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# SOME CONVERGENCE PROPERTIES OF THE SUM OF GAUSSIAN FUNCTIONALS

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**Abstract.** In the paper, some aspects of the convergence of series of dependent Gaussian sequences problem are solved. The necessary and sufficient conditions for the convergence of series of centered dependent indicators are obtained. Some strong convergence results for weighted sums of Gaussian functionals are discussed.

## 1. Introduction

Let $\{X_i\}_{i\geq 1}$ be a normalized Gaussian sequence such that $X_i$, $i = 1, 2, \ldots$ has the standard normal distribution ($N(0;1)$) and let the correlation matrix $\rho_{ij} = E(X_iX_j)$, $i, j = 1, 2, \ldots$, satisfy the following hypothesis:

$$C = \sup_{i\geq 1} \sum_{j=1}^{\infty} |\rho_{ij}| < \infty. \tag{1}$$

It is evident that $C \geq 1$. We denote by $\nu$ the normalized one-dimensional Gaussian measure i.e.

$$\nu(dx) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)dx$$

and use $L^p$ (or $L^p(\nu)$) for $L^p(\mathbb{R}, d\nu)$. The main results of this paper depend on the following lemmas:

**Lemma 1.** ([BC], Borel–Cantelli lemma for Gaussian sequence) *Let the normalized Gaussian sequence $\{X_i\}_{i\geq 1}$ satisfy the hypothesis (1) and let $\{B_i\}_{i\geq 1}$ be a sequence of Borel sets in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} P\{X_i \in B_i\} = \infty$, then $P\{X_i \in B_i \text{ i.o.}\} = 1$. Moreover, if $\sum_{i=1}^{\infty} P\{X_i \in B_i\} < \infty$, then $P\{X_i \in B_i \text{ i.o.}\} = 0$.*

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**Lemma 2.** ([B]) *Let the normalized Gaussian sequence $\{X_i\}_{i\geq 1}$ satisfy the hypothesis* (1) *and let $\{f_i\}_{i\geq 1} \subset L^2(\nu)$. Then for each $n \geq 1$ we have*

$$(2) \qquad (2-C)\sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)) \leq \mathrm{Var}(\sum_{i=1}^{n} f_i(X_i)) \leq C\sum_{i=1}^{n} \mathrm{Var}(f_i(X_i)).$$

**Theorem 1.** ([BC], Rademacher–Menchoff theorem for Gaussian functionals) *Let the normalized Gaussian sequence $\{X_i\}_{i\geq 1}$ satisfy the hypothesis* (1) *and $\{f_i\}_{i\geq 1} \subset L^2(\nu)$ $(Ef_i(X_i) = 0, i \geq 1)$. Suppose additionally that*

$$(3) \qquad \sum_{i=1}^{\infty} (\ln i)^2 E[f_i(X_i)]^2 < \infty.$$

*Then $S_n = \sum_{i=1}^{n} f_i(X_i)$ converges a.s.*

In Section 2, we consider some aspects of the convergence of series of dependent Gaussian sequences problem. There is also a definite result on the necessary and sufficient conditions for the convergence of series of centered dependent indicators. In Section 3, we obtained strong convergence results for weighted sums of Gaussian functionals $f(X_i)$, $i = 1, 2, \dots$, where $f \in L^1(\nu)$, using the same techniques as Xu and Tang [XT] for weighted sums of pairwise negatively quadrant dependent sequences.

## 2. Convergence of the normalized Gaussian sequence

**Theorem 2.** *Let $\{X_i\}_{i\geq 1}$ be a normalized Gaussian sequence satisfying the hypothesis* (1) *and $a = \{a_i\}_{i\geq 1} \in l_2$. Then $\sum_{i=1}^{\infty} a_i X_i$ converges a.s.*

**Proof.** Let us denote

$$(4) \qquad \xi_n = \sum_{j=1}^{n} a_j X_j \quad \text{and} \quad \eta_n = \sum_{j=1}^{n} b_j Y_j,$$

where $b_j = a_j\sqrt{C}$, $j \geq 1$ and $\{Y_j\}_{j\geq 1}$ is a sequence of independent Gaussian variables with $N(0,1)$ distribution.

By Lemma 2 and hypothesis (1) for arbitrary $n \geq m \geq 1$, we have

$$E|\xi_n - \xi_m|^2 = E\Big(\sum_{i=m+1}^{n} a_i X_i\Big)^2 = \sum_{m<i,j\leq n} \rho_{i,j} a_i a_j \leq C \sum_{m<i\leq n} a_i^2$$
$$= \sum_{m<i\leq n} b_i^2 = E\Big(\sum_{i=m+1} b_i Y_i\Big)^2 = E|\eta_n - \eta_m|^2.$$

Hence and by Corollary 3.14 in [LT], we obtain

$$(5) \qquad E \max_{1\leq i\leq N} \xi_i \leq 2E \max_{1\leq i\leq N} \eta_i, \qquad N \geq 1.$$

From (5) and by Levy's inequality, we conclude that

$$\begin{aligned}
E\max_{1\le i\le N}|\xi_i| &\le E|\xi_1| + E\max_{1\le i,j\le N}|\xi_i-\xi_j| = E|\xi_1| + 2E\max_{1\le i,j\le N}\xi_i\\
&\le E|\xi_1| + 4E\max_{1\le i,j\le N}\eta_i \le E|\xi_1| + 4E\max_{1\le i,j\le N}|\eta_i|\\
&\le \sqrt{E\xi_1^2} + 8E|\eta_N| \le \sqrt{E\xi_1^2} + 8\sqrt{E|\eta_N|^2} = (1+8\sqrt{C})\sqrt{\sum_{i=1}^N a_i^2}.
\end{aligned}$$

Hence and by Chebyshev's inequality, we have

$$\begin{aligned}
P\{\sup_{k\ge1}|\xi_{n+k}-\xi_n|>\epsilon\} &= \lim_{N\to\infty} P\{\sup_{1\le k\le N}|\xi_{n+k}-\xi_n|>\epsilon\}\\
&\le \lim_{N\to\infty}\frac{1}{\epsilon}E\sup_{1\le k\le N}|\xi_{n+k}-\xi_n|\\
&\le \lim_{N\to\infty}\frac{1+8\sqrt{C}}{\epsilon}\sqrt{\sum_{k=n+1}^{n+N}a_k^2}\\
&= \frac{1+8\sqrt{C}}{\epsilon}\sqrt{\sum_{k=n+1}^{\infty}a_k^2}\underset{n\to\infty}{\to}0.
\end{aligned}$$

Therefore $\xi_n$ is convergent a.s. and the proof is completed. ∎

It is well known that in case of independent Gaussian sequence $\{X_i\}_{i\ge1}\in N(0,1)$, the condition $\{a_i\}_{i\ge1}\in l_2$ is necessary and sufficient for a.s. convergence of series $\sum_{i=1}^{\infty}a_iX_i$. We have just proved that this condition is also sufficient for Gaussian sequences (not necessary independent) satisfying (1). Necessary condition is satisfied for $1\le C<2$, what is shown by the remark below.

**Remark 1.** Let $\{X_i\}_{i\ge1}$ be a normalized Gaussian sequence satisfying the hypothesis (1), $1\le C<2$ and let $\sum_{i=1}^{\infty}a_iX_i$ converge a.s. Then $a=\{a_i\}_{i\ge1}\in l_2$.

**Proof.** By Lemma 2 with $f_i(x)=a_ix,\quad x\in\mathbb{R}$, we have

$$(2-C)\sum_{i=1}^n a_i^2 = (2-C)\sum_{i=1}^n \mathrm{Var}(a_iX_i) \le \mathrm{Var}(\sum_{i=1}^n a_iX_i).$$

Hence and by implication that $\sum_{i=1}^{\infty}a_iX_i$ converge a.s. then $\sum_{i=1}^{n}\mathrm{Var}(a_iX_i)<\infty$, the proof is completed. ∎

For $C\ge2$ thesis of the Remark 1 is not satisfied. Consider the following example.

Let $\{Y_i\}_{i\geq 1} \subset N(0,1)$ be a sequence of independent Gaussian random variables and define

$$a_n = \frac{1}{\sqrt{\lfloor \frac{n+1}{2} \rfloor}} \qquad \text{and} \qquad X_n = (-1)^n Y_{\lfloor \frac{n+1}{2} \rfloor} \qquad \text{for} \qquad n \geq 1.$$

The series $\sum_{i=1}^{\infty} a_i X_i$ converges a.s. to 0, because the sequences of partial sums $S_{2n}$ and $S_{2n-1}$ converge a.s. to 0. Namely

$$S_{2n} = \sum_{i=1}^{2n} f_i(X_i) = 0 \qquad \text{and} \qquad S_{2n-1} = \sum_{i=1}^{2n-1} f_i(X_i) = -\frac{1}{\sqrt{n}} Y_n \to 0 \ \text{ a.s.}$$

by Borel–Cantelli lemma, where $f_i(x)$ is the same as in the proof of Remark 1.

Then, it is easy to check that

$$C = \sup_{i\geq 1} \sum_{j=1}^{\infty} |\rho_{ij}| = 2,$$

$$\sum_{i=1}^{2n} a_i^2 = 2 \sum_{i=1}^{n} \frac{1}{i} \underset{n\to\infty}{\longrightarrow} \infty.$$

Therefore

$$\sum_{i=1}^{\infty} a_i^2 = \infty.$$

**Lemma 3.** *Let $\{B_n\}_{n\geq 1} \subset B(\mathbb{R})$ be a sequence of Borel subsets and $\{Z_n\}_{n\geq 1}$ be a sequence of random variables. Then*

$$\sum_{n=1}^{\infty} |I_{B_n}(Z_n) - P\{Z_n \in B_n\}| = \sum_{n=1}^{\infty} |I_{B_n'}(Z_n) - P\{Z_n \in B_n'\}| \tag{6}$$

*converges a.s. if and only if*

$$\sum_{n=1}^{\infty} P\{Z_n \in B_n\} P\{Z_n \in B_n'\} < \infty. \tag{7}$$

**Proof.** Let us assume that (7) is satisfied. Then

$$\begin{aligned} E \sum_{n=1}^{\infty} |I_{B_n}(Z_n) - P\{Z_n \in B_n\}| &= \sum_{n=1}^{\infty} E|I_{B_n}(Z_n) - P\{Z_n \in B_n\}| \\ &= 2 \sum_{n=1}^{\infty} P\{Z_n \in B_n\} P\{Z_n \in B_n'\}. \end{aligned}$$

Therefore (6) converges a.s.

On the other hand, (7) is result from the inequality

$$P\{Z_n \in B_n\} P\{Z_n \in B_n'\} \leq |I_{B_n}(Z_n) - P\{Z_n \in B_n\}|. \ \blacksquare$$

**Theorem 3.** *Let $\{X_n\}_{n\geq 1}$ be a normalized Gaussian sequence satisfying the hypothesis* (1) *and $\{B_n\}_{n\geq 1} \subset B(\mathbb{R})$ be a sequence of Borel subsets. Then*

$$\sum_{n=1}^{\infty} (I_{B_n}(X_n) - P\{X_n \in B_n\}) \tag{8}$$

*converges a.s. if and only if*

$$\sum_{n=1}^{\infty} \mathrm{Var}(I_{B_n}(X_n)) = \sum_{n=1}^{\infty} P\{X_n \in B_n\}P\{X_n \in B_n'\} < \infty. \tag{9}$$

**Proof.** (9)$\Rightarrow$ (8) results from Theorem 3. So we only need to prove the implication (8)$\Rightarrow$ (9). Assume that (8) converges a.s. Then there exists a measurable set $\Omega_0 \subset \Omega$ such that $P(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ the series

$$\sum_{n=1}^{\infty} \big(I_{B_n}(X_n(\omega)) - P\{X_n \in B_n\}\big) \tag{10}$$

is convergent. Let $\omega \in \Omega_0$ and denote

$$D(\omega) = \{n \in \mathbb{N} : X_n(\omega) \in B_n\}.$$

In case if $\#D(\omega) < \infty$ or $\#D'(\omega) < \infty$ for some $\omega \in \Omega_0$ thesis is obvious. So, let us assume that $\#D(\omega) = \infty$ and $\#D'(\omega) = \infty$ for all $\omega \in \Omega$. We can order them into increasing sequences $\{n_k(\omega_0)\}_{k\geq 1}$ and $\{n_k'(\omega_0)\}_{k\geq 1}$, respectively where $\omega_0 \in \Omega_0$. Hence and from convergence of series (10), we conclude that

$$\begin{gathered} P\{\omega \in \Omega;\quad X_{n_k(\omega_0)}(\omega) \in B_{n_k(\omega_0)}\} \xrightarrow[k\to\infty]{} 1 \\ \text{and} \\ P\{\omega \in \Omega;\quad X_{n_k'(\omega_0)}(\omega) \in B_{n_k'(\omega_0)}\} \xrightarrow[k\to\infty]{} 0. \end{gathered} \tag{11}$$

Let us point out, that for $\omega_1, \omega_2 \in \Omega_0$ the sets ($\triangle$ means here symmetric difference of sets)

$$D(\omega_1)\triangle D(\omega_2) \quad \text{and} \quad D'(\omega_1)\triangle D'(\omega_2)$$

are finite. Indeed, assume that $D(\omega_1)\triangle D(\omega_2)$ is infinite. Then at least one of the sets $D(\omega_1)\backslash D(\omega_2)$, $D(\omega_2)\backslash D(\omega_1)$ is infinite. Suppose that the first is infinite. Ordering it into the increasing sequence $\{j_i\}_{i\geq 1}$, we see that $\{j_i\}_{i\geq 1}$ is a subsequence of $\{n_k(\omega_1)\}_{k\geq 1}$. Therefore, from (11) we obtain

$$P\{X_{j_i} \in B_{j_i}\} \xrightarrow[i\to\infty]{} 1.$$

On the other hand the sequence $\{j_i\}_{i\geq 1}$ is a subsequence of $\{n_k'(\omega_2)\}_{k\geq 1}$ and from (11) we see that

$$P\{X_{j_i} \in B_{j_i}\} \xrightarrow[i\to\infty]{} 0,$$

is a contradiction. In a similar way we can prove that $D'(\omega_1)\triangle D'(\omega_2)$ is finite. Let $\omega \in \Omega_0$. From assumption the series (10) is convergent. We have two excluding possibilities

$$(12)\qquad \sum_{k=1}^{\infty} P\{X_{n_k(\omega)} \in B'_{n_k(\omega)}\} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} P\{X_{n'_k(\omega)} \in B_{n'_k(\omega)}\} < \infty$$

or

$$(13)\qquad \sum_{k=1}^{\infty} P\{X_{n_k(\omega)} \in B'_{n_k(\omega)}\} = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} P\{X_{n'_k(\omega)} \in B_{n'_k(\omega)}\} = \infty.$$

First, let us notice that the second case (13) does not occur. Indeed, if

$$\sum_{k=1}^{\infty} P\{X_{n_k(\omega)} \in B'_{n_k(\omega)}\} = \infty,$$

then by Lemma 1 we have $P\{X_{n_k(\omega)} \in B'_{n_k(\omega)} \text{ i.o.}\} = 1$. Let $\omega_0 \in \{X_{n_k(\omega)} \in B'_{n_k(\omega)} \text{ i.o.}\} \cap \Omega_0$. Hence, there exists an increasing infinite subsequence $\{n_{k_i}(\omega)\}_{i\geq 1}$ such that $X_{n_{k_i}(\omega)}(\omega_0) \in B'_{n_{k_i}(\omega)}$, $i \geq 1$. It follows that

$$\{n_{k_i}(\omega)\}_{i\geq 1} \subset D(\omega) \cap D'(\omega_0) = D(\omega)\backslash D(\omega_0).$$

But the set $D(\omega)\backslash D(\omega_0)$ is finite, a contradiction. Now, let us return to the first case (12). Then

$$\sum_{n=1}^{\infty} \left| I_{B_n}(X_n(\omega)) - P\{X_n \in B_n\}\right| < \infty.$$

Hence and by the inequality

$$P\{X_n \in B_n\}P\{X_n \in B'_n\} \leq \left|I_{B_n}(X_n(\omega)) - P\{X_n \in B_n\}\right|, \quad n \geq 1$$

we obtain (9), and the proof is complete. ∎

## 3. Strong convergence properties for weighted sums of Gaussian functionals

This section includes two theorems with some sufficient conditions to prove the strong convergence for weighted sums of Gaussian functionals $f(X_i)$, $i = 1, 2, \ldots$, where $f \in L^1(\nu)$. In the proof, we use the same techniques as Xu and Tang [XT] for weighted sums of pairwise negatively quadrant dependent sequences. Hence, here only the main step will be presented. Throughout the section, $a_n \ll b_n$ denotes that there exists a constant $D > 0$ such that $a_n \leq Db_n$ for sufficiently large $n$.

**Theorem 4.** *Let $1 \leq r < 2$, $\{X_n\}_{n\geq 1}$ be a normalized Gaussian sequence satisfying the hypothesis (1) and $f \in L^1(\nu)$. Let $\{a_n\}_{n\geq 1}$ be sequence of positive numbers with $A_n = \sum_{j=1}^{n} a_j \uparrow \infty$. Denote $c_1 = 1$, $c_n = \frac{A_n}{a_n \log n}$ for $n \geq 2$ and $c_n \to \infty$ as $n \to \infty$. Assume that*

$$Ef(X_1) = 0, \quad E|f(X_1)|^r < \infty \tag{14}$$

$$N(n) = \#\{i : c_i \le n\} \ll n^r, \quad n \ge 1, \quad N(0) = 0. \tag{15}$$

*Then*

$$A_n^{-1} \sum_{i=1}^{n} a_i f(X_i) \to 0 \quad a.s. \quad n \to \infty. \tag{16}$$

**Proof.** Define

$$U_n = f(X_n) I(|f(X_n)| \le c_n).$$

To start the proof of (16) note that

$$Ef|(X_1)|^r < \infty \Longleftrightarrow \sum_{i=1}^{\infty} P(|f(X_1)| > c_i) < \infty \Longleftrightarrow P(|f(X_i)| > c_i \ \text{ i.o.}) = 0.$$

Therefore, we have $P(f(X_i) \ne U_i, \ \text{ i.o.}) = 0$ and in order to prove (16), we only need to show

$$A_n^{-1} \sum_{i=1}^{n} a_i U_i \to 0 \qquad \text{a.s.} \qquad n \to \infty. \tag{17}$$

Using the standard techniques, the same as Xu and Tang [XT], it follows from (14) and (15) that

$$\sum_{i=1}^{\infty} \log^2 i \operatorname{Var}\left(\frac{a_i U_i}{A_i}\right) \le \sum_{i=1}^{\infty} c_i^{-2} E[U_i]^2 < \infty.$$

Therefore, by the above inequality, Rademacher–Menchoff type theorem for Gaussian functionals (Theorem 1) and Kronecker's lemma, we have

$$A_n^{-1} \sum_{i=1}^{n} a_i (U_i - EU_i) \to 0 \qquad \text{a.s.}$$

In order to prove (17), it suffices to prove that

$$A_n^{-1} \sum_{i=1}^{n} a_i EU_i \to 0, \qquad n \to \infty. \tag{18}$$

Note that by the Lebesgue dominated convergence theorem

$$E[f(X_i) I(|f(X_i)| \le c_i)] = E[f(X_1) I(|f(X_1)| \le c_i)] \xrightarrow[i \to \infty]{} Ef(X_1) = 0.$$

Therefore

$$|EU_i| = |Ef(X_i) I(|f(X_i)| \le c_i)| \xrightarrow[i \to \infty]{} 0, \quad \text{a.s.}$$

By the Toeplitz's lemma, we have

$$|A_n^{-1} \sum_{i=1}^{n} a_i EU_i| \le A_n^{-1} \sum_{i=1}^{n} a_i |EU_i| \xrightarrow[i \to \infty]{} 0.$$

Hence (18) holds and the proof is completed. ∎

Let $L(x)$ be a slowly varying function at infinity (i.e. $L(x)$ is a positive function defined on $(0,\infty)$ and $L(cx)/L(x) \to 1$ as $x \to \infty$ for all $c > 0$). Then a function $Z(x) = L(x)x^{\gamma}$ varies regularly with exponent $\gamma$ $(-\infty < \gamma < \infty)$. Let us define the functions $Z_p^*(x) = \int_x^\infty y^p Z(y)dy$ and $Z_p(x) = \int_0^x y^p Z(y)dy$, where $x > 0$. In the proof of the next theorem, we will need the following lemma:

**Lemma 4.** ([FE])

a) *If $Z$ varies regularly with exponent $\gamma$ and $Z_p^*$ exists, then*

$$\frac{t^{p+1}Z(t)}{Z_p^*(t)} \to \lambda, \tag{19}$$

*where $\lambda = -(p+\gamma+1) \geq 0$.*

b) *If $Z$ varies regularly with exponent $\gamma$ and if $p \geq -\gamma - 1$, then*

$$\frac{t^{p+1}Z(t)}{Z_p(t)} \to \lambda, \tag{20}$$

*where $\lambda = p + \gamma + 1$.*

**Remark 2.** Let $L(x)$ be a slowly varying function at infinity and let $1 < \alpha < 2$. Then

$$\int_0^x L(t)t^{1-\alpha}\,dt \ll L(x)x^{2-\alpha} \quad \text{and} \quad \int_x^\infty L(t)t^{-\alpha}\,dt \ll L(x)x^{1-\alpha}.$$

**Theorem 5.** *Let $\{X_n\}_{n\geq 1}$ be a normalized Gaussian sequence satisfying the hypothesis (1), $f \in L^1(\nu)$ and*

$$P(|f(X_1)| > x) = \begin{cases} L(x)x^{-\alpha}, & x \geq 1, \\ 1, & x < 1, \end{cases}$$

*where $L(x)$ is a slowly varying function at infinity, $1 < \alpha < 2$ . Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences of positive constants satisfying $b_n \uparrow \infty$. Denote $c_1 = \frac{b_1}{a_1}$, $c_n = \frac{b_n}{a_n \log n}$ for $n \geq 2$. Assume that*

$$\sum_{n=1}^{\infty} P(|f(X_n)| > c_n) < \infty, \tag{21}$$

$$Ef(X_1) = 0, \tag{22}$$

*then*

$$\frac{1}{b_n}\sum_{i=1}^{n} a_k f(X_k) \to 0 \qquad \text{a.s.} \tag{23}$$

**Proof.** By the Borel–Cantelli lemma for Gaussian functionals, (21) implies that

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k f(X_k) I(|f(X_k)| > c_k) \to 0 \qquad \text{a.s.} \qquad n \to \infty. \tag{24}$$

Denote

$$U_k = f(X_k) I(|f(X_k)| \le c_k), \quad k \ge 1.$$

Since

$$\begin{aligned}\frac{1}{b_n}\sum_{k=1}^{n} a_k f(X_k) &= \frac{1}{b_n}\sum_{k=1}^{n} a_k (U_k - EU_k) + \frac{1}{b_n}\sum_{k=1}^{n} a_k EU_k \\ &\quad + \frac{1}{b_n}\sum_{k=1}^{n} a_k f(X_k) I(|f(X_k)| > c_k),\end{aligned}$$

in order to show

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k f(X_k) \to 0 \qquad \text{a.s.},$$

we only need to show that the first two terms above converge to 0 a.s. as $n \to \infty$.

By $E[|X|^q] = q\int_0^\infty t^{q-1} P(|X| > tdt)$, where $q$ denotes a positive real number, Remark 2 and (21), we can get

$$\begin{aligned}\sum_{k=1}^{\infty} \log^2 k \operatorname{Var}\left(\frac{a_k U_k}{b_k}\right) &= \sum_{k=1}^{\infty} c_k^{-2} E[f(X_k)]^2 I(|f(X_k)| \le c_k)] \\ &\le \sum_{k=1}^{\infty} c_k^{-2} \int_0^{c_k} tP(|f(X_k)| > t)dt \\ &\le \sum_{k=1}^{\infty} c_k^{-2} \int_0^{c_k} L(t) t^{1-\alpha} dt \ll \sum_{k=1}^{\infty} L(c_k) c_k^{-\alpha} \\ &\ll \sum_{k=1}^{\infty} P(|f(X_k)| > c_k) < \infty.\end{aligned}$$

Therefore, by Rademacher–Menchoff type theorem for Gaussian functionals (Theorem 1) and Kronecker's lemma, we have

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k (U_k - EU_k) \to 0 \qquad \text{a.s.} \qquad n \to \infty.$$

Again by Remark 2 and (21) and since (22), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\frac{a_k(\log k)EU_k}{b_k}\right| &\leq \sum_{k=1}^{\infty} c_k^{-1}[E|f(X_k)|I(|f(X_k)|>c_k)]\\ &\leq \sum_{k=1}^{\infty} c_k^{-1}\int_{c_k}^{\infty} P(|f(X_k)|>t)dt\\ &\leq \sum_{k=1}^{\infty} c_k^{-1}\int_{c_k}^{\infty} L(t)t^{-\alpha}dt \ll \sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha}\\ &\ll \sum_{k=1}^{\infty} P(|f(X_k)|>c_k)<\infty,\end{aligned}$$

which implies that

$$\sum_{k=1}^{\infty}\frac{a_k EU_k}{b_k} \qquad \text{converges} \qquad \text{a.s.}$$

By Kronecker's lemma, it follows that

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k EU_k \to 0 \qquad \text{a.s.} \qquad n\to\infty.$$

From the statements above we get (23). ∎

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