ENGINEERING TRANSACTIONS • *Engng. Trans.* • **63**, 2, 147–156, 2015
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

Arc-length Algorithm Efficiency in the Analysis of Thermally Loaded Multilayered Shells

Agnieszka SABIK, Ireneusz KREJA

Gdańsk University of Technology
Department of Structural Mechanics
Narutowicza 11/12, 80-233 Gdańsk, Poland
e-mail: {agsa, ikreja}@pg.gda.pl

This paper concerns the efficiency study of the arc-length algorithm in the geometrically non-linear analysis of thermally loaded multilayered shells. The thermal loading is considered as the one-way thermo-mechanical coupling effect. Two implementations of the arc-length method are examined: the path-following technique available in NX-Nastran and the Riks-Wempner-Ramm algorithm adopted in the authors' computer code SHLTH. It is shown that the appropriate unloading condition in each step of the analysis plays the crucial role in obtaining a proper solution. The algorithm offered in NX-Nastran tends to fail, whereas the authors' code enables to find the solution in required temperature range.

Key words: stability, temperature, multilayered shells, arc-length method.

1. Introduction

The arc-length method is a very powerful tool in tracing the non-linear equilibrium paths of structures. Due to application of the mixed load-displacement control parameter [1], this technique enables to pass the load limit and turning points, and consequently to follow the post-critical equilibrium trajectories. For that reason the arc-length method is very often used in the stability analysis. However, the algorithm fails in the vicinity of bifurcation points [2, 3]. Although the snap-through phenomenon, which is associated with the limit point, is one of the common forms of instability, in the theoretical studies, in which ideal structures are considered, there still exists a need to deal with the buckling effect [4]. Taking the foregoing into account, if the structure is sensitive to the buckling, the use of arc-length strategy can be insufficient and suitable improvements are required [3].

In this work, the authors focus on the stability of multilayered shells subjected to the thermal loading, being understood as the one-way thermo-mechanical coupling effect, i.e., the deformation does not influence the temperature distribution in the body. It is assumed, that the material properties remain constant during the temperature increase, thus only geometrically non-linear effects are considered. The analysis is performed with the use of arc-length method. As the thermally loaded shells tend to buckle and bifurcation points on the equilibrium path can be expected, the algorithm can suffer from convergence problems. This was shown in [5] on the basis of a critical examination of the path-following algorithm implemented in NX-Nastran (ver. 7.0). In the current study, the capability of the approach adopted in the authors' finite element program SHLTH is presented and the obtained results are compared with solutions coming from NX-Nastran and those available in the literature.

2. Multilayered shell model

In both compared programs the multilayered shell is modelled in a very similar manner. The basic concept involves the idea of a 'smeared' single layer being statically equivalent to the multilayered cross-section. Such an approach is called the equivalent single layer formulation (ESL) [6, 7]. It is commonly used in analyses concerning the global behaviour of multilayered structures, because in opposition to more sophisticated models like 3D or Layer-Wise approaches [6, 7], the number of unknowns in any ESL model is independent of the number of layers. The computational costs are then significantly reduced, especially if whole structures made of laminates with large number of layers are considered.

There are various types of ESL formulations differing in the underlying kinematical assumptions. The simplest one is the classical lamination theory (CLT) which is based on the Kirchhoff-Love assumptions [8]. This model has rather small application area in analysis of fibre reinforced polymer composites due to their considerable shear flexibility. It works well only if the shell is very thin, otherwise an application of such approach can lead to uncertain results. Notably more adequate theories take into account the transverse shear effect. The straightforward extending of CLT is the first-order shear deformation theory (FOSD) [9] in which the hypothesis of straight normal line remains in force but the line is free to rotate with respect to the reference surface. Other approaches belong to the group of higher-order shear deformation theories (HOSD) [9] in which the through-the-thickness displacement profile is described with higher-order polynomials [9, 10] or trigonometric functions [11]. Worth noticing are also formulations with piecewise linear functions simulating the so-called zig-zag effect [10, 12].

In both, NX-Nastran and authors' program SHLTH the ESL model with FOSD theory is employed. Based on the authors' knowledge and experience, this approach, representing a compromise between the CLT and HOSD models, can be a very effective tool in the analysis of global behaviour of multilayered structures, with the restriction that appropriate shear correction factors are used [13, 14]. They can be given *a priori* as constant local values for materials in each layer or global value for the whole cross-section; however, more advanced models utilize global factors determined numerically. In this case the factors very often result from the assumption that during the presumed cylindrical bending of the plate the transverse shear strain energy is equal to the work done by the total transverse shear force on the averaged shear angle [15]. The values of the factors in two transverse shear planes depend then on the lamination scheme of the structure. Such kind of an approach is adopted in both compared programs.

3. Arc-length algorithm in the authors' program SHLTH

The theoretical basis of the authors' model used in the present study is broadly described in [14, 16]. Moreover, the approach is an extension of the model labelled as LRT5 presented in [6], where only mechanical loadings are considered. Therefore, the detailed information about the kinematics and FEM formulation are here omitted and only the aspects of path tracing algorithm are outlined.

The arc-length parameter ds is a mixing value of the displacement increment vector and the load parameter increment:

$$\left({}^{n}\Delta\mathbf{q}^{(0)}\right)^{T}\left({}^{n}\Delta\mathbf{q}^{(0)}\right)+\left({}^{n}\Delta\lambda_{th}^{(0)}\right)^{2}={}^{n}ds^{2}, \tag{3.1}$$

where (n) stands for the increment number and the right superscript $(i)=0$ is the iteration number within the increment. The value of ${}^{n}ds^{2}$ is established once at the beginning of each step but it can be scaled during the whole incremental process according to the solution's convergence. The solution pair consisting of the displacement increment vector ${}^{n}\Delta\mathbf{q}^{(0)}$ and the load parameter increment ${}^{n}\Delta\lambda_{th}^{(0)}$ obtained at the prediction stage $(i=0)$ in each step composes a tangent vector ${}^{n}\mathbf{t}^{(0)}$

$${}^{n}\mathbf{t}^{(0)}=\left[{}^{n}\Delta\mathbf{q}^{(0)},{}^{n}\Delta\lambda_{th}^{(0)}\right], \tag{3.2}$$

which has to be corrected during the iteration process $(i=1,2..i_{\mathrm{END}})$. In the present approach the solution searching direction is constituted by the condition, that

$${}^{n}\mathbf{t}^{(i-1)}\cdot{}^{n}\delta\Delta^{(i)}=0. \tag{3.3}$$

The components of the vector ${}^{n}\delta\Delta^{(i)}$ are the corrections of displacement vector ${}^{n}\delta\mathbf{q}^{(i)}$ and the load parameter increment ${}^{n}\delta\lambda_{th}^{(i)}$ determined sequentially at the iteration stage, i.e., ${}^{n}\delta\Delta^{(i)} = \left[{}^{n}\delta\mathbf{q}^{(i)}, {}^{n}\delta\lambda_{th}^{(i)}\right]$. The constraint Eq. (3.3) provides that the solution is searched along the direction which is perpendicular to the tangent vector obtained in the previous iteration (${}^{n}\mathbf{t}^{(i-1)}$). It is worth mentioning that the searching direction is updated at every iteration, what is the essence of the Riks-Wempner-Ramm method. The method is therefore called the arc-length control with updated hyper-planes. On the contrary, in the classical Riks-Wempner algorithm the hyper-plane remains constant during the iteration process [6].

Another essential problem is the sign of the load parameter increment (${}^{n}\Delta\lambda_{th}^{(0)}$) which has to be established at the prediction stage ($i = 0$). It determines whether the structure should be loaded or unloaded within the current step of the analysis. There are several conditions enabling to set the sign of the initial load increment [17]. Probably the most popular is the control of the sign of the global stiffness matrix determinant:

$$\operatorname{sgn}\left({}^{n}\Delta\lambda_{th}^{(0)}\right) = \operatorname{sgn}|\mathbf{K}|\,. \tag{3.4}$$

Certainly the stiffness matrix becomes singular at limit points but this also takes place at bifurcation points. The use of the above condition leads then to undesirable oscillations of the solution. In the present approach two unloading conditions are implemented: the first one given by (3.4) and the second described in [17, 18], according to which the sign of the initial load increment (${}^{n}\Delta\lambda_{th}^{(0)}$) should fulfil the following inequality:

$$\left({}^{(n-1)}\Delta\right)^{T} \cdot \left({}^{n}\mathbf{t}^{(0)}\right) > 0, \tag{3.5}$$

where $\left({}^{(n-1)}\Delta\right)$ is a vector with components determined in the previous increment:

$${}^{(n-1)}\Delta = \left[{}^{(n-1)}\Delta\mathbf{q}, {}^{(n-1)}\Delta\lambda_{th}\right]. \tag{3.6}$$

The condition (3.5) prevents the previously mentioned oscillatory problems from occuring and provides that the path tracing follows the direction obtained in the previous increments.

4. Numerical examples

The effectiveness of the proposed formulation implemented in SHLTH and its comparison with the algorithm available in NX-Nastran are presented by two numerical examples. As stated previously, the laminate modelling is very similar in both environments, i.e., the ESL concept together with FOSD theory is employed. The discretization however is performed with different finite elements: in NX-Nastran a four-node flat element (QUAD4) is used while in the authors' program SHLTH the eight-node doubly curved element with uniformly reduced integration technique (8URI) is applied.

4.1. Orthotropic cylindrical shell

Following the proposal from [19] we start with probably the most popular example regarding the composite shells subjected to the uniform thermal loading. A one-layer cylindrical shell is considered. The geometrical data are given as follows: $A = B = R\phi$, $R/A = 5$, $A/H = 200$, (Fig. 1).

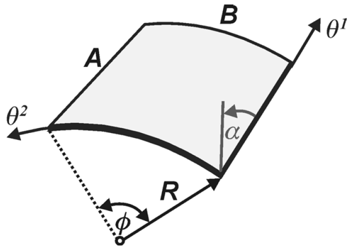

FIG. 1. Geometry of cylindrical shell.

The shell is made of material with the following parameters: $E_a = 138$ GPa, $E_b = 8.28$ GPa, $G_{ab} = G_{ac} = G_{bc} = 6.9$ GPa, $\nu_{ab} = 0.33$, $\alpha_{aa}^{th} = 0.18 \cdot 10^{-6}$ 1/°C, $\alpha_{bb}^{th} = 27 \cdot 10^{-6}$ 1/°C, where E_a and E_b are the Young's modules along (a) and perpendicular (b) to the fibre direction, G_{ab}, G_{ac} and G_{bc} are the shear modules in three orthogonal planes and ν_{ab} is the Poisson's ratio in the layer plane, α_{aa}^{th} and α_{bb}^{th} are the thermal extension coefficients in a and b directions. The fibres are arranged in the circumferential direction ($\alpha = 90°$). All edges are simply supported, fixed against the translations in three directions. In the original paper [19] the symmetry conditions are utilized, whereas in the present study the whole shell is modelled in order not to miss the possible unsymmetrical forms of buckling deformations.

Figure 2 presents the obtained results compared with the reference solution given in [19]. It is a normalized deflection of the central point referred to the actual temperature value. The discretization used in the SHLTH model was 16×16 8URI elements. In NX-Nastran 20×20 QUAD4 elements were employed. It can be observed, that the present model, with the condition (3.5) applied, leads to the proper result in the required range of temperatures, as compared with the reference solution [19]. On the other hand this solution cannot be achieved if the classical condition (3.4) is adopted. This is due to the presence of the bifurcation point, as discussed in details in [14, 16]. By contrast, one

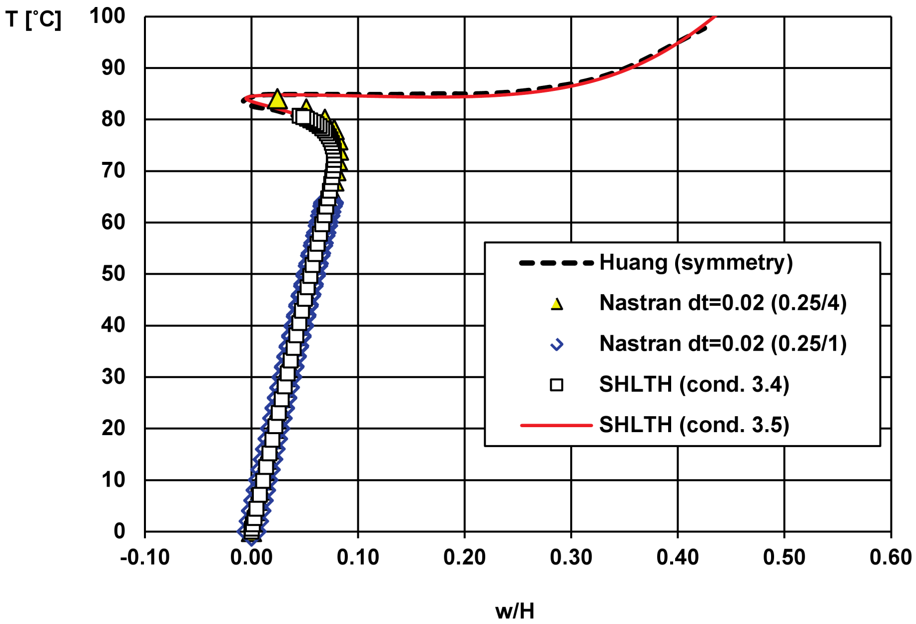

FIG. 2. Central deflection of the orthotropic shell.

cannot obtain the whole path with the use of NX-Nastran. In this case the user has only the possibility to change the constraint equation (Riks, Crisfield and modified Riks) and to control the arc-length parameter. Several options were chosen in the present study [5] and only the exemplary representative results are depicted in Fig. 2. They were obtained with the use of Crisfield condition and initial load parameter value set to $\Delta\lambda_{th,0} = dt = 0.02$, while the scaling of the arc-length was default (0.25/4) or user-defined (0.25/1), whereas the values in the parentheses stand for the scaling factors: the first one represents the shortening and the second the extending factor of the arc-length parameter.

4.2. Cross-ply laminate cylindrical shells

In the second example the cylindrical cross-ply laminate shells subjected to uniform thermal load are studied. The example comes from [20]. The geometry parameters of the panels are determined by $A = B = R\phi$, $\phi = 15°$ (Fig. 1). Two kinds of ratio A/H are examined: $A/H = 200$ and $A/H = 400$. The shell is simply supported with all edges constrained against the three translations. Material data are the same as in the previous example; however, the shell is made of four layers stacked in the sequence [0/90/90/0].

In the authors' program the 10×10 8URI elements were used, whereas in NX-Nastran 40×40 QUAD4 elements were used.

Figure 3 depicts the comparison of the equilibrium paths of the normalized central deflection of the shell with $A/H = 200$. Again, by adopting the condition (3.5) in the present model one is able to obtain the solution which is very close to the reference one. The small discrepancies can be caused by the different laminate modelling used in both cases, i.e., in [20] a layer-wise description was applied. The algorithm fails however if the global stiffness matrix determinant establishes the load increment sign (condition (3.4)). It takes place at the bifurcation temperature level (compare [14]). Similarly as in the previous example the insufficiency of NX-Nastran algorithm is demonstrated here – the solution can be obtained only up to the vicinity of the bifurcation point. The presented path was found with the use of Crisfield condition with initial load increment $\Delta\lambda_{th,0} = dt = 0.02$ and user-defined scaling of arc-length parameter (0.25/1).

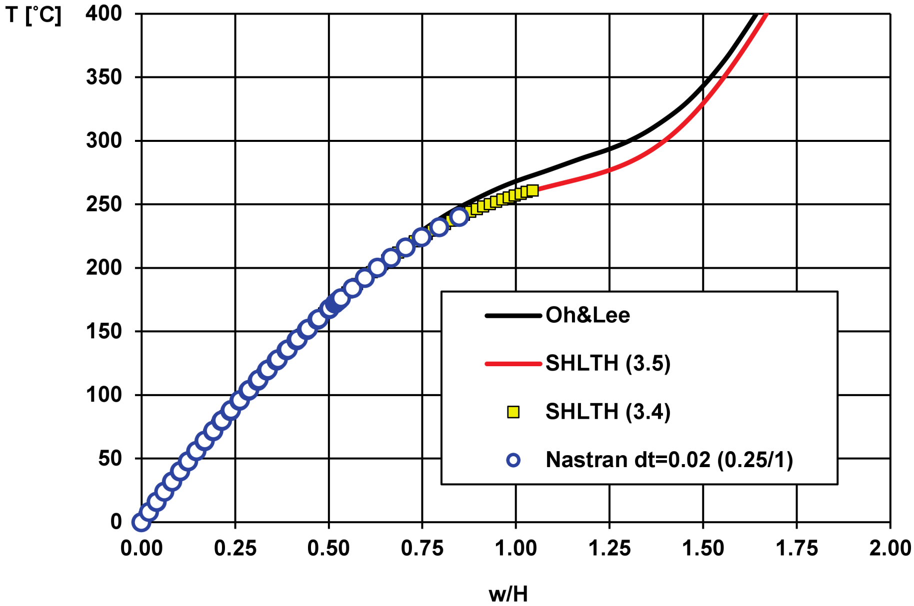

Fig. 3. Central deflection of the cross-ply shell $A/H = 200$.

The equilibrium paths of the normalized central deflection obtained for the thicker shell ($A/H = 400$) are illustrated in the Fig. 4. The same conclusions as in the case of previously analysed shells can be drawn. Only the authors' model, with the condition (3.5) adopted, gives the solution in required range of temperatures; otherwise, oscillatory problems arise around 100°C temperature level. Probably, this is again the result of the presence of the bifurcation point, whose identification is complicated owing to its close location to the load limit point. In this case the results of NX-Nastran are shown for various constraint equations with no scaling of the arc-length parameter (1/1). Regardless the chosen

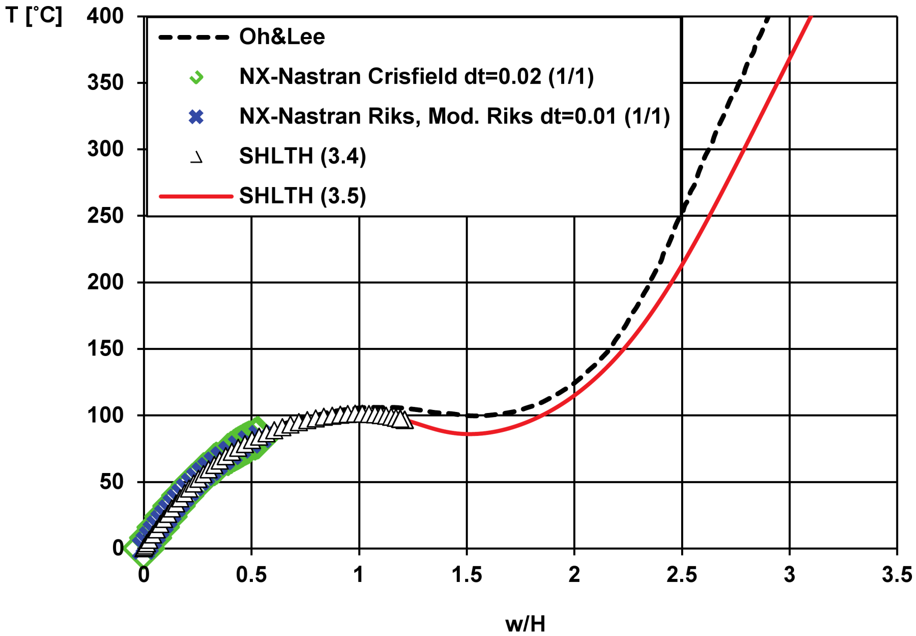

Fig. 4. Central deflection of the cross-ply shell $A/H = 400$.

method convergence problems occur at about 85°C and further computations are impossible.

5. Conclusions

The aim of the work is to present the effectiveness study of the arc-length method in the geometrically non-linear analysis of multilayered shells subjected to the thermal loading. The algorithms in two programs have been compared: the Riks-Wempner-Ramm algorithm in the authors' own program SHLTH and the path-following methods available in NX-Nastran (ver. 7.0). It has been shown that the implementation of the arc-length technique in NX-Nastran is usually unsuccessful if the thermal loads are applied. The authors stress however, that on the other hand this method is very effective when mechanical loadings are considered [21, 22]. On the basis of the own models it has been demonstrated, that the unloading condition, determining whether the structure should be loaded or unloaded, could play an essential role in such a kind of analysis.

Acknowledgment

This contribution has been partly supported by the Polish National Science Centre under grant No. UMO–2011/03/B/ST8/06500.

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Received July 2, 2014; accepted November 14, 2014.