



Edge-coloring of 3-uniform hypergraphs[☆]

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ABSTRACT

We consider edge-colorings of 3-uniform hypergraphs which is a natural generalization of the problem of edge-colorings of graphs. Various classes of hypergraphs are discussed and we make some initial steps to establish the border between polynomial and NP-complete cases. Unfortunately, the problem appears to be computationally difficult even for relatively simple classes of hypergraphs.

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1. Introduction

Let $H = (V, E)$ be a *hypergraph*, where $V = V(H)$ is the set of *vertices* and $E = E(H)$ is a multiset of nonempty subsets of V called *edges* or *hyperedges*. Throughout this paper we let $n = |V|$ and $m = |E|$. We say that a hyperedge e and a vertex v are *incident* if $v \in e$. A d -*edge* e is a hyperedge that contains exactly d vertices. $\Psi(e) = |e|$ denotes the cardinality of an edge e and $\Psi(H) = \max_{e \in E(H)} \Psi(e)$ is the maximum cardinality of an edge in a hypergraph H . If all edges of a hypergraph are of the same cardinality d then we say that the hypergraph is d -*uniform*. A hypergraph is *simple* if each edge is unique. Notice that simple 2-uniform hypergraphs are just graphs.

In this article we mainly discuss 3-uniform hypergraphs. Henceforth, by the term hypergraph we mean a 3-uniform hypergraph, unless otherwise stated.

The *degree* $\deg(v)$ of a vertex $v \in V$ is the number of edges to which v is incident. $\Delta(H) = \max_{v \in V} \deg(v)$ is the *degree* of H . The *neighborhood* $N(e)$ of an edge e is the set of all edges in H that share at least one vertex with e . $N(H)$ is the cardinality of the maximal neighborhood of an edge in H .

A *proper edge-coloring* of a hypergraph H with k colors is a function $c : E(H) \rightarrow \{1, \dots, k\}$ such that no two edges that share a vertex get the same color (number). Any coloring that uses the minimum number of colors is called *optimal*. The *chromatic index* $\chi'(H)$ of H is defined to be the number of colors in an optimal coloring of H .

For a hypergraph H , the *line graph* $L(H)$ is a simple graph representing adjacencies between hyperedges in H . More precisely, each hyperedge of H is assigned a vertex in $L(H)$ and two vertices in $L(H)$ are adjacent if and only if their corresponding hyperedges share a vertex in H . It is easy to notice that an edge-coloring of a hypergraph is equivalent to a vertex coloring of its line graph.

There are some natural applications of edge-colorings of 3-uniform hypergraphs. It can be directly applied to scheduling of unit execution time tasks on dedicated processors, under the assumption that each task requires exactly three processors for its execution. For each scheduling problem a *scheduling hypergraph* may be obtained in the following manner. Associate

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a vertex with each processor. Each task, which is going to be executed on three prespecified processors, is assigned a 3-hyperedge that consists of the three corresponding vertices. Colors in the edge-coloring problem represent time slots, hence each coloring of the scheduling hypergraph corresponds to a timetable. Furthermore, an optimal coloring implies an optimal schedule.

The model of edge-coloring of 3-uniform hypergraphs may be also applied to solving some variants of Clos networks control [4,9].

2. Bounds on the chromatic index

The following fact gives simple bounds for the chromatic index of 3-uniform hypergraphs.

Fact 2.1. For any 3-uniform hypergraph H the following two inequalities hold: $\Delta(H) \leq \chi'(H) \leq 3\Delta(H) - 2$.

The first inequality is obvious. The second one can be justified in the following manner. Let us consider any greedy algorithm for the hypergraph edge-coloring problem. Each edge in a 3-uniform hypergraph has at most $3\Delta(H) - 3$ neighbors. Hence, for a certain edge e even if all the adjacent edges have various colors, the algorithm has always one color available in a palette of $3\Delta(H) - 2$ colors.

The above fact can be easily generalized to $\chi'(H) \leq N(H) + 1$, or due to Brooks theorem [3], even to $\chi'(H) \leq N(H)$ unless $L(H)$ is a complete graph or an odd cycle.

Conjecture 2.1. For any 3-uniform hypergraph H the following upper bound for chromatic index holds: $\chi'(H) \leq 2\Delta(H) + 1$.

Notice that for $\Delta \leq 3$ Fact 2.1 is stronger than the conjecture. However, for bigger values of Δ the problem remains open. The conjecture is inspired by Vizing's Theorem [14] for simple graphs. One additional vertex at each edge (in comparison to simple graphs) generates Δ extra connections. Note that Berge conjectured a similar problem for line graphs (Conjecture B in [1]) and proved it for hypergraphs without repeated 2-edges. Hence, for linear 3-uniform hypergraphs the conjecture is true. In [11] an asymptotic result was proven that for sufficiently large hypergraphs the chromatic index is close to the degree. The gap between this result and our conjecture is significant, which could suggest that the bound may be tightened. However, the conjecture, for example, is sharp for the Fano plane.

3. Chromatic index for bounded degree hypergraphs

At first let us consider a hypergraph H of degree 2. The maximum degree of the line graph $L(H)$ is $\Delta(L(H)) = 3$ since each hyperedge of H has at most three neighbors. It is well known that a vertex coloring of graphs of degree 3 can be obtained in polynomial time. This implies that for 3-uniform hypergraphs of degree 2 an edge-coloring can be obtained in polynomial time.

The problem becomes more complex if hypergraphs of degree $\Delta = 3$ are considered. At first let us observe that each graph can be easily converted to a 3-uniform hypergraph by adding one additional, say 'pendant' vertex to each edge (2-edge). It was proven by Holyer [8] that it is NP-complete to decide whether the chromatic index of a cubic (each vertex of degree 3) graph is 3. This implies that edge-coloring of 3-uniform hypergraphs of degree 3 is also NP-hard.

4. Hypercycles and hypercacti

We say that a hypergraph H has an *underlying (host) graph* G (spanned on the same set of vertices) if each hyperedge from $E(H)$ induces a connected subgraph in G .

We say that a hypergraph that has an underlying tree is a *hypertree*. It was proven, e.g. in [10], that edge-coloring of hypertrees can be done in polynomial time using a modification of the BFS algorithm.

A hypergraph H is a *hypercycle* if and only if there exists a cycle (simple graph) that is a host graph for H . The problem of edge-coloring hypercycles is equivalent to vertex coloring of circular arc graphs. Recall that G is a circular arc graph if there is a finite collection of arcs on a circle such that G is an intersection graph of this model [2]. A hypercycle can be transformed into arcs on a cycle by putting vertices on a cycle in an order corresponding to one in the underlying cycle. Then, the hyperedges can be regarded as arcs crossing equivalent vertices. Note that the opposite transformation is similar. Since coloring of a circular arc graph was proven to be NP-complete [7], edge-coloring of hypercycles is also NP-complete. However, in this article we discuss 3-uniform hypergraphs and we will prove a slightly more general case of $\Psi(H) = 3$ to be polynomial.

Lemma 4.1. Let H be a hypercycle with m edges and $\Psi(H) = 3$. Then edge coloring of H can be accomplished in time $O(m^{3/2})$.

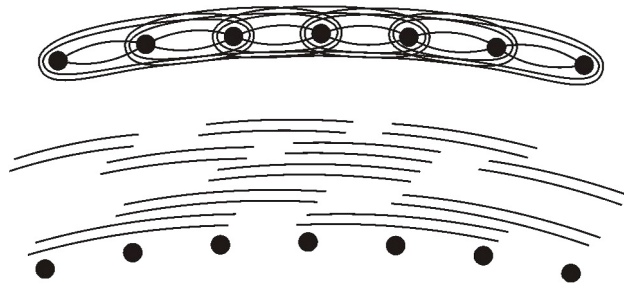


Fig. 1. Part of the hypercycle in which each 2-hyperedge and 3-hyperedge appears twice (upper part of the figure). Proper circular arcs (lower part of the figure) that represent the corresponding part of the hypercycle.

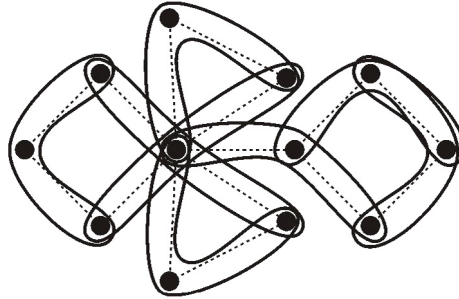


Fig. 2. An example of a 3-uniform hypercactus. Dashed lines stand for a host cactus for the hypercactus.

Proof. In [12] it was shown that a proper circular arc graph (no arc properly contains another) can be colored using Teng and Tucker's approach from [13] in $O(n^{3/2})$ time. A straightforward transformation from a hypercycle to a circular arc graph obviously does not give a proper circular arc graph. However, if we assume that a hypercycle consists of 2-hyperedges and 3-hyperedges only, a transformation to proper circular arcs becomes possible. For a 3-uniform hypercycle, consider any three consecutive vertices a , b and c of the underlying cycle. Arcs related to 3-edges $\{a, b, c\}$ (assume that they are of the same length) cannot be contained one in another, so we shift them slightly. Arcs associated with 2-edges $\{a, b\}$ may also not be contained one in another, thus they are shifted as well. Moreover, they all must start before the arcs $\{a, b, c\}$. A similar tactic is applied to the edges $\{b, c\}$ however here the related arcs must all finish after all $\{a, b, c\}$. The strategy is presented in Fig. 1. Using Teng and Tucker's approach from [13] we get a coloring of the proper circular arc as well as the coloring of the hypercycle. Finally, if we include 1-hyperedges, each of them can be colored with a color missing at the corresponding vertex. Since the transformation to circular arcs and coloring of 1-hyperedges can be both done in linear time, it is the Tucker algorithm that dominates the complexity. ■

Now, we would like to discuss edge-colorings of 3-uniform hypercacti. A *hypercactus* is a hypergraph for which there exist a cactus (simple graph) that underlies it. An exemplary hypercactus is presented in Fig. 2. We remind that a cactus is a graph in which each edge belongs to at most one cycle. In general, coloring hypercacti is NP-complete as they include hypercycles. However, we propose a polynomial time algorithm for edge-coloring of 3-uniform hypercacti.

Theorem 4.1. *Edge-coloring of a 3-uniform hypercactus with m edges can be done in time $O(m^{3/2})$.*

Proof. Let us consider a 3-uniform hypercactus that is not a hypertree. Then there is a hypercycle in it. For example if we take the far left hypercycle of the hypergraph from Fig. 2, we get the hypercycle presented in Fig. 3. Note that hyperedges that are not entirely in the cycle are reduced by missing vertices but are present in the cycle. Since the selected 2-connected component is a hypercycle, by Lemma 4.1, we can color it in time $O(m^{3/2})$. Next, we need to choose another block of the hypercactus that is connected with the colored part. If the current block is a hypercycle it may be colored with Tucker's algorithm otherwise it is a path on two vertices and such hypergraphs can be colored in linear time. To make our approach valid, we need to unify the colors of the two parts. Notice that all hyperedges that appear both in the precolored part and the current block have a vertex in common, hence they get different colors. All we need to do is to permute color labels in the current 2-connected component, so that they match the colors of the precolored part. The procedure of choosing consecutive blocks connected with the colored part and coloring it should be repeated as long as there are uncolored hyperedges. ■

5. 3-partite hypergraphs

Let us define a *3-partite hypergraph* as a 3-uniform hypergraph whose vertices can be partitioned into three non-intersecting sets and each hyperedge is incident to exactly one vertex from each partition.

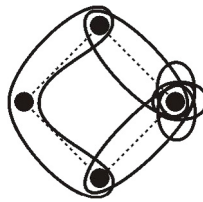


Fig. 3. Hypercycle.

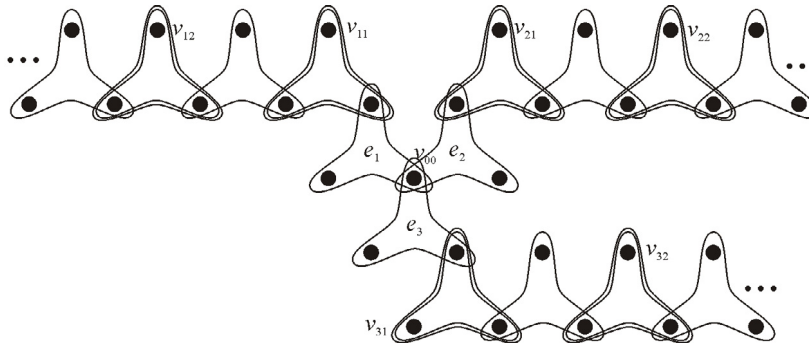


Fig. 4. Gadget used to replace the precoloring function.

Theorem 5.1. *It is NP-complete to decide whether a 3-partite hypergraph of degree 3 is 3-edge-colorable.*

Proof. We will show a polynomial time reduction from the problem of edge precoloring extension to proper 3-coloring for bipartite graphs of degree 3 precolored with at most three colors, which has been proven to be NP-complete by Fiala [6]. However, a similar reduction could be built on a much earlier intractability result for list edge-coloring of subcubic, bipartite graphs [5].

Let us consider a bipartite graph G of degree 3 and its partial precoloring using only three colors. We are going to transform this graph into a 3-partite hypergraph which is three colorable if and only if the precoloring of G can be completed with three colors.

At first let us transform G to a 3-uniform hypergraph by adding one pendant vertex to each edge. Later we will use these pendant vertices to connect edges of a specially created hypergraph (gadget), which will replace the precoloring of the graph G . Obviously the achieved hypergraph is 3-partite as G is bipartite.

The gadget is constructed as follows. Consider one vertex v_{00} and three hyperedges incident with it, namely: e_1 , e_2 and e_3 . e_1 is connected with a hyperpath which consists of one and two parallel hyperedges alternatingly, as shown in Fig. 4. We distinguish vertices v_{11} , v_{12} , etc. that are vertices in the path on the two parallel hyperedges that are not connected to the other hyperedges. Edges e_2 and e_3 also have their equivalent hyperpaths with distinguished vertices. Notice, that if we try to color such a hypergraph (gadget) with 3 colors, the coloring is determined by the colors of the edges e_1 , e_2 and e_3 . Let us assume that e_1 is colored with color 1, e_2 with color 2 and e_3 with color 3. Then the vertices v_{11} , $v_{12} \dots$ are not incident with edges colored with color 1 but the two remaining colors are used. In this way we can get any number of vertices with such a convenient property. However, we need only as many vertices v_{1i} as the number of edges in G precolored with the first color. The situation with vertices v_{21} , $v_{22} \dots$, v_{31} , $v_{32} \dots$ and the second and the third color is analogous.

Now let us remind that we have a bipartite graph G precolored with three colors, that has been modified into a 3-uniform hypergraph. All we need to do is to unify the pendant vertices of edges precolored with the j -th color with one unique vertex v_{ji} of our gadget.

The coloring extension of G with three colors is possible if and only if the obtained hypergraph can be colored with 3 colors.

Since the degree of G is three, the hypergraph is also of degree three. The vertex set of the final hypergraph can be partitioned into three independent sets in the following way. The graph G is bipartite so we have two partitions. The added pendant vertices (unified with vertices v_{ji}) are in the third partition. Also the vertex v_{00} and vertices of degree 1 on paths should be placed in the third partition. The remaining vertices of the gadget (those being part of the paths) are assigned one to the first and another to the second partition accordingly. During the unification process we connect vertices of the third partition in the modified graph G with gadget vertices that are also in the third partition. In this way the result of joining two 3-partite hypergraphs is 3-partite as well. ■

6. Final remarks

In this article, we have shown that edge-coloring of 3-uniform hypergraphs is computationally difficult even for classes of hypergraphs with a relatively simple structure. On the other hand, for some highly-structured classes of hypergraphs the problem is polynomial. Our work makes the first step towards establishing the border between polynomial and intractable instances. In future we would like to direct our research to linear, simple or planar hypergraphs. We also would like to focus our attention on Clos networks and classes of hypergraphs that represent this problem more precisely.

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