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SYSTEMS OF BOUNDARY VALUE PROBLEMS OF ADVANCED DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper considers the existence of extremal solutions to systems of advanced differential equations with corresponding nonlinear boundary conditions. The monotone iterative method is applied to obtain the existence results. An example is provided for illustration.

Key words: Monotone iterative technique, equations with advanced arguments, monotone sequences, existence of solutions.

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1. INTRODUCTION

Throughout this paper, we introduce the following notations:

$$\begin{aligned} F_i(x,y)(t) &= f_i(t,x(t),x(\alpha(t)),y(t),y(\alpha(t))), \quad i=1,2, \\ G_i(x,y) &= g_i(x(0),x(T),y(0),y(T)), \quad i=1,2. \end{aligned}$$

Motivated by [7] and [1], in this paper, we investigate the following system:

$$\text{(1.1)} \qquad \begin{cases} x'(t) = F_1(x,y)(t), & t \in J=[0,T],\ T>0, \\ y'(t) = F_2(y,x)(t), & t \in J, \\ \quad 0 = G_1(x,y), & 0 = G_2(y,x), \end{cases}$$

where $\alpha \in C(J,J)$, $f_i \in C(J\times\mathbb{R}^4,\mathbb{R})$, $g_i \in C(\mathbb{R}^4,\mathbb{R})$, $i=1,2$.

An interesting and fruitful technique for proving existence results for nonlinear differential problems is the monotone iterative method based on lower and upper solutions, for details, see for example [6]. There exists a vast literature devoted to the applications of this method to differential equations with initial or boundary conditions, we indicate only a few, see for example [2]–[6]. We do not know any application of the monotone iterative method to system of advanced differential equations with boundary conditions.

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Using the monotone iterative method, we establish general sufficient conditions when problem (1.1) has extremal solutions in a corresponding region (see Theorem 3.1). Problem (1.1) is discussed under the assumption that $\alpha \in C(J, J)$ and $t \leq \alpha(t) \leq T$ on J. The one–sided Lipschitz condition on f_1, f_2 and g_1, g_2 with corresponding functions K_i, L_i and constants M_1, M_2 respectively, is assumed. The assumption that K_i and L_i are functions of t has the advantage, because K_i and L_i appear in condition (2.5) to guarantee the existence of solutions of problem (1.1). An example will be given to illustrate the results.

2. SOME AUXILIARY RESULTS

Consider the following problem:

$$x'(t) = g(t, x(t), x(\alpha(t))), \quad t \in J, \quad x(T) = k_0 \in \mathbb{R}. \tag{2.1}$$

Theorem 2.1 (see [4]). *Suppose that*

$A_1:$ *$g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$, $t \leq \alpha(t) \leq T$ on J,*
$A_2:$ *there exist nonnegative constants C_1, C_2 such that*

$$|g(t, x_1, x_2) - g(t, \bar{x}_1, \bar{x}_2)| \leq C_1|x_1 - \bar{x}_1| + C_2|x_2 - \bar{x}_2|$$

for $t \in J$, $x_1, x_2, \bar{x}_1, \bar{x}_2 \in \mathbb{R}$.

Then problem (2.1) *has a unique solution $x \in C^1(J, \mathbb{R})$.*

Lemma 2.2. *Assume that: $\alpha(t) \in C(J, J)$, $t \leq \alpha(t) \leq T$, K_1, K_2, $h_1, h_2 \in C(J, \mathbb{R})$, L_1, L_2 are integrable on J, $M_1, M_2 \in \mathbb{R}$ and $M_1 - M_2 \neq 0$, $M_1 + M_2 \neq 0$.*

Then the system

$$\left\{\begin{array}{rcl} y'(t) &=& K_1(t)y(t) + L_1(t)y(\alpha(t)) + K_2(t)z(t) + L_2(t)z(\alpha(t)) + h_1(t), \\ k_1 &=& -M_1 y(T) + M_2 z(T), \quad k_1 \in \mathbb{R}, \\ z'(t) &=& K_1(t)z(t) + L_1(t)z(\alpha(t)) + K_2(t)y(t) + L_2(t)y(\alpha(t)) + h_2(t), \\ k_2 &=& -M_1 z(T) + M_2 y(T), \quad k_2 \in \mathbb{R} \end{array}\right. \tag{2.2}$$

has a unique solution $(y, z) \in C^1(J, \mathbb{R}) \times C^1(J, \mathbb{R})$.

Proof. Put $P = y + z$, $Q = y - z$. Then, in view of (2.2), we have

$$\left\{\begin{array}{rcl} P'(t) &=& [K_1(t) + K_2(t)]P(t) + [L_1(t) + L_2(t)]P(\alpha(t)) + h_1(t) + h_2(t) \\ k &=& -(M_1 - M_2)P(T), \quad k = k_1 + k_2, \end{array}\right. \tag{2.3}$$

$$\left\{\begin{array}{rcl} Q'(t) &=& [K_1(t) - K_2(t)]Q(t) + [L_1(t) - L_2(t)]Q(\alpha(t)) + h_1(t) - h_2(t), \\ \bar{k} &=& -(M_1 + M_2)Q(T), \quad \bar{k} = k_1 - k_2. \end{array}\right. \tag{2.4}$$

Note that problem (2.3) has a unique solution $P \in C^1(J,\mathbb{R})$, by Theorem 2.1. Similarly, problem (2.4) has also a unique solution $Q \in C^1(J,\mathbb{R})$.

Note that: $y + z = P$, $y - z = Q$, so

$$y = \frac{P+Q}{2}, \quad z = \frac{P-Q}{2}$$

is the unique solution of system (2.2). This ends the proof. □

Remark 2.3. If $L_1(t) = L_2(t) = 0$ on J, then

$$\begin{aligned} P(t) &= -\exp\left(-\int_t^T [K_1(s)+K_2(s)]ds\right)\left\{\frac{k_1+k_2}{M_1-M_2}\right. \\ &\quad \left. + \int_t^T [h_1(s)+h_2(s)]\exp\left(\int_s^T [K_1(\tau)+K_2(\tau)]d\tau\right)ds\right\}, \\ Q(t) &= -\exp\left(-\int_t^T [K_1(s)-K_2(s)]ds\right)\left\{\frac{k_1-k_2}{M_1+M_2}\right. \\ &\quad \left. + \int_t^T [h_1(s)-h_2(s)]\exp\left(\int_s^T [K_1(\tau)-K_2(\tau)]d\tau\right)ds\right\}. \end{aligned}$$

Lemma 2.4 (see [4]). *Let $\alpha \in C(J,J)$, $t \le \alpha(t) \le T$ on J. Suppose that $K \in C(J,\mathbb{R})$, $p \in C^1(J,\mathbb{R})$ and*

$$\begin{cases} p'(t) \ge K(t)p(t) + L(t)p(\alpha(t)), & t \in J, \\ p(T) \le 0, \end{cases}$$

where nonnegative function L is integrable on J.

In addition assume that

$$\int_0^T L(t)\exp\left(\int_t^{\alpha(t)} K(s)ds\right)dt \le 1.$$

Then $p(t) \le 0$ on J.

Lemma 2.5. *Let $\alpha \in C(J,J)$, $t \le \alpha(t) \le T$ on J. Suppose that $K_1, K_2 \in C(J,\mathbb{R})$, $K_2(t) \ge 0$, $M_1, M_2 \in \mathbb{R}$, $M_1 < 0$, $M_1 + M_2 > 0$, $M_2 - M_1 > 0$, $p, q \in C^1(J,\mathbb{R})$, and*

$$\begin{cases} p'(t) \ge K_1(t)p(t) + L_1(t)p(\alpha(t)) - K_2(t)q(t) - L_2(t)q(\alpha(t)), \\ M_1 q(T) + M_2 p(T) \le 0, \end{cases}$$

$$\begin{cases} q'(t) \ge K_1(t)q(t) + L_1(t)q(\alpha(t)) - K_2(t)p(t) - L_2(t)p(\alpha(t)), \\ M_1 p(T) + M_2 q(T) \le 0, \end{cases}$$

where nonnegative functions L_1, L_2 is integrable on J and $L_1(t) + L_2(t) \ge 0$, $L_1(t) \ge L_2(t)$, $t \in J$.

In addition, we assume that

$$
\begin{aligned}
&\int_0^T [L_1(t)+L_2(t)] \exp\left(\int_t^{\alpha(t)} [K_1(s)+K_2(s)]ds\right) dt \le 1,\\
&\int_0^T [L_1(t)-L_2(t)] \exp\left(\int_t^{\alpha(t)} [K_1(s)-K_2(s)]ds\right) dt \le 1,
\end{aligned} \tag{2.5}
$$

Then $p(t) \le 0$, $q(t) \le 0$ *on* J.

Proof. Put $P = p + q$. Then

$$
\begin{cases}
P'(t) \ge [K_1(t) - K_2(t)]P(t) + [L_1(t) - L_2(t)]P(\alpha(t)),\\
(M_1 + M_2)P(T) \le 0.
\end{cases}
$$

Hence, $p(t) + q(t) \le 0$ on J, by Lemma 2.4. Now,

$$
\begin{aligned}
p'(t) &\ge K_1(t)p(t) + L_1(t)p(\alpha(t))\\
&\quad -K_2(t)[q(t)+p(t)-p(t)] - L_2(t)[q(\alpha(t)) + p(\alpha(t)) - p(\alpha(t))]\\
&\ge [K_1(t)+K_2(t)]p(t) + [L_1(t)+L_2(t)]p(\alpha(t)),\\
q'(t) &\ge [K_1(t)+K_2(t)]q(t) + [L_1(t)+L_2(t)]q(\alpha(t)),
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\ge M_1[q(T)+p(T)-p(T)] + M_2p(T) \ge (M_2 - M_1)p(T),\\
0 &\ge (M_2 - M_1)p(T).
\end{aligned}
$$

By Lemma 2.4, $p(t) \le 0$, $q(t) \le 0$ on J. This ends the proof. □

3. MAIN RESULTS

Let us introduce the following assumptions:

$H_1:$ $f_1, f_2 \in C(J \times \mathbb{R}^4, \mathbb{R})$, $g_1, g_2 \in C(\mathbb{R}^4, \mathbb{R})$, $\alpha \in C(J, J)$, $t \le \alpha(t) \le T$,

$H_2:$ $y_0, z_0 \in C^1(J, \mathbb{R})$ satisfy the system:

$$
\begin{cases}
y_0'(t) \le F_1(y_0, z_0)(t), \quad z_0'(t) \ge F_2(z_0, y_0)(t) \in J,\\
G_1(y_0, z_0) \le 0, \quad G_2(z_0, y_0) \ge 0.
\end{cases}
$$

Now, we formulate sufficient conditions under which problem (1.1) has extremal solutions.

Theorem 3.1. *Let assumptions* H_1, H_2 *hold and* $z_0(t) \le y_0(t)$, $t \in J$. *In addition, we assume that*

$H_3:$ *there exist functions* $K_1, K_2 \in C(J, \mathbb{R})$ *and functions* L_1, L_2 *integrable on* J, *such that:*

$$
\begin{aligned}
&f_i(t, u_1, u_2, v_1, v_2) - f_i(t, \bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2)\\
&\ge -K_1(t)[\bar{u}_1 - u_1] - L_1(t)[\bar{u}_2 - u_2] - K_2(t)[\bar{v}_1 - v_1] - L_2(t)[\bar{v}_2 - v_2], \quad i = 1, 2,
\end{aligned}
$$

and

$$f_2(t,u_1,u_2,v_1,v_2)-f_1(t,\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2)$$
$$\geq -K_1(t)[\bar{u}_1-u_1]-L_1(t)[\bar{u}_2-u_2]-K_2(t)[\bar{v}_1-v_1]-L_2(t)[\bar{v}_2-v_2],$$

if $z_0(t)\leq u_1\leq \bar{u}_1\leq y_0(t)$, $z_0(\alpha(t))\leq u_2\leq \bar{u}_2\leq y_0(\alpha(t))$, $z_0(t)\leq \bar{v}_1\leq v_1\leq y_0(t)$, $z_0(\alpha(t))\leq \bar{v}_2\leq v_2\leq y_0(\alpha(t))$

$H_4:$ $K_2(t)\geq 0$, $L_2(t)\geq 0$, $L_1(t)+L_2(t)\geq 0$, $L_1(t)-L_2(t)\geq 0$ *and condition* (2.5) *holds,*

$H_5:$ *both* g_1 *and* g_2 *are non-decreasing in the first variable and non-increasing in the third variable; moreover there exist constants* M_1,M_2 *such that* $M_1\leq 0$, $M_1+M_2>0$, $M_2-M_1>0$ *and*

$$g_i(u,u_2,v,\bar{v}_2)-g_i(u,\bar{u}_2,v,v_2)\leq M_1(\bar{u}_2-u_2)+M_2(\bar{v}_2-v_2),\quad i=1,2,$$

and

$$g_2(u,u_2,v,\bar{v}_2)-g_1(u,\bar{u}_2,v,v_2)\leq -M_1(u_2-\bar{u}_2)-M_2(v_2-\bar{v}_2)$$

if $z_0(0)\leq u\leq y_0(0)$, $z_0(0)\leq v\leq y_0(0)$, $z_0(T)\leq u_2\leq \bar{u}_2\leq y_0(T)$, $z_0(T)\leq v_2\leq \bar{v}_2\leq y_0(T)$.

Then problem (1.1) *has extremal solutions in the sector*

$$[z_0,y_0]_*=\{w\in C^1(J,\mathbb{R}): z_0(t)\leq w(t)\leq y_0(t),\quad t\in J\}.$$

Proof. For, $n=0,1,\ldots$, let

$$\begin{cases} y'_{n+1}(t) &= F_1(y_n,z_n)(t)+\mathcal{F}(y_n,y_{n+1},z_n,z_{n+1})(t),\\ 0 &= G_1(y_n,z_n)-M_1[y_{n+1}(T)-y_n(T)]+M_2[z_{n+1}(T)-z_n(T)],\end{cases}$$

$$\begin{cases} z'_{n+1}(t) &= F_2(z_n,y_n)(t)+\mathcal{F}(z_n,z_{n+1},y_n,y_{n+1})(t),\\ 0 &= G_2(z_n,y_n)-M_1[z_{n+1}(T)-z_n(T)]+M_2[y_{n+1}(T)-y_n(T)],\end{cases}$$

where

$$\begin{aligned}\mathcal{F}(x,\bar{x},y,\bar{y})(t) &= K_1(t)[\bar{x}(t)-x(t)]+L_1(t)[\bar{x}(\alpha(t))-x(\alpha(t))]\\ &+ K_2(t)[\bar{y}(t)-y(t)]+L_2(t)[\bar{y}(\alpha(t))-y(\alpha(t))].\end{aligned}$$

In view of Lemma 2.2, functions y_1,z_1 are well defined. First, we show that

$$z_0(t)\leq z_1(t)\leq y_1(t)\leq y_0(t),\quad t\in J. \tag{3.1}$$

Put $p=z_0-z_1$, $q=y_1-y_0$. This and Assumptions H_3,H_5 yield

$$\begin{aligned}p'(t) &\geq F_2(z_0,y_0)(t)-F_2(z_0,y_0)(t)-\mathcal{F}(z_0,z_1,y_0,y_1)(t)\\ &= K_1(t)p(t)+L_1(t)p(\alpha(t))-K_2(t)q(t)-L_2(t)q(\alpha(t)),\\ q'(t) &\geq F_1(y_0,z_0)(t)-F_1(y_0,z_0)(t)+\mathcal{F}(y_0,y_1,z_0,z_1)(t)\\ &= K_1(t)q(t)+L_1(t)q(\alpha(t))-K_2(t)p(t)-L_2(t)p(\alpha(t)),\end{aligned}$$

and

$$\begin{aligned} 0 &\leq G_2(z_0,y_0) - G_2(z_0,y_0) + M_1[z_1(T) - z_0(T)] - M_2[y_1(T) - y_0(T)] \\ &= -M_1q(T) - M_2p(T), \\ 0 &\geq G_1(y_0,z_0) - G_1(y_0,z_0) + M_1[y_1(T) - y_0(T)] - M_2[z_1(T) - z_0(T)] \\ &= M_1p(T) + M_2q(T). \end{aligned}$$

By Lemma 2.5, we have $z_0 \leq z_1,\ y_1 \leq y_0$.

Now, we put $p = z_1 - y_1$. Then

$$\begin{aligned} p'(t) &= F_2(z_0,y_0)(t) + \mathcal{F}(z_0,z_1,y_0,y_1)(t) - F_1(y_0,z_0)(t) - \mathcal{F}(y_0,y_1,z_0,z_1)(t) \\ &\geq [K_1(t) - K_2(t)]p(t) + [L_1(t) - L_2(t)]p(\alpha(t)), \\ 0 &= G_2(z_0,y_0) - M_1[z_1(T) - z_0(T)] + M_2[y_1(T) - y_0(T)] - G_1(y_0,z_0) \\ &+ M_1[y_1(T) - y_0(T)] - M_2[z_1(T) - z_0(T)] \\ &\leq -(M_1 + M_2)p(T), \end{aligned}$$

by Assumptions H_3 and H_5. Hence, $z_1 \leq y_1$, by Lemma 2.4, so condition (3.1) holds.

Next, let us assume that

$$z_{k-1} \leq z_k \leq y_k \leq y_{k-1} \tag{3.2}$$

for some integer $k > 1$. Put $p = z_k - z_{k+1}$, $q = y_{k+1} - y_k$. Then

$$\begin{aligned} p' &= F_2(z_{k-1}, y_{k-1}(t) + \mathcal{F}(z_{k-1}, z_k, y_{k-1}, y_k)(t) \\ &- F_2(z_k, y_k)(t) - \mathcal{F}(z_k, z_{k+1}, y_k, y_{k+1})(t) \\ &\geq K_1(t)p(t) + L_1(t)p(\alpha(t)) - K_2(t)q(t) - L_2(t)q(\alpha(t)), \\ q'(t) &= F_1(y_k, z_k)(t) + \mathcal{F}(y_k, y_{k+1}, z_k, z_{k+1})(t) \\ &- F_1(y_{k-1}, z_{k-1})(t) - \mathcal{F}(y_{k-1}, y_k, z_{k-1}, z_k)(t) \\ &\geq K_1(t)q(t) + L_1(t)q(\alpha(t)) - K_2(t)p(t) - L_2(t)p(\alpha(t)), \end{aligned}$$

and

$$\begin{aligned} 0 &= G_2(z_{k-1}, y_{k-1}) - M_1[z_k(T) - z_{k-1}(T)] + M_2[y_k(T) - y_{k-1}(T)] \\ &- G_2(z_k, y_k) + M_1[z_{k+1}(T) - z_k(T)] - M_2[y_{k+1}(T) - y_k(T)] \\ &\leq -M_1p(T) - M_2q(T), \\ 0 &= G_1(y_k, z_k) - M_1[y_{k+1}(T) - y_k(T)] + M_2[z_{k+1}(T) - z_k(T)] \\ &- G_1(y_{k-1}, z_{k-1}) + M_1[y_k(T) - y_{k-1}(T)] - M_2[z_k - z_{k-1}(T)] \\ &\leq -M_1q(T) - M_2p(T), \end{aligned}$$

by Assumptions H_3 and H_5.

Hence, $z_k \leq z_{k+1}$, $y_{k+1} \leq y_k$, by Lemma 2.5.

Put $p = z_{k+1} - y_{k+1}$. Then

$$\begin{aligned} p'(t) &= F_2(z_k, y_k)(t) + \mathcal{F}(z_k, z_{k+1}, y_k, y_{k+1})(t) \\ &\quad - F_1(y_k, z_k)(t) - \mathcal{F}(y_k, y_{k+1}, z_k, z_{k+1})(t) \\ &\geq [K_1(t) - K_2(t)]p(t) + [L_1(t) - L_2(t)]p(\alpha(t)), \\ 0 &= G_2(z_k, y_k) - M_1[z_{k+1}(T) - z_k(T)] + M_2[y_{k+1}(T) - y_k(T)] - G_1(y_k, z_k) \\ &\quad + M_1[y_{k+1}(T) - y_k(T)] - M_2[z_{k+1}(T) - z_k(T)] \\ &\leq -(M_1 + M_2)p(T). \end{aligned}$$

Hence, $z_{k+1} \leq y_{k+1}$, by Lemma 2.4, so $z_k \leq z_{k+1} \leq y_{k+1} \leq y_k$.

Hence, using the mathematical induction, we have

$$z_0(t) \leq z_1(t) \leq \cdots \leq z_n(t) \leq z_{n+1}(t) \leq y_{n+1}(t) \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t)$$

for $t \in J$ and $n = 1, 2, \ldots$. Employing standard arguments we see that the sequences $\{y_n, z_n\}$ converge to their limit functions y, z, respectively. Indeed, y and z are solutions of problem (1.1) and $z_0(t) \leq z(t) \leq y(t) \leq y_0(t)$ on J.

To show that z, y are the minimum and maximum solutions of (1.1) we have to prove that if $(u, v) \in [z_0, y_0]_* \times [z_0, y_0]_*$ is any solution of (1.1), then $z(t) \leq u(t) \leq y(t)$, $z(t) \leq v(t) \leq y(t)$ on J. To do this, we assume that $z_m(t) \leq u(t) \leq y_m(t)$, $z_m(t) \leq v(t) \leq y_m(t)$, $t \in J$ for some integer m. Let $p = z_{m+1} - u$, $q = v - y_{m+1}$, $P = u - y_{m+1}$, $Q = z_{m+1} - v$. Then, in view of Assumptions H_3 and H_5, we have

$$\begin{aligned} cllp'(t) &= F_2(z_m, y_m)(t) + \mathcal{F}(z_m, z_{m+1}, y_m, y_{m+1})(t) - F_1(u, v)(t) \\ &\geq K_1(t)p(t) + L_1(t)p(\alpha(t)) - K_2(t)q(t) - L_2(t)q(\alpha(t)), \\ q'(t) &= F_2(v, u)(t) - F_1(y_m, z_m)(t) - \mathcal{F}(y_m, y_{m+1}, z_m, z_{m+1})(t) \\ &\geq K_1(t)q(t) + L_1(t)q(\alpha(t)) - K_2(t)p(t) - L_2(t)p(\alpha(t)), \\ 0 &= G_2(z_m, y_m) - M_1[z_{m+1}(T) - z_m(T)] + M_2[y_{m+1}(T) - y_m(T)] - G_2(u, v) \\ &\leq -M_1p(T) - M_2q(T), \\ 0 &= G_2(v, u) - G_1(y_m, z_m) + M_1[y_{m+1}(T) - y_m(T)] - M_2[z_{m+1}(T) - z_m(T)] \\ &\leq -M_1q(T) - M_2p(T), \end{aligned}$$

and

$$\begin{aligned} P'(t) &= F_1(u, v)(t) - F_1(y_m, z_m)(t) - \mathcal{F}(y_m, y_{m+1}, z_m, z_{m+1})(t) \\ &\geq K_1(t)P(t) + L_1(t)P(\alpha(t)) - K_2(t)Q(t) - L_2(t)Q(\alpha(t)), \\ Q'(t) &= F_2(z_m, y_m)(t) + \mathcal{F}(z_m, z_{m+1}, y_m, y_{m+1})(t) - F_2(v, u)(t) \\ &\geq K_1(t)Q(t) + L_1(t)Q(\alpha(t)) - K_2(t)P(t) - L_2(t)P(\alpha(t)), \\ 0 &= G_2(u, v) - G_1(y_m, z_m) + M_1[y_{m+1}(T) - y_m(T)] - M_2[z_{m+1}(T) - z_m(T)] \end{aligned}$$

$$\le -M_1P(T) - M_2Q(T),$$
$$0 = G_2(z_m, y_m) - M_1[z_{m+1}(T) - z_m(T)] + M_2[y_{m+1}(T) - y_m(T)] - G_2(v, u)$$
$$\le -M_1Q(T) - M_2P(T).$$

This and Lemma 2.5 give $z_{m+1}(t) \le u(t) \le y_{m+1}(t)$, $z_{m+1}(t) \le v(t) \le y_{m+1}(t)$ on J. This proves by induction that $z_n(t) \le u(t) \le y_n(t)$, $z_n(t) \le v(t) \le y_n(t)$ on J for all n. Taking the limit as $n \to \infty$, we conclude the assertion. The proof is complete. □

Example 3.2. Consider the problem

$$(3.3)\qquad \begin{cases} x'(t) &=& t\left[x(\sqrt{t})) + \dfrac{1}{2}y(\sqrt{t})\right] \equiv F_1(x,y)(t), \quad t \in J = [0,1], \\ y'(t) &=& t\left[y(\sqrt{t})) + \dfrac{1}{2}x(\sqrt{t})\right] \equiv F_2(y,x)(t), \quad t \in J, \\ 0 &=& x(1) + y(1) \equiv \mathcal{G}_1(x,y), \quad 0 = y(1) + x(1) \equiv \mathcal{G}_2(y,x). \end{cases}$$

Put $y_0 = 1$, $z_0 = -1$. Then Assumption H_2 holds. Indeed, $K_1(t) = K_2(t) = 0$, $L_1(t) = t$, $L_2(t) = \frac{1}{2}t$, $t \in J$ and $M_1 = 1$, $M_2 = 2$. Moreover, condition (2.5) holds too. In view of Theorem 3.1, problem (3.3) has extremal solutions in the sector $[z_0, y_0]_*$.

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