) \text{ and } \bar{H} \in (H) \text{ such that } \bar{K} \subset \bar{H}.$$

Throughout the whole paper we consider only compact Lie groups and their closed subgroups. Given a subgroup $H \subset G$ let $N(H)$ be the normalizer of H. The Weyl group of H is the quotient $W(H) := N(H)/H$. Let us define the set

$$\Phi_0(G) := \{(H) \in \Phi(G) : \dim W(H) = 0\}.$$

A *G-set* is a pair (X, ξ), where X is a set and $\xi : G \times X \to X$ is an action of G on X, i.e., a map such that:

(i) $\xi(g_1, \xi(g_2, x)) = \xi(g_1g_2, x)$ for $g_1, g_2 \in G$ and $x \in X$,
(ii) $\xi(e, x) = x$ for $x \in X$, where $e \in G$ is the group unit.

In the sequel we write gx instead of $\xi(g, x)$. For every subgroup $H \subset G$ the set G/H is a G-set by the action $g(\tilde{g}H) = g\tilde{g}H$. If ξ is continuous, we call (X, ξ) a *G-space*. We say that a real (resp. complex) Banach space $\mathbb{E}$ is a *real* (resp. *complex*) *Banach representation of G* if $\mathbb{E}$ is G-space and, for each $g \in G$, the map $\xi_{\mathbb{E}}(g, \cdot) : \mathbb{E} \ni x \mapsto gx$ is linear and bounded.

For $x \in X$, the subgroup $G_x = \{g \in G : gx = x\}$ is called the *isotropy group* of X of the point x. The conjugacy class of an isotropy group is called an *isotropy type*. Denote by $\mathrm{Iso}(X)$ the set of all isotropy types in X. The set $Gx = \{gx : g \in G\}$ is called an *orbit* through x.

For a given subgroup $H \subset G$ we specify several subspaces of a given G-space X: $X_H = \{x \in X : H = G_x\}$, $X_{(H)} = \{x \in X : (H) = (G_x)\}$, $X^H = \{x \in X : H \subset G_x\}$, $X^{(H)} = \{x \in X : (H) \leqslant (G_x)\}$.

Now we define the *Burnside ring* of G as follows (cf. [1] for details and examples):

As a group $A(G)$ is a free abelian group generated by $(H) \in \Phi_0(G)$, i.e., an element $a \in A(G)$ is a finite sum $a = n_{H_1}(H_1) + \ldots + n_{H_m}(H_m)$ with $n_{H_i} \in \mathbb{Z}$ and $(H_i) \in \Phi_0(G)$.

The operation of multiplication in $A(G)$ is a bit more sophisticated. Let $(H), (K) \in \Phi_0(G)$. Consider the diagonal action of G on $G/H \times G/K$. Then for any $(L) \in \Phi_0(G)$, the spaces G/H^L and G/K^L consist of finitely many $W(L)$-orbits. Therefore the space $(G/H \times G/K)_{(L)}/G$ is finite. Let $n_L(H, K)$ denote the number of elements of this space. Define

$$(H) \cdot (K) := \sum_{(L) \in \Phi_0(G)} n_L(H, K)(L).$$

A free abelian group $U(G) = \mathbb{Z}(\Phi(G))$ can also be equipped with a natural multiplicative structure and it is called then an *Euler ring* of G in [6].

3. Local flows and flow generators. Let X be a space.

Definition 3.1. A *flow* on X is a map $\varphi : X \times \mathbb{R} \to X$ such that

- $\varphi(x, 0) = x$ for all $x \in X$;
- $\varphi(\varphi(x, t), s) = \varphi(x, s + t)$ for all $x \in X$, $t, s \in \mathbb{R}$.

A *local flow* is defined on an open subset

$$\operatorname{dom} \varphi = \{(x, t) : t \in (a_x, b_x),\ a_x < 0 < b_x\} \subset X \times \mathbb{R},$$

with the above properties whenever φ is defined.

Let U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ a vector field which is at least locally Lipschitz (we consider here smooth vector fields for simplicity). Then F *generates* a local flow η on U by the rule that $\eta(x, t)$ is the value of a unique solution to the Cauchy problem

$$\begin{cases} \dot\eta(x, \cdot) = F(\eta(x, \cdot)), \\ \eta(x, 0) = x \end{cases}$$

at the time t.

Throughout the whole paper we denote by V a finite-dimensional orthogonal representation of a compact Lie group G. Let $U \subset V$ be an open G-invariant subset. A local vector field $F : U \to V$ is *G-equivariant* if $F(gx) = gF(x)$ for all $g \in G$, $x \in U$. It is easy to observe that the local flow η_F generated by an equivariant vector field is also equivariant, i.e., $g(\eta_F(x, t)) = \eta_F(gx, t)$ for each $g \in G$, whenever defined.

Given a local flow η on U, we define a *maximal invariant* part of U:

$$\operatorname{Inv}_\eta(U) := \{x \in U : \eta(x, t) \in U \text{ for all } (x, t) \in \operatorname{dom} \eta\}.$$

One easily observes that for a G-equivariant local flow η the above set is a G-invariant subset of U.

Definition 3.2. An open set $W \subseteq U$ is an *isolating domain* for a local flow η, if $\operatorname{Inv}_\eta(W) \subset W$ is a compact subset.

Observe that if W is an isolating domain then there exists an open relatively compact neighbourhood W' of $\operatorname{Inv}_\eta(W)$ such that $\operatorname{Inv}_\eta(W) = \operatorname{Inv}_\eta(\overline{W'}) \subset W' \subset \overline{W'} \subset W$. The compact set $\overline{W'}$ is usually called an *isolating neighbourhood* in Conley index theory (cf. [5]). A compact set S is an *isolated invariant set*, if $S = \operatorname{Inv}_\eta(W) \subset W$ for some isolating domain W.

Definition 3.3. Let S be an isolated invariant set. A pair (X, A) of compact G-invariant sets is called a *G-index pair* for S, if

- $\operatorname{int}(X \setminus A)$ is an isolating domain and $S = \operatorname{Inv}(X \setminus A) \subset \operatorname{int}(X \setminus A)$;
- A is positively invariant in X, i.e., for each $x \in A$: if $\eta(x, [0, t]) \subset X$ for some $t > 0$ then $\eta(x, [0, t]) \subset A$;
- A is an *exit set* from X: if $x \in X$ and $\eta(x, t) \notin X$ for some $t > 0$, then there exists $s \in [0, t)$ such that $\eta(x, s) \in A$.

The following existence result is an easy consequence of the non-equivariant case (see [17], comp. [9], [10]).

THEOREM 3.4. *If $U \subset V$ is a G-invariant isolating domain, then there exists a G-index pair for the compact isolated set $S = \mathrm{Inv}_\eta(U)$.*

If (X, A) is a pair of compact G-spaces, then X/A denotes the pointed compact G-space obtained by identifying all points of A with the distinguished point. In case of empty set A we take $X/A = X^+$, where X^+ denotes the pointed space with a separate base point added. Denote by $[X/A]$ the G-equivariant homotopy class of the pointed space X/A. The following proposition is true.

PROPOSITION 3.5. *If (X_1, A_1), (X_2, A_2) are two G-index pairs in an isolating domain U, then $[X_1/A_1] = [X_2/A_2]$.*

The proof can be also carried from [17], Theorem 4.10, and it is enough to observe that all maps defined in [17] are G-equivariant if the local flow is G-equivariant.

DEFINITION 3.6. If $U \subset V$ is a G-invariant isolating domain for a G-equivariant local flow η, then the *G-equivariant Conley index* is defined to be

$$\mathcal{C}IG(\eta, U) := [X/A],$$

where (X, A) is a G-index pair in U.

The independence of the index pair is assured by Proposition 3.5. In fact one can define the index for isolated invariant sets because of the following obvious localization property.

PROPOSITION 3.7. *Let $U \subset V$ be an isolating domain and let $U_1 \subset U$ be open and G-invariant with $\mathrm{Inv}_\eta(U) \subset U_1$. Then U_1 is an isolating domain and $\mathcal{C}IG(\eta, U_1) = \mathcal{C}IG(\eta, U)$.*

DEFINITION 3.8. A *local equivariant flow generator* is a pair (F, U), where $U \subset V$ is open and G-invariant subset of V, and $F : U \to V$ is a G-equivariant vector field generating a local flow η on U with $\mathrm{Inv}_\eta(U) \subset U$ compact.

We consider here only local flows generated by vector fields. Motivated by [3] we introduce a convenient relation of *otopy* which plays a role of continuation. If V is a representation of G then $V \times [0, 1]$ is a G-space with the action $g(x, t) = (gx, t)$. If Ω is an open (G-invariant) subset of $V \times [0, 1]$, then for each $\tau \in [0, 1]$ we put $\Omega_\tau := \{x \in V : (x, \tau) \in \Omega\}$. A map (family of vector fields) $h : \Omega \to V$ generates a family of local flows η_τ, i.e., $h_\tau = h_{|\Omega_\tau} : \Omega_\tau \to V$ generates η_τ.

DEFINITION 3.9. An (equivariant) *otopy* is a pair (h, Ω), such that $\Omega \subset V \times [0, 1]$ is open and G-invariant, and the equivariant map $h : \Omega \to V$ generates a family of local flows η_τ such that the invariant part

$$S := \bigcup_{\tau \in [0,1]} \mathrm{Inv}_{\eta_\tau}(\Omega_\tau) \subset V$$

is compact.

We admit that Ω_τ is empty for some $\tau \in [0, 1]$. Note that h_τ are local flow generators. We say then also that h_1, h_0 are *otopic local flow generators.* This defines an equivalence relation among local (equivariant) flow generators.

Proposition 3.10. *Let (h,Ω) be an otopy generating a family of equivariant flow generators of $\{\eta_\tau\}_{\tau\in[0,1]}$. Then $\mathcal{CIG}(\eta_0,\Omega_0)=\mathcal{CIG}(\eta_1,\Omega_1)$.*

Proof. This is an immediate consequence of Proposition 3.7 and the continuation property of the Conley index (see e.g. [10], Proposition 5.5). ∎

Obviously all the properties are valid for non-equivariant Conley index, when a trivial action of G is considered not necessarily on subsets of $\mathbb{R}^n$, but on locally compact metric spaces [5].

4. Normalization. For an invariant subset $X\subset V$ of a finite-dimensional representation of G we define

$$\operatorname{Iso}(X):=\{(H)\in\Phi(G):X_{(H)}\neq\emptyset\},$$

where $X_{(H)}:=\{x\in X:(G_x)=(H)\}$. For every closed subgroup $H\subset G$ the set $M=V_{(H)}$ is a submanifold of V (in fact it is a linear subspace). Then

$$\nu(M):=\big\{(x,v)\in M\times V:x\in M,\ v\in N_x=(T_xM)^\perp\big\}$$

denotes a normal bundle over M. We have the map $\mathcal{N}:\nu(M)\to V$, $\mathcal{N}(x,v):=x+v$.

We shall use the following version of the equivariant tubular neighbourhood theorem (see [3], Theorem 3.1, or [14], Theorem 4.8 for a proof).

Theorem 4.1. *There exists an open G-invariant subset T* (tubular neighbourhood) *containing M and such that the map $\mathcal{N}$ restricted to $\mathcal{N}^{-1}(T)$ is a homeomorphism.*

Definition 4.2. Let $(H)\in\operatorname{Iso}(V)$. A local vector field (f,U) is *H-normal* if there is an open G-invariant subset $U_0\subset V_{(H)}$ and $\varepsilon>0$ such that

- $T=\{x+v:x\in U_0,\ v\in N_x,\ |v|<\varepsilon\}\subset U$, where T is a tubular neighbourhood of $V_{(H)}$;
- $f(x+v)=f(x)+v$ for $x\in U_0$, $v\in N_x$, $|v|<\varepsilon$.

Lemma 4.3. *Let (f,U) be a local equivariant flow generator and let (H) be a maximal orbit type in $\operatorname{Iso}(U)$. Then there exist two equivariant local flow generators $(f_1,U_1),(f_2,U_2)$ such that*

- *$U_1\subset U$, $U_2\subset U\setminus U_{(H)}$, $U_1\cap U_2=\emptyset$,*
- *(f_1,U_1) is (H)-normal and $f_1(x)=f(x)$ for all $x\in(U_1)_{(H)}$,*
- *(f,U) is otopic to the disjoint union (f_1,U_1) and (f_2,U_2).*

Proof. First we find an open bounded set $U_0\subset U_{(H)}$ such that $\overline{U_0}\subset U_{(H)}$ and

$$\operatorname{Inv}_\eta(U)_{(H)}=\operatorname{Inv}_\eta(U)\cap U_{(H)}\subset U_0.$$

Given $\rho>0$, we define two sets

$$X(\rho):=\{u+v\in V:u\in\overline{U_0},\ v\in N_u,\ |v|\leqslant\rho\},$$
$$Y(\rho):=\{u+v\in V:u\in\partial_{(H)}U_0,\ v\in N_u,\ |v|<\rho\}.$$

Let T be a tubular neighbourhood of $U_{(H)}$ in V. Then there exists $\varepsilon>0$ such that $X(4\varepsilon)\subset T$ and $\operatorname{Inv}(U,\eta)\cap Y(4\varepsilon)=\emptyset$.

Next we find a smooth function $\alpha : [0, 4\varepsilon] \to [0, 1]$ such that $\alpha(t) = 0$ for $t \in [0, 2\varepsilon]$, $\alpha(t) = 1$ for $t \in [3\varepsilon, 4\varepsilon]$ and $\alpha'(t) > 0$ for $t \in (2\varepsilon, 3\varepsilon)$. Define $r : X(4\varepsilon) \to X(4\varepsilon)$ by the formula $r(x) := u + \alpha(|v|)v$, where $x = u + v,\ u \in \overline{U}_0,\ v \in N_u$.

Let $\widehat{U} := U \setminus Y(4\varepsilon)$, and $\widehat{f} : \widehat{U} \to V$ be defined by

$$\widehat{f}(x) := \begin{cases} f(r(x)) & \text{for } x \in X(4\varepsilon) \setminus Y(4\varepsilon), \\ f(x) & \text{for } x \in \widehat{U} \setminus X(4\varepsilon). \end{cases}$$

Clearly $(f, \widehat{U})$ and $(\widehat{f}, \widehat{U})$ are otopic local flow generators. Observe that $f_{|\widehat{U} \setminus \operatorname{int} X(4\varepsilon)} = \widehat{f}_{|\widehat{U} \setminus \operatorname{int} X(4\varepsilon)}$.

Take another smooth function $\theta : [0, 3\varepsilon] \to [0, 1]$ such that $\theta(t) = 1$ for $t \leqslant \varepsilon$ and $\theta(t) = 0$ for all $t \in [2\varepsilon, 3\varepsilon]$. Define a vector field $g : \widehat{U} \to V$ by

$$g(x) := \begin{cases} \theta(|v|)v & \text{for } x = u + v \in X(3\varepsilon) \setminus Y(3\varepsilon), \\ 0 & \text{for } x \in \widehat{U} \setminus X(3\varepsilon). \end{cases}$$

Consider a homotopy $h : \widehat{U} \times [0, 1] \to V$ given by $h(x, t) := \widehat{f}(x) + tg(x)$. Observe that $g_{|V_{(H)}} \equiv 0$ and $g_{\widehat{U} \setminus X(2\varepsilon)} \equiv 0$. Therefore h defines an otopy relation between $(\widehat{f}, \widehat{U})$ and $(\widehat{f} + g, \widehat{U})$.

Now let

$$U_1 := \{x = u + v : u \in U_0,\ v \in N_u,\ |v| < \varepsilon\}, \quad U_2 := \widehat{U} \setminus [X(\varepsilon) \cup U_{(H)}].$$

Define f_i to be a restriction of $\widehat{f} + g$ to U_i, $i = 1, 2$. Since $\operatorname{Inv}(\widehat{U}, \eta_{\widehat{f}+g})$ is a compact subset of $U_1 \cup U_2$, the generators $(\widehat{f} + g, \widehat{U})$ and $(f_1 \sqcup f_2, U_1 \sqcup U_2)$ are otopic. The proof is complete. ∎

A similar procedure as in Lemma 4.3 can be applied to otopies and we obtain the following:

Lemma 4.4. *Let (h, Ω) be an otopy and (H) a maximal orbit type in Ω. Then there exist two otopies $(k, \widehat{\Omega}), (l, \widetilde{\Omega})$ such that*

- $\widehat{\Omega} \subset \Omega,\ \widetilde{\Omega} \subset \Omega \setminus \Omega_{(H)},\ \widehat{\Omega} \cap \widetilde{\Omega} = \emptyset$;
- *$(k_t, \widehat{\Omega})$ is (H)-normal for all $t \in [0, 1]$;*
- *$k(x, t) = h(x, t)$ for all $(x, t) \in \widehat{\Omega}_{(H)}$;*
- *$(h, \Omega)_i$ is otopic to the disjoint union $(k, \widehat{\Omega})_i \sqcup (l, \widetilde{\Omega})_i$ for $i = 0, 1$.*

Thus Lemma 4.4 gives the uniqueness of the decomposition in Lemma 4.3 up to an otopy, i.e., it is invariant under equivariant continuation.

Now let $\operatorname{Iso}(V) = \{(H_1), (H_2), \ldots, (H_k)\}$. Let the order be such that $(H_i) < (H_j)$ implies $i < j$.

Theorem 4.5. *Let (f, U) be an equivariant local flow generator. Then there exists a collection of local flow generators (f_i, U_i), $i = 1, 2, \ldots, k$, such that f_i is (H_i)-normal for each $i = 1, 2, \ldots, k$ and (f, U) is otopic to the disjoint sum $(f_1, U_1) \sqcup (f_2, U_2) \sqcup \ldots \sqcup (f_k, U_k)$.*

Proof. The proof is by induction. We start from (H_k). Applying Lemma 4.3 we obtain that (f, U) is otopic to $(f_1, U_1) \sqcup (f_2, U_2)$, where U_2 is disjoint from $V_{(H_k)}$. Thus (H_{k-1}) is maximal in $\mathrm{Iso}(U_2)$ and thus we can use Lemma 4.3 again. After k steps we obtain the desired collection. ∎

We call a collection obtained in Theorem 4.5 *a normal collection of local equivariant flow generators*. We have just proved the existence of a normal collection in the otopy class of any equivariant local flow generator. The uniqueness up to homotopy follows from the following:

THEOREM 4.6. *Assume we are given two normal collections $\{(f_i^\alpha, U_i^\alpha)\}$ of local equivariant flow generators ($\alpha = 1, 2$, $i = 1, 2, \ldots, k$). Then there exists a normal collection of otopies $\{(h_i, U_i)\}$ such that, for every $i = 1, 2, \ldots, k$, (h_i, U_i) is an otopy between (f_i, U_i^1) and (f_i, U_i^2).*

Proof. An induction argument is the same as in the proof of Theorem 4.5. We only have to apply Lemma 4.4 instead of Lemma 4.3. ∎

COROLLARY 4.7. *Let (f, U) be an equivariant local flow generator. Then its equivariant Conley index is homotopy equivalent to a G-CW complex which can be described as a union*:

$$\mathcal{CIG}(\eta, U) = \mathcal{CIG}(\eta, U_1) \vee \mathcal{CIG}(\eta, U_2) \vee \ldots \vee \mathcal{CIG}(\eta, U_k)$$

where U_i are domains of a normal collection of local equivariant flow generators which is otopic to (f, U).

5. Relative cup-length. Throughout this section we assume that $A \subset X \subset Y$ are compact metric spaces and denote by H^* the Alexander–Spanier cohomology with the coefficients in a fixed abelian group G.

The cup product (see e.g. [20], Section 5.6)

$$\smile : H^k(X) \times H^l(X, A) \to H^{k+l}(X, A),$$

endows $H^*(X, A)$ with a structure of an $H^*(X)$-module. If $k : X \to Y$ denotes the inclusion map, then the formula

$$\beta \cdot \alpha := k^*(\beta) \smile \alpha$$

defines on $H^*(X, A)$ a structure of an $H^*(Y)$-module. The following remark is a simple consequence of the naturality property of the cup product (see e.g. [12], Proposition 3.10).

REMARK 5.1. If $B \subset A$ is compact, then

$$H^*(X, A) \to H^*(X, B) \to H^*(A, B)$$

is an exact sequence of $H^*(Y)$-modules, where the maps are induced by inclusions.

DEFINITION 5.2. Let $\beta \in H^p(Y)$, $p > 0$, $\beta \neq 0$, and $A \subset X \subset Y$ be CW-complexes. The *relative cup-length* of β with respect to (X, A) is the number $\chi(\beta; X, A) \in \mathbb{N}$ defined as follows:

- $\chi(\beta; X, A) = 0$ if $H^*(X, A) = 0$;
- $\chi(\beta; X, A) = 1$ if $H^*(X, A) \neq 0$ and $\beta \cdot \alpha = 0$ for every $\alpha \in H^*(X, A)$;
- $\chi(\beta; X, A) = k \geqslant 2$ if there exists $\alpha_0 \in H^*(X, A)$ such that $\beta^{k-1} \cdot \alpha_0 \neq 0$ and $\beta^k \cdot \alpha = 0$ for every $\alpha \in H^*(X, A)$.

Definition 5.3. The *relative cup-length* of the $H^*(Y)$-module $H^*(X,A)$ is the number given by
$$\Upsilon(X,A;Y) := \max\bigl\{\chi(\beta;X,A) : 0 \neq \beta \in H^k(Y),\ k>0\bigr\}.$$
If $H^k(Y)$ are trivial for all $k>0$, but $H^*(X,A)$ is non-zero, we set $\Upsilon(X,A;Y)=1$; and if $H^l(X,A)$ are trivial for all $l \geqslant 0$, then $\Upsilon(X,A;Y) := 0$.

Lemma 5.4. *If $B \subset A \subset X \subset Y$, then*
$$\Upsilon(X,B;Y) \leqslant \Upsilon(X,A;Y) + \Upsilon(A,B;Y).$$

Proof. Let $k_1 := \Upsilon(X,A;Y)$, $k_2 := \Upsilon(A,B;Y)$ and
$$0 \neq \alpha \in H^p(X,B),\ p \geqslant 0,\ 0 \neq \beta \in H^q(Y),\ q>0.$$
Let also
$$i : (X,B) \to (X,A), \quad j : (A,B) \to (X,B)$$
be inclusions.

Since $k_2 = \Upsilon(A,B;Y)$ we have $j^*(\beta^{k_2} \cdot \alpha) = 0$.

By Remark 5.1 there exists $\gamma \in H^*(X,A)$ such that $\beta^{k_2} \cdot \alpha = i^*(\gamma)$. Therefore
$$\beta^{k_1+k_2} \cdot \alpha = i^*(\beta^{k_1} \cdot \gamma).$$
But $\beta^{k_1} \cdot \gamma = 0$ by definition of k_1, and thus $\beta^{k_1+k_2} \cdot \alpha = 0$. This means that
$$\Upsilon(X,B;Y) \leqslant k_1 + k_2,$$
which ends the proof. ∎

Lemma 5.5. *If $A \subset X \subset Y_1 \subset Y_2$, then*
$$\Upsilon(X,A;Y_2) \leqslant \Upsilon(X,A;Y_1)$$

Proof. Let us denote inclusions by
$$s : X \hookrightarrow Y, \quad k : A \hookrightarrow X, \quad t : A \hookrightarrow Y.$$
If $\beta \in H^q(Y_2)$, $q>0$ and $\alpha \in H^*(X,A)$, then $\beta \cdot \alpha = t^*(\beta) \smile \alpha = k^*\bigl(s^*(\beta)\bigr) \smile \alpha$. Therefore $\chi(\beta;X,A) = \chi(s^*(\beta);X,A)$ for all $\beta \in H^q(Y_2)$, $q>0$. But $t = k \circ s$, thus the condition $t^*(\beta) \smile \alpha \neq 0$ implies $s^*(\beta) \smile \alpha \neq 0$, and our inequality follows. ∎

Recall that the *cross product* is defined by the formula
$$a \times b := p_1^*(a) \smile p_2^*(b)$$
where p_1, p_2 denote projections $(X,A) \times (Y,B)$ onto (X,A) and (Y,B). For algebraic properties of the maps $\times : H^k(X) \times H^l(Y) \to H^{k+l}(X \times Y)$ and $\times : H^k(X,A) \times H^l(Y,B) \to H^{k+l}(X \times Y, X \times B \cup A \times Y)$ we refer to [12].

Let σ be a generator of $H^1(I, \partial I)$, where $I := [-1,1]$. The formula
$$\mathfrak{S}(a) := a \times \sigma$$
defines a mapping
$$\mathfrak{S} : H^k(X,A) \to H^{k+1}((X,A) \times (I,\partial I)) = H^{k+1}(X \times I, X \times \partial I \cup A \times I).$$
The following lemma holds (comp. [12], Theorem 3.21 for more general version).

Lemma 5.6. *If $X \subset Y$ then $\mathfrak{S}$ is an isomorphism of $H^*(Y)$-modules. More exactly*
$$\mathfrak{S}(b \cdot a) = p^*(b) \cdot \mathfrak{S}(a),$$
where p denotes the projection $Y \times I$ onto Y.

Proof. Let $b \in H^*(Y)$, $a \in H^*(X,A)$. Consider the following projections:

$$p_1 : (X \times I, A \times I) \to (X,A),\ p_2 : (X \times I, X \times \partial I) \to (I, \partial I),\ \overline{p}_1 : X \times I \to X.$$

The following diagram is commutative ($i_1(x,t) = (i(x),t)$):

$$\begin{array}{ccc} X \times I & \xrightarrow{i_1} & Y \times I \\ \big\downarrow{\scriptstyle \overline{p}_1} & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{i} & Y. \end{array}$$

Using this diagram together with the naturality and associativity properties of the cup product we obtain

$$\begin{aligned} \mathfrak{S}(b \cdot a) = (b \cdot a) \times \sigma = p_1^*(i^*(b) \smile a) \smile p_2^*(\sigma) = \overline{p}_1^*(i^*(b)) \smile p_1^*(a) \smile p_2^*(\sigma) \\ = \overline{p}_1^*(i^*(b)) \smile \mathfrak{S}(a) = i_1^*(p^*(b)) \smile \mathfrak{S}(a) = p^*(b) \cdot \mathfrak{S}(a), \end{aligned}$$

which ends the proof. ∎

THEOREM 5.7.

$$\Upsilon\big((X,A) \times (I, \partial I); Y\big) = \Upsilon(X,A;Y).$$

Proof. Let us notice that formally $X \times I \subset Y \times I$ and thus $H^*(X \times I, X \times \partial I \cup A \times I)$ is an $H^*(Y \times I)$-module, but $p^* : H^*(Y) \to H^*(Y \times I)$ is an isomorphism which gives the naturally isomorphic $H^*(Y)$-module structure: $b \odot a := p^*(b) \cdot a$ for $b \in H^*(Y)$ and $a \in H^*(X \times I, X \times \partial I \cup A \times I)$. By using this into account the desired equality follows directly from Lemma 5.6. ∎

Now we apply the above notion to the Conley index, first in the nonequivariant case. It is useful to consider the *cohomology Conley index* defined by

$$CH^*(S) := H^*(N,L) = H^*(N/L),$$

where H^* denotes the Alexander–Spanier cohomology and (N,L) is an index pair for the isolated invariant set S. The last equality is understood that we identify $H^*(N,L)$ and $H^*(N/L)$ via the isomorphism induced by the quotient map.

It is convenient to extend the index to an index of isolating neighbourhoods: if N is an isolating neighbourhood for η then the *homotopy* (resp. *cohomology*) *Conley index* of N is defined to be

$$h(N) = h(N,\eta) := h(\mathrm{Inv}(N,\eta)),\ \text{resp.}\ \ CH^*(N) = CH^*(N,\eta) := CH^*(\mathrm{Inv}(N,\eta)).$$

Before giving the definition of the relative cup-length of Conley index we need some useful lemmas. If (N_0, N_1) is an index pair and $t \geqslant 0$ then, following [18], we set

$$\begin{aligned} N_1^t &:= \{x \in N_1 : \eta(x, [-t,0]) \subset N_1\}, \\ N_0^{-t} &:= \big\{x \in N_1 : \text{there is a point } x' \in N_0 \text{ and } t' \in [0,t] \\ &\qquad\qquad \text{with } \eta(x', [-t',0]) \subset N_1 \text{ and } \eta(x't) = x\big\}. \end{aligned}$$

For $t \geqslant 0$ define a map of pointed spaces

$$g : (N_1/N_0^{-t}, *) \to (N_1^t / N_0 \cap N_1^t, *)$$

by

$$g([x]) := \begin{cases} [\eta(x,t)] & \text{if } \ \eta(x,[0,t]) \subset N_1 \setminus N_0, \\ * & \text{otherwise.} \end{cases}$$

It is known ([18], Lemma 23.14) that g is a homeomorphism. Therefore g induces an isomorphism

$$g^* : H^*(N_1^t, N_0 \cap N_1^t) \to H^*(N_1, N_0^{-t}).$$

LEMMA 5.8. *Assume that N is an isolating neighbourhood for η and (N_1, N_0) is an index pair for $S \subset N$. If $N_1 \subset N$ then the inclusion $i : (N_1, N_0 \cap N_1^t) \to (N_1, N_0^{-t})$ induces an isomorphism*

$$i^* = (g^*)^{-1} : H^*(N_1, N_0^{-t}) \to H^*(N_1, N_0 \cap N_1^t).$$

Proof. Consider the following diagram, where the vertical arrows denote the quotient maps:

$$\begin{array}{ccc} (N_1, N_0^{-t}) & \xleftarrow{\ i\ } & (N_1, N_0 \cap N_1^t) \\ \downarrow & & \downarrow \\ N_1/N_0^{-t} & \xrightarrow{\ g\ } & N_1/(N_0 \cap N_1^t). \end{array}$$

From the definition of g it is obvious that the diagram is homotopy commutative and the conclusion follows. ∎

DEFINITION 5.9. Let N be an isolating neighbourhood for the flow η. We define the *relative cup-length* of η with respect to N to be

$$\Upsilon(\eta, N) := \Upsilon(N_1, N_0; N),$$

where (N_1, N_0) is an index pair for S.

The following lemma states that $\Upsilon(\eta, N)$ is well defined.

LEMMA 5.10. *Let N be an isolating neighbourhood for η and let $S \subset N$ be an isolated invariant set. If (N_1, N_0) and $(\overline{N}_1, \overline{N}_0)$ are index pairs for S such that $N_1, \overline{N}_1 \subset N$ then*

$$\Upsilon(\overline{N}_1, \overline{N}_0; N) = \Upsilon(N_1, N_0; N).$$

Proof. As in the proof of Lemma 23.17 in [18], we consider the following sequence of maps, where $j, \hat{\imath}, \hat{\imath}_1$ are defined by inclusion maps of pairs of spaces and $g, \hat{g}$ are as above. All of them are homotopy equivalences of pointed spaces, as is in details proved in [18].

$$\begin{aligned} N_1/N_0 \xrightarrow{\ j\ } N_1/N_0^{-t} \xrightarrow{\ g\ } N_1^t/(N_0 \cap N_1^{-t}) \xrightarrow{\ \hat{\imath}_1\ } \overline{N}_1/\overline{N}_0^{\,-t} \\ \xrightarrow{\ \hat{g}\ } N_1^t/(\overline{N}_0 \cap \overline{N}_1^{\,t}) \xrightarrow{\ \hat{\imath}\ } N_1^t/(\overline{N}_0 \cap \overline{N}_1^t). \end{aligned}$$

By Lemma 5.8 and definition of j it follows that the following sequence of isomorphisms

$$\begin{aligned} H^*(N_1, N_0) \xleftarrow{\ \approx\ } H^*(N_1, N_0^{-t}) \xleftarrow{\ \approx\ } H^*(N_1^t, N_0 \cap N_1^{-t}) \xrightarrow{\ \approx\ } H^*(\overline{N}_1, \overline{N}_0^{\,-t}) \\ \xrightarrow{\ \approx\ } H^*(N_1^t, \overline{N}_0 \cap \overline{N}_1^{\,t}) \xrightarrow{\ \approx\ } H^*(N_1^t, \overline{N}_0 \cap \overline{N}_1^{\,t}) \end{aligned}$$

all are induced by inclusions. Therefore they all are isomorphisms of $H^*(N)$-modules and the conclusion follows. ∎

The continuation property holds for the relative cup-length.

LEMMA 5.11. *Consider a continuous family of flows $\eta_\lambda : X \times \mathbb{R} \to X$, $\lambda \in [0, 1]$. Let $N \subset X$ be an isolating neighbourhood for all flows η_λ. Then*

$$\Upsilon(\eta_0, N) = \Upsilon(\eta_1, N).$$

Proof. Similarly as in the proof of Lemma 5.10 we shall use parts of the proof of Theorem 23.31 in [18]. Given $\mu \in [0,1]$, there exists a neighbourhood W of μ in $[0,1]$ with the property that for all $\lambda \in W$ we can find pairs $(N_1, N_0) \subset (P_1^\lambda, P_0^\lambda) \subset (\overline{N_1}, \overline{N_0})$ such that $(N_1, N_0), (\overline{N_1}, \overline{N_0})$ are index pairs for η_μ in N, and $(P_1^\lambda, P_0^\lambda)$ is an index pair for η_λ in N (see Lemma 23.28 in [18]). Then it is shown in the proof of Theorem 23.31 in [18] that the inclusion $i : (N_1, N_0) \to (P_1^\lambda, P_0^\lambda)$ induces a homotopy equivalence of pointed spaces N_1/N_0 and P_1^λ/P_0^λ. The same argument applies to show that $i^* : H^*(P_1^\lambda, P_0^\lambda) \approx H^*(N_1, N_0)$ is an isomorphism of $H^*(N)$-modules. Therefore $\Upsilon(\eta_\lambda, N) = \Upsilon(\eta_\mu, N)$. Since $[0,1]$ is compact and connected, this completes the proof. ∎

One easily sees that the relative cup length is also invariant under otopy (the proof is practically the same).

Now we turn back to the equivariant case. Let V be a finite-dimensional orthogonal representation of a compact Lie group G and let $\mathrm{Iso}(V) = \{(H_1), (H_2), \ldots, (H_k)\}$. Consider an equivariant flow generator which is already a collection of local flow generators $(f_1, U_1) \sqcup (f_2, U_2) \sqcup \ldots \sqcup (f_k, U_k)$ such that f_i is (H_i)-normal, $i = 1, 2, \ldots, k$.

Consider one component (f_i, U_i) which is (H_i)-normal. Choose a representative $H \in (H_i)$. Then V^H is a linear subspace of V and it is a representation of the Weyl group WH. The set $U_i^H = U_i \cap V^H$ is WH-invariant and $f^h : U_i^H \to V^H$ is a WH-equivariant local flow generator (comp. [1]). Therefore we can consider the local flow $\widehat{\eta}$ defined on the quotient space $N := U_i^H/WH \subset V^H/WH$. Then the relative cup-length $\Upsilon(\widehat{\eta}, N)$ is well-defined. It is easy to observe that for any other representative $\widetilde{H} = gHg^{-1}$ in the same conjugacy class we obtain a conjugated local flow and therefore it gives the same number $\Upsilon(\widehat{\eta}, N)$. Therefore we are ready to define a G-equivariant otopy invariant.

Definition 5.12. Let us consider a G-equivariant local flow generator (f, U), which generates a local flow η. We define

$$\Upsilon_G(\eta, U) := \sum_{(H_i) \in \Phi(G)} \Upsilon(\widehat{\eta}_i, N) \cdot (H_i) \in U(G),$$

where $\widehat{\eta}_i, N)$ are as above.

Because of Lemma 5.11 this notion is well defined. Now, by use of Theorem 4.5 for an arbitrary equivariant flow generator we find in the otopy class a collection of local flow generators which is normal and apply the above definition. Thus we obtain an invariant with immediate properties:

Theorem 5.13.

a) *If two equivariant local flow generators $(f_0, U_0), (f_1, U_1)$ are otopic then*

$$\Upsilon_G(\eta_0, U_0) = \Upsilon_G(\eta_1, U_1),$$

where η_i are the local flows generated by f_i, respectively.

b) *If the H-component of $\Upsilon_G(\eta, U)$ is non-zero, then* $\mathrm{Inv}(\eta, U^H) \neq \emptyset$. *Moreover, if $f = \nabla\varphi$ is a gradient, then this coefficient is a lower bound of critical WH-orbits of φ.*

Proof. The first statement is a consequence of the continuation property of each coefficient $\Upsilon(\widehat{\eta}_i, N)$, see Lemma 5.11. As for the second one, we can apply Theorem 4.1 of [7] for the local flow $\widehat{\eta}$, defined by a H-normal component of the normal collection of local flow generators in the otopy class of (f, U). ■

6. Mountain Pass type theorems. In this section we give simple applications of the module structure described above. We start with a classic result.

Let M be a smooth closed manifold and let $f : M \to \mathbb{R}$ be a function of class C^1. Assume that f has only a finite number of critical points. Let $c_1 < c_2 < \ldots < c_p$ denote critical values of f. We choose numbers $a_0, a_1, \ldots, a_p$ such that

$$a_0 < c_1 < a_1 < c_2 < \ldots < c_p < a_p$$

As usual, we consider sublevel sets $f^a := \{x \in M : f(x) \leqslant a\}$.

Denote by $\varphi : M \times \mathbb{R} \to M$ the flow generated by a vector field $-\nabla f : M \to TM$. The following is well-known.

Theorem 6.1. *For every $i = 1, 2, \ldots, p$ the sets f^{a_i} are isolating neighbourhoods, and $(f^{a_i}, f^{a_{i-1}})$ are index pairs for φ.*

We can assume that f^{a_i} are CW-complexes (a_i are regular values). One observes that $f^{a_p} = M$, because c_p is a maximum of f. Let R be any ring of coefficients.

Lemma 6.2. *Let $\beta \in H^k(M; R)$, $k > 0$. Then for every $i = 1, 2, \ldots, p$ we have the inequality $\chi(\beta; f^{a_i}, f^{a_{i-1}}) \leqslant 1$.*

Proof. Since $\mathrm{Inv}(\overline{f^{a_i} \setminus f^{a_{i-1}}}) \subset f^{-1}(c_i)$ is finite, the set $A = \{x_1, x_2, \ldots, x_s\}$, which corresponds to one critical level c_i and each of the singletons $\{x_j\}$, is an isolated invariant set. Moreover as an isolating neighbourhood we can choose a small disc D_j which is contained in $f^{a_i} \setminus f^{a_{i-1}}$ and is disjoint with the other discs D_k. Thus the disjoint sum $D = \bigcup D_j$ is an isolating neighbourhood for A. We find an index pair (Y, Z) for A in D. Since D_j are contractible, we have $\chi(\beta; Y, Z) \leqslant 1$. On the other hand, both pairs (Y, Z) and $(f^{a_i}, f^{a_{i-1}})$ are index pairs for A in $(f^{a_i} \setminus f^{a_{i-1}})$, thus the inclusion gives an isomorphism of $H^*(Y, Z)$ and $H^*(f^{a_i}, f^{a_{i-1}})$ as $H^*(M)$-modules. ■

Proposition 6.3. *For every $i = 0, 1, 2, \ldots, p$ and $\beta \in H^k(M)$ we have $\chi(\beta; f^{a_i}) \leqslant i$.*

Proof. The set f^{a_0} is empty thus we start the induction. The inequality

$$\chi(\beta; f^{a_i}) \leqslant \chi(\beta; f^{a_i}, f^{a_{i-1}}) + \chi(\beta; f^{a_{i-1}})$$

can be proved identically to Lemma 5.6 (with $B = \emptyset$). Then we apply Lemma 4.3 to complete the proof. ■

Therefore we have proved

Theorem 6.4. *If M is a smooth closed manifold and a C^1-function $f : M \to \mathbb{R}$ has a finite number of critical points on at most p levels, then $\chi(\beta; M) \leqslant p$ for every $\beta \in H^k(M)$, $k > 0$.*

Example 6.5. Let M be an n-dimensional real projective space $\mathbb{R}P^n$. We have $H^*(M; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^{n+1}$, where $\alpha \in H^1(M; \mathbb{Z}_2)$. This means that $\chi(\alpha; M) = n + 1$. Therefore each smooth function $f : \mathbb{R}P^n \to \mathbb{R}$ has at least $n + 1$ critical points.

The following example is a straightforward consequence of the last one.

EXAMPLE 6.6. Let S^{n-1} be a unit sphere in $\mathbb{R}^n$ and consider an even function $f : S^{n-1} \to \mathbb{R}$ of class C^1 with a finite number of critical points. Then f has at least n pairs of antipodal critical points with different values.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an even function of class C^1 such that the vector field $-\nabla f$ generates a flow φ on $\mathbb{R}^n$. Assume that the annulus $\Omega := \{x \in \mathbb{R}^n : r \leqslant \|x\| \leqslant R\}$ is an isolating neighbourhood for φ. Moreover, assume

(H1) $f(x) \leqslant 0$ for $x \in \partial\Omega$ and $\partial\Omega$ is an exit set (e.g. $-\nabla f$ is directed outward of Ω at points in $\partial\Omega$).

(H2) There exists $\rho \in (r, R)$ such that for x with $\|x\| = \rho$ we have $f(x) \geqslant \alpha > 0$.

THEOREM 6.7. *Under the above assumptions f has at least n pairs of critical points in Ω.*

Let us begin with some notation. First, since the antipodal action of the group $\mathbb{Z}_2$ on Ω is free, the quotient space $M = \Omega/\mathbb{Z}_2$ is a compact manifold with boundary. Indeed, M is diffeomorphic to the product $\mathbb{R}P^{n-1} \times [r, R]$, and the boundary $\partial M \approx \mathbb{R}P^{n-1} \times \{r, R\}$. If we denote by σ the generator of the group $H^1([r, R], \{r, R\}; \mathbb{Z}_2)$, then by Lemma 5.6 $H^*(M, \partial M; \mathbb{Z}_2)$ is a $H^*(M)$-module with the generator $\beta = 1 \times \sigma \in H^1(M, \partial M; \mathbb{Z}_2)$. Nontrivial elements are of the form $\beta \cdot \gamma^k \in H^{k+1}(M, \partial M; \mathbb{Z}_2)$, where $\gamma \in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$.

We denote with the same letter f the induced map $f : M \to \mathbb{R}$. As before, we consider the sublevel sets $f^a = \{x \in M : f(x) \leqslant a\}$. We have natural inclusions $i_a : f^a \hookrightarrow M$.

DEFINITION 6.8. Let $\xi \in H^*(M, \partial M; \mathbb{Z}_2)$. The *depth* of ξ is the number

$$\nu(\xi) = \inf\{a : i_a^*(\xi) \neq 0\}.$$

Notice that the depth of any element is always a critical level of f. Indeed, if a is a regular value of f, then for some $\varepsilon > 0$ the interval $[a - \varepsilon, a]$ consists of regular values. By Deformation Lemma, $f^{a-\varepsilon}$ is a deformation retract of f^a. Thus, if $i_a^*(\xi) \neq 0$ then also $i_{a-\varepsilon}^*(\xi) \neq 0$.

We start with

LEMMA 6.9. *$\nu(\beta) \geqslant \alpha$, where α is from* (H2).

Proof. Let $0 < a < \alpha$. Consider the commutative diagram

$$\begin{tikzcd}
H^0(M) \arrow[r, "\text{mono}"] \arrow[d, "i_a^*"] & H^0(\partial M) \arrow[r, "\delta_1"] \arrow[d, "id^*"] & H^1(M, \partial M) \arrow[d, "i_a^*"] \\
H^0(f^a) \arrow[r, "k"] & H^0(\partial M) \arrow[r, "\delta"] & H^1(f^a, \partial M)
\end{tikzcd}$$

We have to prove that $i_a^*(\beta) = 0$. Since δ_1 is an epimorphism, there exists $\overline{\beta}$ such that $\beta = \delta_1(\overline{\beta})$. Therefore $i_a^*(\beta) = i_a^*(\delta_1(\overline{\beta})) = \delta(\mathrm{id}^*(\overline{\beta}))$.

On the other hand, by our assumptions $\partial M \subset f^a$ has two connected components and the generators of $H^0(\partial M)$ correspond to the generators of $H^0(f^a)$ given by different components of f^a, containing them. Therefore $\mathrm{id}^*(\overline{\beta}) \in \mathrm{Im}(k)$ and thus $\delta(\mathrm{id}^*(\overline{\beta})) = 0$. This completes the proof. ∎

LEMMA 6.10. *Assume that f has only a finite number of critical points. Let $c = \nu(\beta \cdot \gamma^k)$ be the only critical value in the interval $[a_1, a_2]$. Then $\nu(\beta \cdot \gamma^{k+1}) > \nu(\beta \cdot \gamma^k)$.*

Proof. We have $i^*_{a_2}(\beta \cdot \gamma^k) \neq 0$ and $i^*_{a_1}(\beta \cdot \gamma^k) = 0$. We can repeat the argument from Lemma 6.2. The set of critical points $A \subset f^{-1}(c)$ is finite and (f^{a_2}, f^{a_1}) is an index pair of it. We have then $\chi(\gamma; f^{a_2}, f^{a_1}) \leqslant 1$. On the other hand, applying the proof of Lemma 5.4 with $B = \partial M \subset A = f^{a_1} \subset X = f^{a_2} \subset Y = M$ we obtain the inequalities

$$k \leqslant \chi(\gamma; f^{a_2}, \partial M) \leqslant \chi(\gamma; f^{a_2}, f^{a_1}) + \chi(\gamma; f^{a_1}, \partial M) \leqslant 1 + (k-1) = k.$$

Therefore $i^*_{a_2}(\beta \cdot \gamma^{k+1}) = 0$, which ends the proof. ∎

Proof of Theorem 6.7. Now the proof of Theorem 6.7 is immediate. If the number of critical points is finite, then the above lemmas give us n different critical levels of f, which are greater than α. ∎

A more general abstract result of this type can be found in [7], Theorem 4.1.

7. Elliptic BVP. Consider the following family of Dirichlet boundary problems with a parameter $\lambda \in \mathbb{R}$:

$$\Delta + \lambda u = g(u) \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{in } \partial\Omega \tag{2}$$

where

- $\Omega \subset \mathbb{R}^n$ is an open Lipschitzian domain;
- $g \in C^1(\mathbb{R}, \mathbb{R})$ defines a C^1-operator $\mathcal{G} : L^2(\Omega) \to L^2(\Omega)$, $\mathcal{G}(u)(x) := g(u(x))$ such that $\mathcal{G}(u) = o(\|u\|)$ when $u \to 0$;
- $g = \gamma'$, where $\gamma \in C^2(\mathbb{R}, \mathbb{R})$.

It is clear that $u \equiv 0$ is a solution of the above problem for every $\lambda \in \mathbb{R}$.

We are interested in the existence and multiplicity of bifurcation for this problem. To this aim we formulate an appropriate problem in the Hilbert space $L^2(\Omega)$. Let $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\Omega)$. Consider the Sobolev space $H^1(\Omega) \subset L^2(\Omega)$ with the inner product

$$\langle u, v \rangle_1 = \sum_{i=1}^{n} \langle u'_{x_i}, v'_{x_i} \rangle + \langle u, v \rangle,$$

where derivatives are weak derivatives. By $H^1_0(\Omega)$ we denote the closure of a subspace $C^\infty_0(\Omega) \subset H^1(\Omega)$ in the norm $\|\cdot\|_1$.

A variational reformulation of the problem (1)(2) is the following integral equation

$$t(u, v) - \langle \lambda u, v \rangle + \langle \mathcal{G}(u), v \rangle = 0 \quad \forall v \in H^1_0(\Omega), \tag{3}$$

where

$$t(u, v) = \sum_{i=1}^{n} \langle u'_{x_i}, v'_{x_i} \rangle$$

is a bilinear form on $H^1_0(\Omega)$. Then solutions to (3) are called weak solutions to the Dirichlet problem (1)(2). Since t is densely defined, symmetric, closed and bounded from below, there exists a closed linear operator T on $L^2(\Omega)$ with the domain $D(T) \subset H_0^1(\Omega)$ and

$$t(u,v) = \langle Tu, v\rangle \quad \forall u \in D(T),\ v \in H_0^1(\Omega).$$

The operator T is positive and invertible, both T and $T^{-1} : L^2(\Omega) \to L^2(\Omega)$ are selfadjoint and closed. Moreover, T is completely continuous by the Rellich–Kondrashov theorem (comp. [8]).

The spectrum $\sigma(T)$ of T consists of a sequence of real eigenvalues with finite multiplicities $0 < \lambda_1 < \lambda_2 < \ldots$, and $\lambda_k \to \infty$ when $k \to \infty$.

Thus the equality

$$t(u,v) = \left\langle T^{1/2}u, T^{1/2}v\right\rangle \quad \forall u,v \in H_0^1(\Omega),$$

defines a selfadjoint operator in $L^2(\Omega)$ such that $T = (T^{1/2})^2$. $T^{1/2}$ is an isomorphism of spaces $H_0^1(\Omega)$ and $L^2(\Omega)$ and its inverse is completely continuous. Now we can write our problem (3) in the form

$$\left\langle T^{1/2}u, T^{1/2}v\right\rangle - \langle \lambda u, v\rangle + \langle \mathcal{G}(u), v\rangle = 0 \quad \forall v \in H_0^1(\Omega), \tag{4}$$

or, equivalently,

$$\langle w, \zeta\rangle - \left\langle \lambda T^{-1/2}w, T^{-1/2}\zeta\right\rangle + \left\langle T^{-1/2}\mathcal{G}(T^{-1/2}w), \zeta\right\rangle = 0 \quad \forall \zeta \in L^2(\Omega), \tag{5}$$

where $w = T^{1/2}u$, $\zeta = T^{1/2}v$. Since $T^{1/2}$ is selfadjoint, we obtain an equivalent form

$$\langle w, \zeta\rangle - \langle \lambda T^{-1}w, \zeta\rangle + \langle T^{-1/2}\mathcal{G}(T^{-1/2}w), \zeta\rangle = 0 \quad \forall \zeta \in L^2(\Omega). \tag{6}$$

That is, we have an equation in $L^2(\Omega)$:

$$w - \lambda T^{-1}w + f(w) = 0, \tag{7}$$

where $f : L^2(\Omega) \to L^2(\Omega)$ is given by $f(w) = T^{-1/2}\mathcal{G}(T^{-1/2}w)$. This map is of class C^1 and $f(w) = o(\|w\|)$, whenever $w \to 0$; hence $w = 0$ is a solution to (7). Bifurcation of nontrivial solutions can happen only for $\lambda = \lambda_k \in \sigma(T)$, when $\operatorname{Ker}(\mathrm{id} - \lambda T^{-1}) \neq 0$.

Defining a selfadjoint and compact operators $A_\lambda := \mathrm{id} - (\lambda_k + \lambda)T^{-1}$ for $\lambda \in \mathbb{R}$ we write (7) in the form

$$A_\lambda w + f(w) = 0.$$

Since λ_k is an isolated eigenvalue of T with finite multiplicity, we have $0 < \dim \operatorname{Ker} A_0 < \infty$ and for $\lambda \neq 0$ A_λ is an isomorphism onto $R(A_\lambda)$ in a small neighbourhood of $\lambda = 0$. It is easy to check that $F : L^4(\Omega) \times \mathbb{R} \to L^2(\Omega)$ given by $F(w,\lambda) := A_\lambda w + f(w)$ is a family of gradient vector fields with potentials for the components given by $a_\lambda(w) = \frac{1}{2}\langle A_\lambda w, w\rangle$, $\varphi(w) = \int_\Omega \gamma\big(T^{-1/2}w(x)\big)\,dx$, respectively. In this way we obtain a bifurcation problem in the sense of [7], Section 5:

$$F(w,\lambda) = 0. \tag{8}$$

That is, for some interval $0 \in (\lambda_1, \lambda_2) \subset [\lambda_1, \lambda_2]$ we have $F(0,\lambda) = 0$ for all λ, and the derivatives with respect to ω, $D_\omega F(0,\lambda_1)$ and $D_\omega F(0,\lambda_2)$ are isomorphisms. We can now apply a finite-dimensional reduction:

Assume that we have two Banach spaces embedded continuously in a Hilbert space $E_1 \subset E_0 \subset H$. They all can be representations of the group G.

Theorem 7.1 ([7], Theorem 5.1). *Let a G-equivariant mapping $F : \Omega_F \to E_0$ define a bifurcation problem on $[\lambda_1, \lambda_2]$. If there exist decompositions*

$$E_1 = V \oplus W_1,\ E_0 = V \oplus W_0,\ F(x,y,\lambda) = (f_1(x,y,\lambda), f_2(x,y,\lambda)),$$

such that

$$Df_2(0,\lambda)_{|W_1} : W_1 \approx W_0 \ \textit{for} \ \ \lambda \in [\lambda_1, \lambda_2]$$

then there exist:

(1) *an open G-invariant subset $\Omega \subset \Omega_f$, with $\{0\} \times [\lambda_1, \lambda_2] \subset \Omega$;*
(2) *a map $h : \Omega_h \to E_0$, defining a bifurcation problem on $[\lambda_1, \lambda_2]$*

such that

(a) *$F_{|\Omega}$ defines a bifurcation problem on $[\lambda_1, \lambda_2]$ equivalent to that defined by h;*
(b) *$h(V \cap \Omega_h) \subset V$ and $h^{-1}(0) \subset V$;*
(c) *if $D_1 f_2(0,0,\lambda) = 0$ then $D_1 h(0,0,\lambda) = D_1 f_1(0,0,\lambda)$.*

Here the finite-dimensional subspace V is $\operatorname{Ker} A_0$.

Remark. In fact, in order to apply the above theorem we use a small perturbation argument near $\lambda = 0$ (comp. [13], Section II.4.2). We omit the details (see also [21]). Observe that the reduction procedure works for equivariant maps.

In a finite-dimensional case $V = (\mathbb{R}^n, \varphi)$ is an orthogonal representation of a compact Lie group G, i.e. $\varphi : G \to O(n)$ is a group homomorphism. Let $S(V) := \{x \in V : |x| = 1\}$, and $S(\mathbb{R}^n, \epsilon) := \{x \in \mathbb{R}^n : |x| = \epsilon\}$. We can use the following

Lemma 7.2 ([7], Lemma 6.2). *Let $f : \Omega_f \to \mathbb{R}^n$ be a gradient equivariant map defining a bifurcation problem on $[-1, 1]$ and $A_\lambda := D_x f(0,\lambda)$, $\lambda \in [-1, 1]$. Assume that there is $C > 0$ such that*

$$\langle A_1(x), x\rangle \geqslant C|x|^2 \quad \textit{for} \quad x \in \mathbb{R}^n$$

and

$$\langle A_{-1}(x), x\rangle \leqslant -C|x|^2 \quad \textit{for} \quad x \in \mathbb{R}^n.$$

Then for a sufficiently small ϵ the number of zero G-orbits of f in $S(\mathbb{R}^n, \epsilon) \times (-1, 1)$ is not less than the cup-length of $S(V)/G$.

Applying the above results to the trivial group $G = \{e\}$, we have the following:

Theorem 7.3. *For each $k \in \mathbb{N}$ the point $(0, \lambda_k)$ is a bifurcation point of (7) of order at least 2, i.e. for $\lambda = \lambda_k$ there exist at least two solutions on each sufficiently small sphere in $L^2(\Omega)$.*

Proof. It is enough to observe that the cup-length of a sphere is 2 and use Lemma 7.2. ∎

Assume now that our domain is symmetric and the function g in problem (1) is odd $g(-x) = -g(x)$. Then

$$F(-w, \lambda) = -F(w, \lambda).$$

Theorem 7.4. *If g is an odd function and Ω is a symmetric domain in $\mathbb{R}^n$ with respect to the antipodal action, then each point $(0, \lambda_k)$ is a bifurcation point of order $2l$, where l is the multiplicity of λ_k.*

Proof. One observes that the reduction procedures preserve the equivariance property. An action of $\mathbb{Z}_2$ on $L^2(\Omega) \times \mathbb{R}$ is given by $-1 \cdot (f(x), \lambda) = (f(-x), \lambda)$. The main ingredient is that the cup-length of $S^{l-1}/\mathbb{Z}_2$ is equal to l. ∎

The last two results are not new, in fact. Similar results one can find e.g. in [2]. They are described here as a simple illustration of the technique. More complicated symmetries may be considered (comp. an example in [21]). The author is also convinced that problems with p-Laplacians can be considered in a similar way. An application to periodic solutions of Hamiltonian systems is given in [7], where a natural action the group $G = S^1$ on the space of periodic functions is used.

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