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THE PAIRED-DOMINATION AND THE UPPER PAIRED-DOMINATION NUMBERS OF GRAPHS

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Abstract. In this paper we continue the study of paired-domination in graphs. A paired–dominating set, abbreviated PDS, of a graph G with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a PDS of G. The upper paired–domination number of G, denoted by $\Gamma_p(G)$, is the maximum cardinality of a minimal PDS of G. Let G be a connected graph of order $n \geq 3$. Haynes and Slater in [*Paired-domination in graphs*, Networks 32 (1998), 199–206], showed that $\gamma_p(G) \leq n-1$ and they determine the extremal graphs G achieving this bound. In this paper we obtain analogous results for $\Gamma_p(G)$. Dorbec, Henning and McCoy in [*Upper total domination versus upper paired-domination*, Questiones Mathematicae 30 (2007), 1–12] determine $\Gamma_p(P_n)$, instead in this paper we determine $\Gamma_p(C_n)$. Moreover, we describe some families of graphs G for which the equality $\gamma_p(G) = \Gamma_p(G)$ holds.

Keywords: paired-domination, paired-domination number, upper paired-domination number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [5, 6]. Paired-domination in graphs was introduced by Haynes and Slater [7] as a model for assigning backups to guard for security purposes. This concept of domination is well studied (see [1–4, 8–10]).

Let $G = (V, E)$ be a graph which is finite, undirected, without loops, multiple edges and isolated vertices. The number of vertices of G is called the *order* of G and is denoted by $n(G)$. When there is no confusion we use the abbreviation $n(G) = n$. Let H be a connected graph. Then we denote by mH, $m \geq 1$, the graph consisting of

m components $H_1, \ldots, H_m$, where $H_i = H$ for $i = 1, \ldots, m$. A *matching* in a graph G is a set of independent edges in G. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M. A *paired-dominating set*, abbreviated PDS, of a graph G is a set $S = \{u_1, \ldots, u_t, v_1, \ldots, v_t\}$ of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $\langle S \rangle$ induced by S contains a perfect matching $M = \{e_1, \ldots, e_t\}$, where $e_i = u_i v_i$ for $i = 1, \ldots, t$. Two vertices u_i and v_i joined by an edge of M are said to be *paired*. Let S_p be the set of paired vertices in S, that is $S_p = \{\{u_i, v_i\} |$ where $i = 1, \ldots, t\}$. The *paired-domination number* of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a PDS. A PDS S of G is *minimal* if no proper subset of S is a PDS of G. The *upper paired-domination number* of G, denoted by $\Gamma_p(G)$, is the maximum cardinality of a minimal PDS. A minimal PDS of G of cardinality $\Gamma_p(G)$ we call a $\Gamma_p(G)$*–set*.

2. GRAPHS WITH EQUAL γ_p AND Γ_p

The aim of this section is describing graphs G for which $\gamma_p(G) = \Gamma_p(G) = n - i$ for $i = 0, 1, 2$.

We start from the following statement.

Observation 2.1. *For a graph G, $\Gamma_p(G) = n$ if and only if G is mK_2.*

Proof. Obviously, $\Gamma_p(mK_2) = 2m = n$, since for $G = mK_2$ the unique PDS of G is $V(G)$.

Now, suppose that $\Gamma_p(G) = n$ and $G \neq mK_2$. Then, n is even and all the vertices of G are paired in S_p. Since $G \neq mK_2$, without loss of generality we may assume that vertex u_j is adjacent to vertex v_k, where $j \neq k$. But then the vertices of $V(G) - \{v_j, u_k\}$ form a paired-dominating set, which is a contradiction with minimality of $S = V(G)$. □

The *subdivided star* $K^*_{1,t}$ is the graph obtained from a star $K_{1,t}$ by subdividing every edge once. In [7] we have the following notation and statements. Let $\mathcal{F}$ be the collection of graphs C_3,C_5 and the subdivided stars $K^*_{1,t}$.

Theorem 2.2 ([7]). *If G is a connected graph of order $n \geq 3$, then $\gamma_p(G) \leq n - 1$. Furthermore $\gamma_p(G) = n - 1$ if and only if $G \in \mathcal{F}$.*

We can reformulate Corollary 8 of [7] and then we obtain the following statement.

Corollary 2.3. *Let G be a graph with $n \geq 3$. Then $\gamma_p(G) = n - 1$ if and only if $G = H \cup rK_2$, where $H \in \mathcal{F}$ and $r \geq 0$.*

Let $K^{*\Delta}_{1,t}$ be the graph obtained by attaching zero or more triangles to the central vertex of $K^*_{1,t}$ (see Figure 1). Now let $\mathcal{F}^{\Delta} = \{C_3, C_5, K^{*\Delta}_{1,t}\}$.

Now we establish a bound on $\Gamma_p(G)$ for connected graphs G. Moreover, we determine extremal graphs achieving this bound.

Theorem 2.4. *If G is a connected graph of order $n \geq 3$, then $\Gamma_p(G) \leq n - 1$. Furthermore, $\Gamma_p(G) = n - 1$ if and only if $G \in \mathcal{F}^{\Delta}$.*

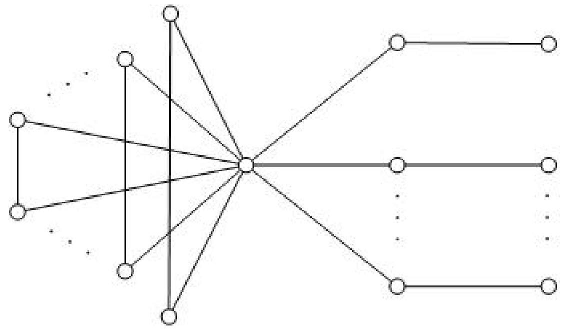


Fig. 1. The graph $K^{*\Delta}_{1,t}$.

Proof. Since G is a connected graph with $n \geq 3$, by Observation 2.1 we have that $\Gamma_p(G) \leq n-1$. It is easy to see that $\Gamma_p(C_3) = 2$, $\Gamma_p(C_5) = 4$ and $\Gamma_p(K^{*\Delta}_{1,t}) = n-1$, and so $\Gamma_p(G) = n-1$ for $G \in \mathcal{F}^{\Delta}$.

Now assume that G is a connected graph with $n \geq 3$ such that $\Gamma_p(G) = n-1$. Let S be a $\Gamma_p(G)$-set and let $V - S = \{x\}$. Since S dominates G, x has at least one neighbour in S, say u_1. If $\Gamma_p(G) = 2$, then G is ether $P_3 = K^*_{1,1}$ or C_3, so $G \in \mathcal{F} \subseteq \mathcal{F}^{\Delta}$. Thus we may assume that $\Gamma_p(G) \geq 4$. Now we state that $S - \{u_1, v_1\}$ induces an independent set of edges. Let us assume that there is on the contrary. Then without loss of generality we may suppose that vertex u_i is adjacent to vertex v_k, where $2 \leq i < k$. It follows that $S - \{v_i, u_k\}$ is a PDS of G with $S_p - \{\{u_i, v_i\}, \{u_k, v_k\}\} \cup \{\{u_i, v_k\}\}$ as a set of paired vertices, that contradicts the minimality of S. Furthermore, if the pair $\{u_i, v_i\} \in S_p - \{\{u_1, v_1\}\}$ has a common neighbour in S, then $S - \{u_i, v_i\}$ is a PDS, contradicting the minimality of S. Suppose that u_1 is adjacent to u_i, where $i \geq 2$. Then, $S_p - \{\{u_1, v_1\}, \{u_i, v_i\}\} \cup \{\{u_1, u_i\}\}$ is a set of paired vertices for a PDS which is $S - \{v_1, v_i\}$, again contradicting the minimality of S. Hence $N(u_1) = \{x, v_1\}$. By connectivity, exactly one vertex from each pair $\{u_i, v_i\} \in S_p - \{u_1, v_1\}$ must be adjacent to v_1 or vertices from $\{u_i, v_i\}$ must be adjacent to x.

Now assume that v_1 is adjacent to u_i for $i \geq 2$. If $N(x) \cap (S - \{u_1, v_i\}) \neq \emptyset$, then the vertices in the pairs of $S_p - \{\{u_1, v_1\}, \{u_i, v_i\}\} \cup \{\{u_i, v_1\}\}$ form a PDS of G, a contradiction. Hence, if $xv_i \in E(G)$ then $N(v_1) = \{u_1, u_i\}$ and $N(x) = \{u_1, v_i\}$ and $G = C_5$.

Thus we have the remaining cases:

(1) exactly one vertex from each pair $\{u_i, v_i\} \in S_p - \{\{u_1, v_1\}\}$ is adjacent to v_1 and we obtain $G = K^*_{1,t}$

and

(2) at least one vertex from $\{u_i, v_i\}$ is adjacent to x and then we obtain $G = K^{*\Delta}_{1,t}$.

This completes the proof of the theorem. □

Corollary 2.5. *Let G be a graph with $n \geq 3$. Then $\Gamma_p(G) = n-1$ if and only if $G = H \cup rK_2$, where $H \in \mathcal{F}^{\Delta}$ and $r \geq 0$.*

Now, let us consider the following problem: for which graphs G the equality $\gamma_p(G) = \Gamma_p(G)$ holds? In this paper we present a solution of the above problem for large parameters.

By Theorem 6 of [7] and Observation 2.1 we obtain the following statement.

Fact 2.6. *Let G be a graph. Then $\gamma_p(G) = \Gamma_p(G) = n$ if and only if $G = mK_2$.*

Since $\mathcal{F} \subseteq \mathcal{F}^{\Delta}$, by Corollary 2.3 and Corollary 2.5, we obtain the following result.

Corollary 2.7. *Let G be a graph satisfying $n \geq 3$. Then $\gamma_p(G) = \Gamma_p(G) = n - 1$ if and only if $G = H \cup rK_2$, where $H \in \mathcal{F}$ and $r \geq 0$.*

Now we determine graphs G for which $\gamma_p(G) = \Gamma_p(G) = n - 2$.

In [10] we showed that only the graphs in family $\mathcal{G}$ (see Figure 2) are connected and satisfy the condition $\gamma_p(G) = n - 2$.

Theorem 2.8. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_p(G) = n - 2$ if and only if $G \in \mathcal{G}$.*

Corollary 2.9. *If G is a graph of order $n \geq 4$, then $\gamma_p(G) = n - 2$ if and only if:*

1) *exactly two of the components of G are isomorphic to graphs of the family $\mathcal{F}$ given in Theorem 2.2 and every other component is K_2 or*
2) *exactly one of the components of G is isomorphic to a graph of the family $\mathcal{G}$ given in Theorem 2.8 and every other component is K_2.*

Next, we describe graphs with the paired-domination and the upper paired-domination numbers two less than their order.

Corollary 2.10. *If G is a graph of order $n \geq 4$, then $\gamma_p(G) = \Gamma_p(G) = n - 2$ if and only if G is a graph given in Theorem 2.8 and Corollary 2.9.*

Proof. It follows from the former theorems that the condition $\gamma_p(G) = n - 2$ holds if and only if $G \in \mathcal{G}$ or G is the graph described in Corollary 2.9. It follows the necessity. Now let $G \in \mathcal{G}$ or G be a graph from Corollary 2.9. Since G is a graph of even order, moreover $\Gamma_p(G) \geq \gamma_p(G)$ and $G \neq mK_2$, then by Observation 2.1 we conclude that $\Gamma_p(G) = \gamma_p(G)$. But then by Theorem 2.8 and Corollary 2.9 we obtain the sufficiency. $\square$

3. Γ_p FOR PATHS AND CYCLES

Dorbec *et al.* [2] established the upper paired-domination number of the path.

Proposition 3.1. *For $n \geq 2$ an integer,*

$$\Gamma_p(P_n) = 8\lfloor (n+1)/11 \rfloor + 2\lfloor ((n+1) \bmod 11)/3 \rfloor.$$

In this paper we determine the upper paired-domination number for the cycle.

Proposition 3.2. *For $n \geq 3$ an integer,*

$$\Gamma_p(C_n) = 8\lfloor n/11 \rfloor + 2\lfloor (n \bmod 11)/3 \rfloor$$

for $n \neq 5$ and $\Gamma_p(C_5) = 4$.

Proof. For $3 \leq n \leq 12$ we can detemine the values of $\Gamma_p(C_n) = 2, 2, 4, 4, 4, 4, 6, 6, 8, 8$, respectively. Thus, the statement holds.

For $n \geq 13$, let $f(n) = 8\lfloor (n+1)/11 \rfloor + 2\lfloor ((n+1) \bmod 11)/3 \rfloor$.

Then we proceed with the following statement.

Claim 1. *For $n \geq 3$ an integer, $f(n-1) \geq 2\lfloor n/3 \rfloor$.*
Proof of Claim 1. Let $n = 33k + r$, where $0 \leq r < 33$. Then $f(n-1) = 24k + r_1$, $2\lfloor n/3 \rfloor = 22k + r_2$ and $r_1 \geq r_2$. Hence we can obtain the desired result.

Now, for the path P_n of order n, we costruct a special $\Gamma_p(P_n)$ – set.

Claim 2. *Let P_n be the path $v_1, v_2, \ldots, v_n$ of order n, where $n \geq 2$ and $n \neq 4$. Then there exists a $\Gamma_p(P_n)$–set S such that $v_1 \in S$.*
Proof of Claim 2. Assume that $v_1, v_2, \ldots, v_n$ are consecutive vertices on the path P_n. We construct a set S as follows. Let $S_p = A_n$ be a set of paired vertices in S for the path P_n. First we define A_n for $2 \leq n \leq 10$. Let

$$\begin{aligned} &A_2 = A_3 = \{\{v_1, v_2\}\}, && A_4 = \{\{v_2, v_3\}\}, \\ &A_5 = \{\{v_1, v_2\}, \{v_4, v_5\}\}, && A_6 = A_7 = \{\{v_1, v_2\}, \{v_5, v_6\}\}, \\ &A_8 = A_9 = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_7, v_8\}\}, && A_{10} = A_8 \cup \{\{v_9, v_{10}\}\}. \end{aligned}$$

Now, we determine A_n for $n = 11k + r$, where $k \geq 1$ and $0 \leq r < 11$. At first we define the sets B_i for $i \geq 0$ as follows:

$$B_i = \{\{v_{1+11i}, v_{2+11i}\}, \{v_{3+11i}, v_{4+11i}\}, \{v_{7+11i}, v_{8+11i}\}, \{v_{9+11i}, v_{10+11i}\}\}.$$

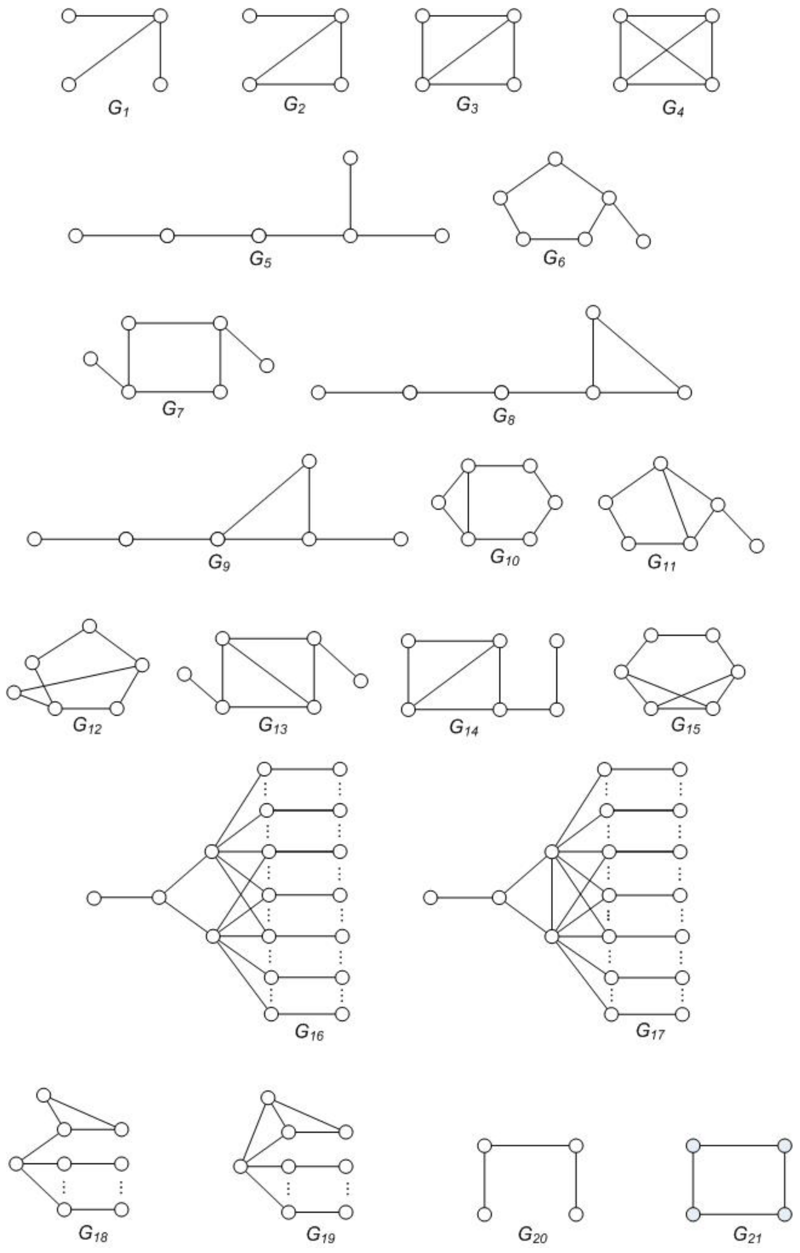
Next, we define A_n as follows: $A_n = \bigcup_{i=0}^{k-1} B_i$ for $r = 0$, $A_n = \bigcup_{i=0}^{k-1} B_i \cup A_r$ for $r \geq 2$ and

$$A_n = \bigcup_{i=0}^{k-1} B_i - \{\{v_{11k-2}, v_{11k-1}\}\} \cup \{\{v_{11k}, v_{11k+1}\}\} \quad \text{for} \quad r = 1.$$

It is clear that the above set S is a PDS of P_n. Now we show the minimality of S. For this purpose suppose that $S' \subseteq S$ and $S' \neq S$, next consider two possibilities. If $S' = S - \{v_j, v_{j+1}\}$, where $\{v_j, v_{j+1}\} \in S_p = A_n$, then S' is not a PDS of P_n. Now assume that $\{v_j, v_{j+1}\}, \{v_{j+2}, v_{j+3}\} \in S_p$. Then $S' = S - \{v_j, v_{j+3}\}$ with paired vertices v_{j+1} and v_{j+2}, is not a PDS of P_n again. Now we calculate the size of S. Let $n = 11k + r$, where $k \geq 1$ and $0 \leq r < 11$. Then consider the following cases.

Case A. $r = 0$. Then we have $|S| = (8/11)n = 8k$, but on the other hand

$$f(n) = 8\lfloor (11k+1)/11 \rfloor + 2\lfloor ((11k+1) \bmod 11)/3 \rfloor = 8k.$$


G_1
G_2
G_3
G_4
G_5
G_6
G_7
G_8
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G_{10}
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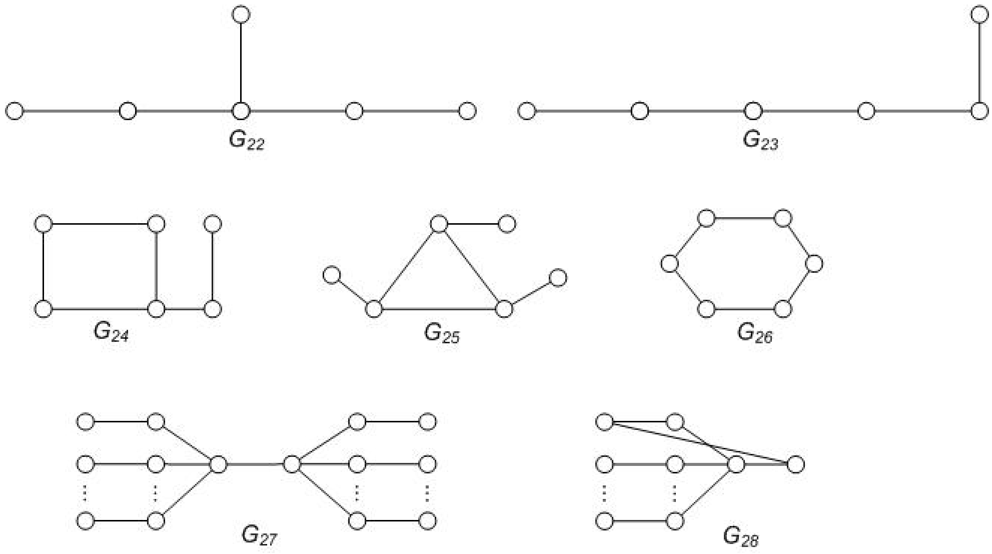

Fig. 2. Graphs in family $\mathcal{G}$

Case B. $r \geq 2$. Now we obtain

$$|S| = 8k + f(r) = 8k + 8\lfloor (r+1)/11 \rfloor + 2\lfloor ((r+1) \bmod 11)/3 \rfloor.$$

Case B.1. $r < 10$. Then

$$|S| = 8k + 2\lfloor ((r+1) \bmod 11/3 \rfloor = f(n).$$

Case B.2. $r = 10$. Then

$$|S| = 8k + f(r) = 8k + 8 = 8\lfloor (n+1)/11 \rfloor = f(n).$$

Case C. $r = 1$. In this case we have $|S| = 8k$, but on the other hand

$$f(n) = 8\lfloor (11k+2)/11 \rfloor + 2\lfloor ((11k+2) \bmod 11)/3 \rfloor = 8k.$$

Thus, in every case we have that $|S| = f(n)$ and S is a $\Gamma_p(P_n)$–set.

Now let $v_1, \ldots, v_n$ are consecutive vertices on the cycle C_n and consider the path $P_{n-1} = C_n - v_n$. By Claim 2, we conclude that there exists a $\Gamma_p(P_{n-1})$–set S such that $v_1 \in S$. It is obvious that S is a PDS of C_n. Now suppose that there exists a proper subset S' of S such that S' is a PDS of C_n. Since $v_n \notin S'$, then S' would be a PDS of P_{n-1}, contradicting the minimality of S. Therefore, S is a minimal PDS of C_n.

Hence we obtain

$$\Gamma_p(P_{n-1}) \leq \Gamma_p(C_n).$$

Now we show that $\Gamma_p(C_n) \leq \Gamma_p(P_{n-1})$.

At first assume that there exists a $\Gamma_p(C_n)$–set S' such that for all vertices v_i, v_{i+1} paired in S', $v_{i-1} \notin S'$ and $v_{i+2} \notin S'$. Then we have $\Gamma_p(C_n) \leq 2\lfloor n/3 \rfloor$. Hence and by Claim 1 we obtain $\Gamma_p(P_{n-1}) \geq \Gamma_p(C_n)$.

So we may assume that for every $\Gamma_p(C_n)$–set S' there exist vertices v_i, v_{i+1} paired in S' and such that at least one vertex v_{i-1}, v_{i+2} is in S'.
Without loss of generality we may assume that vertices v_{n-1}, v_n are paired in S' and at least one among vertices v_{n-2}, v_1 is in S'. It follows from the minimality of S' that exactly one of v_{n-2}, v_1 belongs to S'. Let $v_1 \notin S'$ and $v_{n-2} \in S'$. Hence $v_{n-3} \in S'$. Note that $v_{n-4} \notin S'$, because vertices v_{n-4}, v_{n-5} would be paired in the opposite case, which contradicts the minimality of S'.

Now consider the following cases.

Case 1. $v_2 \in S'$. Then $v_{n-5} \notin S'$, because the set $S' - \{v_{n-3}, v_n\}$ would be a PDS of C_n in the opposite case, which contradicts the minimality of S'. Now S' is a minimal PDS of $P_{n-1} = C_n - v_1$. Really, suppose that S'', where $S'' \subseteq S'$ and $S'' \neq S'$, is a PDS of P_{n-1}. Then S'' must include vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n$, therefore S'' would be a PDS of C_n, a contradiction.

Case 2. $v_2 \notin S'$. Then $v_3 \in S'$.

Case 2.1. $v_{n-5} \in S'$. Then consider the path $P_{n-1} = C_n - v_{n-4}$. By reasoning similar to that in Case 1 we conclude that S' is a minimal PDS of P_{n-1}.

Case 2.2. $v_{n-5} \notin S'$. Then S' is a minimal PDS of $P_{n-1} = C_n - v_1$. Really, suppose that S'', where $S'' \subseteq S'$ and $S'' \neq S'$, is a PDS of P_{n-1}. Then S'' must include vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n$ and v_3, therefore S'' would be a PDS of C_n, a contradiction.

In all cases we have that S' is a minimal PDS of P_{n-1} and so $\Gamma_p(C_n) \leq \Gamma_p(P_{n-1})$.

This completes the proof of the statement. □

Now let us consider the problem when

$$\gamma_p(G) = \Gamma_p(G)$$

for $G = P_n$ or $G = C_n$.

Since $\gamma_p(P_n) = \gamma_p(C_n) = 2\lceil n/4 \rceil$ (see [7]), by Proposition 3.1 and by Proposition 3.2 one can obtain the following statements.

Proposition 3.3. $\gamma_p(P_n) = \Gamma_p(P_n)$ *if and only if* $n = 2, 3, 4, 5, 6, 7$ *or* 9.

Proposition 3.4. $\gamma_p(C_n) = \Gamma_p(C_n)$ *if and only if* $n = 3, 4, 5, 6, 7, 8, 9, 10$ *or* 13.

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