

Anisotropic Orlicz-Sobolev spaces of vector valued functions and Lagrange equations

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Abstract

In this paper we study some properties of anisotropic Orlicz and Orlicz-Sobolev spaces of vector valued functions for a special class of G-functions. We introduce a variational setting for a class of Lagrangian Systems. We give conditions which ensure that the principal part of variational functional is finitely defined and continuously differentiable on Orlicz-Sobolev space.

Keywords: anisotropic Orlicz space, anisotropic Orlicz-Sobolev space, Lagrange equations, variational functional

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1. Introduction

In this paper we make some preliminary steps for variational analysis in anisotropic Orlicz-Sobolev spaces of vector valued functions. We consider the Euler-Lagrange equation

$$\frac{d}{dt}L_v(t, u(t), \dot{u}(t)) = L_x(t, u(t), \dot{u}(t)), \quad t \in (a, b) \quad (1)$$

where Lagrangian is of the form $L(t, x, v) = F(t, x, v) + V(t, x)$.

If $F(v) = \frac{1}{2}|v|^2$ then the equation (1) reduces to $\ddot{u}(t) + \nabla V(t, u(t)) = 0$. One can consider more general case $F(v) = \phi(|v|)$, where ϕ is convex and nonnegative. In the above cases F does not depend on v directly but rather on its norm $|v|$ and the growth of F is the same in all directions, i.e. F has isotropic growth. Equation (1) with Lagrangian $L(t, x, v) = \frac{1}{p}|v|^p + V(t, x)$ has been studied by

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many authors under different conditions. The classical reference is [1]. The isotropic Orlicz-Sobolev space setting was considered in [2].

We are interested in anisotropic case. This means that F depends on all components of v not only on $|v|$ and has different growth in different directions. A simple example of such function is $F(v) = \sum_{i=1}^N |v_i|^{p_i}$ or $F(v) = \sum_{i=1}^N \phi_i(|v_i|)$, where ϕ_i are N-functions. We wish to consider more general situation. We assume that $F: [a, b] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies

$$(F_1) \quad F \in C^1,$$

$$(F_2) \quad |F(t, x, v)| \leq a(|x|)(b(t) + G(v)),$$

$$(F_3) \quad |F_x(t, x, v)| \leq a(|x|)(b(t) + G(v)),$$

$$(F_4) \quad G^*(F_v(t, x, v)) \leq a(|x|)(c(t) + G^*(\nabla G(v))),$$

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$ and $G: \mathbb{R}^N \rightarrow \mathbb{R}$ is a G-function. Conditions (F_1) – (F_4) are direct generalization of standard growth conditions from [1] (see also [2]). We show (see Theorem 5.7) that under these conditions the functional $\mathcal{I}: \mathbf{W}^1 \mathbf{L}^G \rightarrow \mathbb{R}$ given by

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u}) dt$$

is continuously differentiable.

We restrict our considerations to a special class of G-functions. Here $G: \mathbb{R}^n \rightarrow [0, \infty)$ is convex, $G(-x) = G(x)$, supercoercive, $G(0) = 0$ and satisfies Δ_2 and ∇_2 conditions. We define the anisotropic Orlicz space to be

$$\mathbf{L}^G(I, \mathbb{R}^N) = \{u: I \rightarrow \mathbb{R}^N: \int_I G(u) dt < \infty\}.$$

The Orlicz space \mathbf{L}^G equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0: \int_I G\left(\frac{u}{\alpha}\right) dt \leq 1 \right\}$$

is a reflexive Banach space. An important example of Orlicz space is classical Lebesgue \mathbf{L}^p space, defined by $G(x) = \frac{1}{p}|x|^p$. In this case, the Luxemburg norm and the standard \mathbf{L}^p norm are equivalent. Therefore, Orlicz spaces can be viewed as a straightforward generalization of \mathbf{L}^p spaces.

Properties of N-functions and Orlicz spaces of real-valued functions has been studied in great details in monographs [3, 4, 5] and [6]. The standard references for vector-valued case are [7, 8, 9] and [10, 11] for Banach-space valued functions. In [7, 8] author considers a class of G-functions together with a uniformity conditions



which, for example, excludes the function $G(x) = \sum |x_i|^{p_i}$ unless $1 < p_1 = \dots = p_N < \infty$. Moreover G is not necessarily assumed to be an even function. As was pointed out in [11], if G is not even then \mathbf{L}^G is no longer a vector space (see also [10, Example 2.1]).

Our strong conditions on G allow us to work in Orlicz spaces without worry about some technical difficulties arising in general case. For example, it is well known that the set $\mathbf{L}^G(I, \mathbb{R}^N)$ is a vector space if and only if G satisfies Δ_2 condition. Otherwise \mathbf{L}^G is only a convex set. Another difficulty is the convergence notion. In Lebesgue spaces $\|u_n - u\|_{\mathbf{L}^p} \rightarrow 0$ means simply $\int |u_n - u|^p \rightarrow 0$. For arbitrary G-function G , convergence in Luxemburg norm is not equivalent to $\int G(u_n - u) dt \rightarrow 0$ unless G satisfies Δ_2 . The Δ_2 condition is also crucial for separability and reflexivity of \mathbf{L}^G .

The main consequence of anisotropic nature of G is the lack of monotonicity of the norm. It is no longer true that $|u| \leq |v|$ implies $\|u\|_{\mathbf{L}^G} \leq \|v\|_{\mathbf{L}^G}$. In anisotropic case, standard dominance condition $|u_n| \leq f$ does not implies convergence in \mathbf{L}^G norm and must be replaced by $G(u_n) \leq f$ (see Theorem 3.17).

For every G there exist $p, q \in (1, \infty)$ such that $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$. If $G(x) = \sum |x_i|^{p_i}$ then \mathbf{L}^G can be identified with the product of \mathbf{L}^{p_i} but in many cases an anisotropic Orlicz Space is not equal to the space $\mathbf{L}^{p_1} \times \mathbf{L}^{p_2} \times \dots \times \mathbf{L}^{p_N}$ (see Example 3.7).

To give a proper variational setting for equation (1) we introduce a notion of an anisotropic Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G$ of vector-valued functions. It is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^N) = \{u \in \mathbf{L}^G(I, \mathbb{R}^N) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^N)\}$$

with the norm

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

To the authors best knowledge there is no reference for the case of anisotropic norm and vector-valued functions of one variable. The references for other cases are [2, 9, 12, 13, 14, 15, 16, 17, 18, 19].

In [9] and [18] the space $H^0(G, \Omega)$, $\Omega \subset \mathbb{R}^n$ is defined as a completion of $C_0^1(\Omega, \mathbb{R}^n)$ under norm $\|u\|_{H^0(G, \Omega)} = \|Du\|_{G, \Omega}$. It is classical result due to Trudinger $H^0(G, \Omega) \hookrightarrow L_A(\Omega)$, where A is some N-function (see also Cianchi [14]).

In [17] and [19] the anisotropic Orlicz-Sobolev space $W^1 L_G$ is defined for G-function $G : \mathbb{R}^{n+1} \rightarrow [0, \infty]$ as a space of weakly differentiable functions $u : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}$ such that $(u, D_1 u, D_2 u, \dots, D_n u)$ belongs to the Orlicz space generated by G . A norm for $W^1 L_G$ is given by

$$\|u\|_{1, G, \Omega} = \|(u, Du)\|_{G, \Omega}.$$

In [12] we can find definition of isotropic Orlicz-Sobolev space of real valued functions

$$W_A^1(\Omega) = \{u \in \Omega \rightarrow \mathbb{R} \text{ measurable} : u, |\nabla u| \in L_A\},$$

where L_A is Orlicz Space and A is an N-function.

In [2] the isotropic Orlicz-Sobolev space of vector-valued functions is defined to be a space of absolutely continuous functions $u : [0, T] \rightarrow \mathbb{R}^d$ such that u and \dot{u} belongs to Orlicz space generated by an N-function. Similar treatment can be found in [20].

2. G-functions

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^N and $|\cdot|$ is the induced norm. We assume that $G : \mathbb{R}^N \rightarrow [0, \infty)$ satisfies the following conditions:

$$(G_1) \quad G(0) = 0,$$

$$(G_2) \quad G \text{ is convex,}$$

$$(G_3) \quad G \text{ is even,}$$

$$(G_4) \quad G \text{ is supercoercive:}$$

$$\lim_{|x| \rightarrow \infty} \frac{G(x)}{|x|} = \infty,$$

$$(G_5) \quad G \text{ satisfies the } \Delta_2 \text{ condition:}$$

$$\exists_{K_1 \geq 2} \exists_{M_1 > 0} \forall_{|x| \geq M_1} G(2x) \leq K_1 G(x), \quad (\Delta_2)$$

$$(G_6) \quad G \text{ satisfies the } \nabla_2 \text{ condition:}$$

$$\exists_{K_2 \geq 1} \exists_{M_2 > 0} \forall_{|x| \geq M_2} G(x) \leq \frac{1}{2K_2} G(K_2 x). \quad (\nabla_2)$$

A function G is a G-function in the sense of Trudinger [9]. In general, G-function can be unbounded on bounded sets and need not satisfy conditions (G_4) – (G_6) but only $\lim_{x \rightarrow \infty} G(x) = \infty$. A G-function of one variable is called N-function. Some typical examples of G are:

1. $G_p(x) = \frac{1}{p}|x|^p, 1 < p < \infty,$
2. $G(x) = \sum_{i=1}^N G_{p_i}(x_i), 1 < p_i < \infty,$
3. $G(x_1, x_2) = (x_1 - x_2)^2 + x_2^4.$

A function G can be equal to zero in some neighborhood of 0. So that a function

$$G(x) = \begin{cases} 0 & |x| \leq 1 \\ |x|^2 - 1 & |x| > 1 \end{cases}$$

is also admissible. Condition Δ_2 implies that G is of polynomial growth (see Lemma 2.4 below and [3]). A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ $f(x) = e^{|x|} - |x| - 1$ does not satisfy Δ_2 .

Since G is convex and finite on \mathbb{R}^n , G is locally Lipschitz and therefore continuous. Note that for every $x \in \mathbb{R}^N$

$$\begin{aligned} G(\alpha x) &\leq \alpha G(x), \text{ if } 0 \leq \alpha \leq 1, \\ \alpha G(x) &\leq G(\alpha x), \text{ if } 1 \leq \alpha. \end{aligned}$$

We get immediately that G is non-decreasing along any half-line through the origin i.e. for every $x \in \mathbb{R}^N$

$$0 < \alpha \leq \beta \implies G(\alpha x) \leq G(\beta x). \quad (2)$$

Our assumptions on G imply that for every $x_0 \in \mathbb{R}^N$ there exists $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$

$$\langle a, x_0 \rangle + b = G(x_0) \text{ and } \langle a, x \rangle + b \leq G(x).$$

From this, we can easily obtain the Jensen integral inequality. Let $I \subset \mathbb{R}$ be a finite interval and let $u \in \mathbf{L}^1(I, \mathbb{R}^N)$. Then

$$G\left(\frac{1}{\mu(I)} \int_I u \, dt\right) \leq \frac{1}{\mu(I)} \int_I G(u) \, dt.$$

We will often make use of the following simple observation.

Proposition 2.1. For all $\alpha \in \mathbb{R}$ there exists $K_1(\alpha) > 0$ such that

$$G(\alpha x) \leq K_1(\alpha)G(x)$$

for all $|x| \geq M_1$.

In fact, the above proposition provides a characterization of Δ_2 (see [7, 11]). It follows that for every $\alpha \in \mathbb{R}$ there exists $C_\alpha > 0$ such that for $x \in \mathbb{R}^N$

$$G(\alpha x) \leq C_\alpha + K_1(\alpha)G(x).$$

We recall a notion of Fenchel conjugate. Define $G^* : \mathbb{R}^N \rightarrow [0, \infty)$ by

$$G^*(y) := \sup_{x \in \mathbb{R}^N} \{\langle x, y \rangle - G(x)\}.$$

A function G^* is called Fenchel conjugate of G . As an immediate consequence of definition we have the so called Fenchel inequality:

$$\forall_{x,y \in \mathbb{R}^N} \langle x, y \rangle \leq G(x) + G^*(y).$$

Consider arbitrary $f: \mathbb{R}^N \rightarrow [0, \infty)$. It is obvious that the conjugate function f^* is always convex. But in general f^* need not be continuous, finite or coercive, even if f is. From the other hand, it is well known that if f is convex and l.s.c. then $f^* \neq \infty$ and $(f^*)^* = f$.

Example 2.2.

1. If

$$g(x) = \begin{cases} 0 & |x| \leq 1 \\ \infty & |x| > 1 \end{cases}$$

then $g^*(x) = |x|$. Note that g and g^* are G-functions but do not satisfy our assumptions.

2. If $G_p(x) = \frac{1}{p}|x|^p$, then $G_p^*(x) = \frac{1}{q}|x|^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

3. If $G(x) = \sum_{i=1}^N G_{p_i}(x_i)$, then $G^*(x) = \sum_{i=1}^N G_{p_i}^*(x_i)$,

4. If $G(x, y) = (x - y)^2 + y^4$, then

$$G^*(x, y) = \frac{1}{4}x^2 + \frac{3}{4}(x + y) \left(\frac{x + y}{4} \right)^{\frac{1}{3}}.$$

More information on general theory of conjugate functions can be found in standard books on convex analysis, see for instance [21, 22].

If a function $G: \mathbb{R}^n \rightarrow [0, \infty)$ satisfies conditions (G_1) – (G_6) then the same is true for its conjugate G^* . This is main reason we want to restrict class of considered functions.

Theorem 2.3. If G satisfies conditions (G_1) – (G_6) then G^* also satisfies (G_1) – (G_6) and $(G^*)^* = G$.

Proof. It is evident that G^* satisfies (G_1) , (G_2) and (G_3) . It is well known that, under our conditions, G^* is finite (proposition 1.3.8, [21]), G^* is supercoercive (proposition 1.3.9, [21]) and G^* satisfies (G_5) and (G_6) (remark 2.3, [10]). Corollary [21, cor. 1.3.6] gives $(G^*)^* = G$. \square

In order to compare growth rate of G-functions we define two relations. Let G_1 and G_2 be G-functions. Define

$$G_1 \prec G_2 \iff \exists_{M \geq 0} \exists_{K > 0} \forall_{|x| \geq M} G_1(x) \leq G_2(Kx)$$

and

$$G_1 \prec\prec G_2 \iff \forall \alpha > 0 \lim_{|x| \rightarrow \infty} \frac{G_2(\alpha x)}{G_1(x)} = \infty.$$

For conjugate functions we have (see [3, thm. 3.1])

$$G_1 \prec G_2 \Rightarrow G_2^* \prec G_1^*.$$

Obviously $G_1 \prec\prec G_2$ implies $G_1 \prec G_2$. Assumption (G_4) implies $|x| \prec\prec G$. It is true that $|x| \prec G$ holds under weaker assumption: $G(x) \rightarrow \infty$. Note that, if $p > 1$ then $|x| \prec\prec |x|^p$. Hence, if $|x|^p \prec G$ then $|x| \prec\prec G$. Since G satisfies (G_5) and (G_6) we have the following bounds for the growth of G .

Lemma 2.4 ([10, Lemma 2.4]). There exist $p, q \in (1, \infty)$ such that

$$|x|^p \prec G \prec |x|^q.$$

The exponents p and q depend on the constants in the ∇_2 and Δ_2 conditions respectively. Immediately from the above we get $|x|^{\frac{q}{q-1}} \prec G^* \prec |x|^{\frac{p}{p-1}}$.

3. Orlicz spaces

Let $I \subset \mathbb{R}$ be a finite interval. The Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^n)$ is defined to be

$$\mathbf{L}^G(I, \mathbb{R}^n) = \left\{ u: I \rightarrow \mathbb{R}^n: u \text{ - measurable, } \int_I G(u) dt < \infty \right\}.$$

As usual, we identify functions equal a.e. For $u \in \mathbf{L}^G$ define:

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0: \int_I G\left(\frac{u}{\alpha}\right) dt \leq 1 \right\}.$$

The function $\|\cdot\|_{\mathbf{L}^G}$ is called the Luxemburg norm. It is easy to see that

$$\int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1,$$

since G satisfies Δ_2 . Moreover

$$\int_I G\left(\frac{u}{k}\right) dt \leq 1 \iff \|u\|_{\mathbf{L}^G} \leq k.$$

Using Fenchel's inequality we obtain the Hölder inequality

$$\int_I \langle u, v \rangle dt \leq 2\|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^*}}, \quad u \in \mathbf{L}^G \text{ and } v \in \mathbf{L}^{G^*}$$



Similarly to [3] and [8] one can show that \mathbf{L}^G is a linear ([3, thm. 8.2]) and normed space ([8, thm. 2.3]). Completeness and separability of \mathbf{L}^G can be obtained in the same way as in [11, thm. 6.1, thm. 6.3, cor. 6.1]. Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^p \hookrightarrow \mathbf{L}^{p_0} \hookrightarrow \mathbf{L}^1$ (see propositions 3.3 and 3.4 below) and $1 < p_0 < p$, it follows that \mathbf{L}^G is reflexive space. The proof, in more general case, can be found in [11].

According to above remarks, we have the following theorem.

Theorem 3.1. If $G: \mathbb{R}^n \rightarrow [0, \infty)$ satisfies (G_1) – (G_6) , then $(\mathbf{L}^G(I, \mathbb{R}^n), \|\cdot\|_{\mathbf{L}^G})$ is a separable, reflexive Banach space.

Remark 3.2.

1. All properties of \mathbf{L}^G remains true for \mathbf{L}^{G^*} , since G and G^* belongs to the same class of functions.
2. For an arbitrary G-function $f: \mathbb{R}^n \rightarrow [0, \infty)$ which does not satisfies Δ_2 the set \mathbf{L}^f is not a linear space but only a convex set. In fact, it is well known that the set \mathbf{L}^f is linear space if and only if a G-function f satisfies Δ_2 condition.
3. It was pointed out by Schappacher [11, example 3.1] that if f is not bounded on bounded sets (i.e. we allow $f(x) = +\infty$ for some $x \in \mathbb{R}^n$) then \mathbf{L}^f need not be a linear space, even if f satisfies Δ_2 condition (see [3, 11]).
4. It is well known that if G-function does not satisfies Δ_2 condition then \mathbf{L}^G is not separable. One can define a subspace E^G as the closure of bounded functions under Luxemburg norm. In this case, the space E^G is a proper subset of \mathbf{L}^G and is always separable (see [3, 11]).
5. For every $F \in (\mathbf{L}^G)^*$ there exists unique $v \in \mathbf{L}^{G^*}$ such that for every $u \in \mathbf{L}^G$

$$Fu = \int_I \langle u, v \rangle dt.$$

As a consequence we obtain that $\mathbf{L}^{G^*} \simeq (\mathbf{L}^G)^*$. Since $G^{**} = G$, we also get $\mathbf{L}^G \simeq (\mathbf{L}^{G^*})^*$ (see [3, 8, 11]).

6. If G-function does not satisfies Δ_2 and ∇_2 conditions, then \mathbf{L}^G is not reflexive and $(\mathbf{L}^G)^*$ is not isomorphic to \mathbf{L}^{G^*} (see [3, 11]).

An important example of Orlicz space is a classical Lebesgue space $(\mathbf{L}^p, \|\cdot\|_{\mathbf{L}^p})$, $p \in (1, \infty)$ defined by $G(x) = \frac{1}{p}|x|^p$. It is easy to check that in this case $\mathbf{L}^G = \mathbf{L}^p$ and the Luxemburg norm and standard \mathbf{L}^p norm are equivalent. Two important examples of Lebesgue spaces are not covered in our setting, namely \mathbf{L}^1 and \mathbf{L}^∞ .

The space \mathbf{L}^1 is generated by $f(x) = |x|$ and the space \mathbf{L}^∞ generated by f^* . We exclude these two spaces because we want to have only reflexive spaces in the class of Orlicz spaces we consider.

We will use the symbols \hookrightarrow and \hookleftrightarrow for, respectively, continuous and compact embeddings. Using the same methods as in [6, th. 8.12, 8.24] we obtain basic embedding theorems for anisotropic Orlicz spaces.

Proposition 3.3. Assume that $F \prec G$. Then $L^G \hookrightarrow L^F$ and

$$\|u\|_{\mathbf{L}^F} \leq K(C\mu(I) + 1)\|u\|_{\mathbf{L}^G}$$

for some $C > 0$.

Proposition 3.4. If $F \prec\prec G$ then $\mathbf{L}^G \hookleftrightarrow \mathbf{L}^F$.

Directly from Lemma 2.4 we obtain that Orlicz spaces can be viewed as a spaces between two Lebesgue spaces determined by constants in Δ_2 and ∇_2 conditions.

Proposition 3.5. For every G there exist $p, q \in (1, \infty)$ such that

$$\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p.$$

In particular $\mathbf{L}^\infty \hookrightarrow \mathbf{L}^G \hookleftrightarrow \mathbf{L}^1$.

In some cases \mathbf{L}^G is simply a product of $\mathbf{L}^{p_i}(I, \mathbb{R})$, but there exist Orlicz spaces which are not in the form $\mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R})$ (cf. [9, pp. 18-20]).

Example 3.6. Consider the Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$ generated, by $G(x) = |x_1|^{p_1} + |x_2|^{p_2}$, $p_1, p_2 > 0$. If $u = (u_1, u_2) \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$, then

$$\int_I G(u) dt = \int_I |u_1|^{p_1} dt + \int_I |u_2|^{p_2} dt < \infty.$$

Conversely, if $u = (u_1, u_2) \in \mathbf{L}^G$ then

$$\int_I |u_1|^{p_1} dt \leq \int_I G(u) dt < \infty \text{ and } \int_I |u_2|^{p_2} dt \leq \int_I G(u) dt < \infty.$$

Hence $u \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$.

Example 3.7. Consider the Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$ generated, by $G(x) = (x_1 - x_2)^4 + x_2^2$. From Lemma 2.4 and Proposition 3.5 we obtain that $\mathbf{L}^4(I, \mathbb{R}^2) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}^2)$. Let u_1 be a function in $\mathbf{L}^2(I, \mathbb{R})$ such that $u_1 \notin \mathbf{L}^p(I, \mathbb{R})$, for $p > 2$. Set $u = (u_1, u_1)$, then

$$\int_I G(u) dt = \int_I |u_1|^2 dt < \infty$$



but

$$\int_I |u|^p dt = \infty.$$

Therefore for every $p > 2$ there exists $u \in \mathbf{L}^G$ such that $u \notin \mathbf{L}^p(I, \mathbb{R}^2)$. Moreover, $u \notin \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$ for any $p > 2$. From the other hand if $u = (u_1, u_2) \in \mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R})$ then $u \in \mathbf{L}^G$. Therefore

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$$

but \mathbf{L}^G cannot be identified with any

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R}) \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R}).$$

3.1. Convergence

Now we investigate relations between Luxemburg norm and the integral

$$R_G(u) := \int_I G(u) dt.$$

A functional R_G is called modular. Theory of modulars is well known and is developed in more general setting than ours. More information can be found in [23, 5].

For Lebesgue spaces a notions of modular and norm are indistinguishable because modular $\int_I |u|^p dt$ is equal to $\|u\|_{\mathbf{L}^p}^p$. But in Orlicz spaces relation between R_G and $\|\cdot\|_{\mathbf{L}^G}$ is more complex.

There is remarkable difference between isotropic and anisotropic spaces. It is clear that if $u, v \in \mathbf{L}^p$ (or more generally in isotropic Orlicz space) then $|u(t)| \leq |v(t)|$ a.e. implies $\|u\|_{\mathbf{L}^p} \leq \|v\|_{\mathbf{L}^p}$. In anisotropic case it is no longer true, even if $G(u(t)) < G(v(t))$. Next two examples illustrates this point.

Example 3.8. Let $G(x, y) = (x - y)^2 + y^4$, $I = [0, 1]$, $u(t) = (2, 0)$ and $v(t) = (2, 3/2)$. Then $|u(t)| < |v(t)|$, $G(u(t)) < G(v(t))$ and $R_G(u) \leq R_G(v)$, but $2 = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.6$.

Example 3.9. Let $G(x, y) = x^2 + y^4$, $u(t) = (1, 0)$ and $v(t) = \frac{11}{10}(\cos t, \sqrt{\sin t})$. In $\mathbf{L}^G([0, \pi], \mathbb{R}^2)$ we have $\sqrt{\pi} = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.7$, but $|u(t)| < |v(t)|$, $G(u(t)) < G(v(t))$ for all $t \in [0, \pi]$ and $R_G(u) < R_G(v)$.

Definition 3.10. We say that a subset $K \subset \mathbf{L}^G$ is modular bounded if there exists $C > 0$ such that

$$R_G(u) \leq C, \text{ for all } u \in K.$$

Modular boundedness is sometimes called mean boundedness. It is evident that $R_G(u) \leq \|u\|_{\mathbf{L}^G}$ if $\|u\|_{\mathbf{L}^G} \leq 1$ and $R_G(u) > \|u\|_{\mathbf{L}^G}$ if $\|u\|_{\mathbf{L}^G} > 1$.

Lemma 3.11. Let $u \in \mathbf{L}^G$.

1. If $R_G(u) \leq C$ then $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$.
2. If $\|u\|_{\mathbf{L}^G} \leq C$ then $R_G(u) \leq \mu(I)\tilde{C} + K_1(C)$ for some $\tilde{C} > 0$.

Moreover, a set $K \subset \mathbf{L}^G$ is modular bounded if and only if is norm bounded.

Proof. Assume that $R_G(u) \leq C$. If $C \leq 1$ then $\|u\|_{\mathbf{L}^G} \leq 1$. If $C > 1$ then

$$\int_I G\left(\frac{u}{C}\right) dt \leq \frac{1}{C} \int_I G(u) dt \leq 1.$$

This implies $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$. For the second statement, assume $\|u\|_{\mathbf{L}^G} \leq C$. Then

$$R_G(u) = \int_{I_1} G(u) dt + \int_{I \setminus I_1} G\left(C \frac{u}{C}\right) dt \leq \mu(I_1)\tilde{C} + K_1(C) \int_I G\left(\frac{u}{C}\right) dt,$$

where $I_1 = \{t \in I: |u(t)| \leq M_1 C\}$ and $\tilde{C} > 0$. To finish the proof observe that

$$\int_I G\left(\frac{u}{C}\right) dt \leq \int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1.$$

□

Definition 3.12. We say that a sequence of functions $u_k \in \mathbf{L}^G$ is modular convergent to $u \in \mathbf{L}^G$ if $R_G(u_k - u) \rightarrow 0$ as $k \rightarrow \infty$.

Modular convergence is sometimes called mean convergence. Norm convergence always implies modular convergence. Let $\|u_k\|_{\mathbf{L}^G} \rightarrow 0$ as $k \rightarrow \infty$. We can assume that $\forall_k \|u_k\|_{\mathbf{L}^G} \leq 1$, then

$$\frac{1}{\|u_k\|_{\mathbf{L}^G}} R_G(u_k) \leq R_G\left(\frac{u_k}{\|u_k\|_{\mathbf{L}^G}}\right) = 1.$$

Hence $0 \leq R_G(u_k) \leq \|u_k\|_{\mathbf{L}^G}$. In general, converse is not true unless G satisfies Δ_2 condition (see [3, 11]).

Theorem 3.13. Norm convergence is equivalent to modular convergence.

Proof. We need only to prove that modular convergence implies norm convergence. Fix $\varepsilon > 0$ and assume that $\{u_k\}$ is modular convergent to 0. Define

$$I_{1,k} = \{t \in I: |u_k(t)| \leq M_1\}.$$

Since G satisfies Δ_2 , for all $k > 0$ we have

$$\begin{aligned} \int_I G(u_k/\varepsilon) dt &\leq \mu(I_{1,k}) C_{1/\varepsilon} + K_1(1/\varepsilon) \int_{I \setminus I_{1,k}} G(u_k) dt \leq \\ &\leq \mu(I) C_{1/\varepsilon} + K_1(1/\varepsilon) \int_I G(u_k) dt. \end{aligned}$$

For sufficiently large k we have

$$\int_I G(u_k) dt \leq \frac{1}{K_1(1/\varepsilon)}$$

and

$$\int_I G(u_k/\varepsilon) dt \leq \mu(I) C_{1/\varepsilon} + 1 = C.$$

Finally, Lemma 3.11 shows that $\|u_k\|_{\mathbf{L}^G} \leq C\varepsilon$ and hence $\|u_k\|_{\mathbf{L}^G} \rightarrow 0$. \square

It is standard result due to Riesz that for $f_n, f \in \mathbf{L}^p$

$$f_n \rightarrow f \text{ a.e.} \implies (\|f_n\|_{\mathbf{L}^p} \rightarrow \|f\|_{\mathbf{L}^p} \iff \|f_n - f\|_{\mathbf{L}^p} \rightarrow 0).$$

Following lemmas establish Orlicz space version of this fact.

Lemma 3.14. For every $k > 1$ and $0 < \varepsilon < \frac{1}{k}$ and $x, y \in \mathbb{R}^n$

$$|G(x+y) - G(x)| \leq \varepsilon |G(kx) - kG(x)| + 2G(C_\varepsilon y)$$

where $C_\varepsilon = \frac{1}{\varepsilon(k-1)}$

The proof can be found in [24] (see also [25]).

Lemma 3.15. If $u_n \rightarrow u$ in \mathbf{L}^G then $R_G(u_n) \rightarrow R_G(u)$.

Proof. In Lemma 3.14 set $x+y = u_n$, $x = u$, $k = 2$. Then $\varepsilon < 1/2$, $C_\varepsilon = \frac{1}{\varepsilon}$ and

$$|G(u_n) - G(u)| \leq \varepsilon |G(2u) - 2G(u)| + 2G\left(\frac{u_n - u}{\varepsilon}\right).$$

Since $u_n \rightarrow u$ in \mathbf{L}^G , there exists n_0 such that for $n > n_0$ we have $\|u_n - u\|_{\mathbf{L}^G} < \varepsilon^2 \leq \varepsilon < 1$. Thus

$$\int_I G\left(\frac{u_n - u}{\varepsilon}\right) dt \leq \frac{1}{\varepsilon} \|u_n - u\|_{\mathbf{L}^G} < \varepsilon.$$

From this and inequality above we obtain

$$|R_G(u_n) - R_G(u)| \leq \varepsilon \int_I |G(2u) - 2G(u)| dt + 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ we have $R_G(u_n) \rightarrow R_G(u)$. \square

According to the above lemma, if $u_n \rightarrow u$ in \mathbf{L}^G then:

1. Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^1$ (see Lemma 3.5 below), we can extract a subsequence u_{n_k} such that

$$u_{n_k} \rightarrow u \text{ a.e and } |u_{n_k}| \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

2. Since $R_G(u_n - u) \rightarrow 0$, $G(u_n - u) \rightarrow 0$ in \mathbf{L}^1 . Thus we can extract a subsequence $\{u_{n_k}\}$ such that

$$G(u_{n_k} - u) \rightarrow 0 \text{ a.e and } G(u_{n_k} - u) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

3. Since $R_G(u_n) \rightarrow R_G(u)$, $G(u_n) \rightarrow G(u)$ in \mathbf{L}^1 . Hence there exists a subsequence $\{u_{n_k}\}$ such that

$$G(u_{n_k}) \rightarrow G(u) \text{ a.e and } G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

Lemma 3.16. Let $\{u_n\} \subset \mathbf{L}^G$ and $u \in \mathbf{L}^G$. Suppose that

1. $u_n \rightarrow u$ a.e.,
2. $R_G(u_n) \rightarrow R_G(u)$.

Then $u_n \rightarrow u$ in \mathbf{L}^G .

Proof. This lemma was proved in [4, p. 83] for N-functions. Since G is convex, we get $\frac{1}{2}(G(u_n(t)) + G(u(t))) - G\left(\frac{u_n(t) - u(t)}{2}\right) \geq 0$. Continuity of G and $u_n \rightarrow u$ a.e. implies

$$\frac{1}{2}(G(u_n(t)) + G(u(t))) - G\left(\frac{u_n(t) - u(t)}{2}\right) \rightarrow G(u) \text{ a.e.}$$

So that by the Fatou Lemma, we have

$$\begin{aligned} \int_I G(u) dt &\leq \liminf_{n \rightarrow \infty} \int_I \frac{1}{2}(G(u_n) + G(u)) dt - G\left(\frac{u_n - u}{2}\right) dt \leq \\ &\leq \lim_{n \rightarrow \infty} \int_I \frac{1}{2}(G(u_n) + G(u)) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt = \\ &= \int_I G(u) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt. \end{aligned}$$

This implies that

$$\int_I G\left(\frac{u_k(t) - u(t)}{2}\right) dt \rightarrow 0$$

and $\|u_k - u\|_{\mathbf{L}^G} \rightarrow 0$ by Theorem 3.13. □

As a consequence we obtain dominated convergence theorem for anisotropic Orlicz spaces:

Theorem 3.17. Suppose that $\{u_n\} \subset \mathbf{L}^G$ and

1. $u_n \rightarrow u$ a.e.
2. there exists $h \in \mathbf{L}^1(I, \mathbb{R})$ such that $G(u_n) \leq h$ a.e.

Then $u \in \mathbf{L}^G$ and $u_n \rightarrow u$ in \mathbf{L}^G .

Proof. Since G is continuous and $u_n \rightarrow u$ a.e., $G(u_n) \rightarrow G(u)$ a.e. It follows that $G(u) \leq h$ a.e. Thus $G(u) \in \mathbf{L}^1(I, \mathbb{R})$ and hence $u \in \mathbf{L}^G$. In a standard way we get $R_G(u_n) \rightarrow R_G(u)$. Hence $u_n \rightarrow u$ in \mathbf{L}^G , by the Lemma 3.16. \square

In the above theorem, assumption $G(u_n) \leq h$ can be replaced by $G(u_n) \leq G(h)$, $h \in \mathbf{L}^G$. Consider a sequence $\{u_n\} \subset \mathbf{L}^G$ convergent pointwise to measurable function u . Under standard dominance condition (i.e. $|u_n| \leq |g|$, $g \in \mathbf{L}^G$) it is not true in general that $u_n \rightarrow u \in \mathbf{L}^G$.

Example 3.18. Let $G(x, y) = x^2 + y^4$, $I = (0, 1)$, $u(t) = (0, t^{-1/4})$ and $h(t) = (t^{-3/8}, 0)$. Define

$$u_n(t) = \begin{cases} u(t) & |u(t)| \leq n \\ 0 & |u(t)| > n \end{cases}$$

Then $u_n \rightarrow u$ a.e., $u_n, h \in \mathbf{L}^G$ and $|u_n| \leq |h|$ for every t . But $G(u(t)) = t^{-1} \notin \mathbf{L}^1(I, \mathbb{R})$. Hence $u \notin \mathbf{L}^G$.

Remark 3.19. Modular R_G is called monotone modular if $|x| \leq |y|$ implies $R_G(x) \leq R_G(y)$. If R_G is monotone modular then $u_k \rightarrow u$ a.e and $|u_k| \leq |g|$, $g \in \mathbf{L}^G$ implies $u \in \mathbf{L}^G$ and $\|u_k - u\|_{\mathbf{L}^G} \rightarrow 0$. We refer the reader to [25] for more details.

4. Orlicz-Sobolev spaces

The Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G = \mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n)$ is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) := \{u \in \mathbf{L}^G(I, \mathbb{R}^n) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^n)\}.$$

For $u \in \mathbf{W}^1 \mathbf{L}^G$ we define

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} := \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

Define $\mathbf{W}_0^1 \mathbf{L}^G = \mathbf{W}_0^1 \mathbf{L}^G(I, \mathbb{R}^n)$ as the closure of $C_0^1(I, \mathbb{R}^n)$ in $\mathbf{W}^1 \mathbf{L}^G$ with respect to the $\|\cdot\|_{\mathbf{W}^1 \mathbf{L}^G}$.



Theorem 4.1. The space $(\mathbf{W}^1 \mathbf{L}^G, \|\cdot\|_{\mathbf{W}^1 \mathbf{L}^G})$ is a separable reflexive Banach space.

Proof is standard and will be omitted, see for instance [26]. If $G(x) = \frac{1}{p}|x|^p$, then the Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G$ coincides with the Sobolev space $\mathbf{W}^{1,p}(I, \mathbb{R}^n)$. Observe that $u_n \rightarrow u$ in $\mathbf{W}^1 \mathbf{L}^G$ is equivalent to $R_G(u_n - u) \rightarrow 0$ and $R_G(\dot{u}_n - \dot{u}) \rightarrow 0$.

Since there exist $p, q \in (1, \infty)$ such that $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$, the following continuous embeddings exist

$$\mathbf{W}^{1,q} \hookrightarrow \mathbf{W}^1 \mathbf{L}^G \hookrightarrow \mathbf{W}^{1,p}$$

Using standard results from the theory of Sobolev spaces we get

1. $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow \mathbf{W}^{1,1}$,
2. $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow \mathbf{L}^q$, for all $1 \leq q \leq \infty$,
3. $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow C(\bar{I})$.

As a consequence we have

Theorem 4.2. A function $u \in \mathbf{W}^1 \mathbf{L}^G$ is absolutely continuous. Precisely, there exists absolutely continuous representative of u such that for all $a, b \in I$

$$u(b) - u(a) = \int_a^b \dot{u}(t) dt.$$

Directly from definition of $\mathbf{W}_0^1 \mathbf{L}^G$ we obtain important property of functions in $\mathbf{W}_0^1 \mathbf{L}^G$.

Theorem 4.3. If $u \in \mathbf{W}_0^1 \mathbf{L}^G$, then $u = 0$ on ∂I .

Using embeddings mentioned above we have for every $u \in \mathbf{W}^1 \mathbf{L}^G$

$$\|u\|_{\mathbf{L}^\infty} \leq C \|u\|_{\mathbf{W}^1 \mathbf{L}^G} \quad (3)$$

Theorem 4.4 (Sobolev inequality). For every function $u \in \mathbf{W}^1 \mathbf{L}^G$

$$\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I) \|\dot{u}\|_{\mathbf{L}^G}$$

where $u_I = \frac{1}{\mu(I)} \int_I u$.

Proof. Since u is absolutely continuous, there exists $t_0 \in I$ such that $u(t_0) = \frac{1}{\mu(I)} \int_I u$ and for every $t \in I$ we have

$$u(t) - u(t_0) = \int_{t_0}^t \dot{u} dt.$$

By Jensen's inequality,

$$\begin{aligned} G\left(\frac{u(t) - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) &= G\left(\frac{1}{|t - t_0|} \int_{t_0}^t \frac{\dot{u}}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}} dt\right) \leq \\ &\leq \frac{1}{|t - t_0|} \int_{t_0}^t G\left(\frac{\dot{u}}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)} \int_I G\left(\frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)}. \end{aligned}$$

Integrating both sides over I we get

$$\int_I G\left(\frac{u - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq 1.$$

Thus $\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$. □

In similar way we get

Theorem 4.5 (Poincaré inequality). For every $u \in \mathbf{W}_0^1 \mathbf{L}^G$

$$\|u\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$$

It follows that one can introduce equivalent norm in $\mathbf{W}_0^1 \mathbf{L}^G$:

$$\|u\|_{\mathbf{W}_0^1 \mathbf{L}^G} = \|\dot{u}\|_{\mathbf{L}^G}.$$

Every linear functional F on $\mathbf{W}_0^1 \mathbf{L}^G$ can be represented in the form

$$F(u) = \int_I \langle u, v_0 \rangle + \langle \dot{u}, v_1 \rangle dt,$$

where $v_0, v_1 \in \mathbf{L}^{G^*}$. Moreover, $\|F\| = \max\{\|v_0\|_{\mathbf{L}^{G^*}}, \|v_1\|_{\mathbf{L}^{G^*}}\}$. In the case of Sobolev space $\mathbf{W}^{1,p}$ the proof is given in [26, proposition 8.14], but it remains the same for Orlicz-Sobolev spaces. As was pointed out in [26], the first assertion of the above proposition holds for every linear functional on $\mathbf{W}^1 \mathbf{L}^G$.

5. Variational setting

In this section we examine the principal part

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u}) dt \quad (4)$$

of the variational functional associated with Euler-Lagrange equation

$$\frac{d}{dt} F_v(t, u, \dot{u}) = F_x(t, u, \dot{u}) + \nabla V(t, u), \quad t \in I$$

where $u: I \rightarrow \mathbb{R}^N$ and the Lagrangian $L: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is given by $L(t, x, v) = F(t, x, v) + V(t, x)$.

In definition of the Orlicz space we need not to assume that G is differentiable, but when we consider the functional \mathcal{I} we need it to show that $\mathcal{I} \in C^1$. Throughout this section we will assume, in addition to (G_1) – (G_6) , that G satisfies (G_7) G is of a class C^1 .

Remark 5.1. Differentiability of f is not sufficient to differentiability of f^* . But if f is finite, strictly convex, 1-coercive and differentiable then so is f^* . This result is in close relation with Legendre duality (see [21, p. 239] and [1] for more details).

It is well known that if G is continuously differentiable then for all $x, y \in \mathbb{R}^n$

$$G(x) - G(x - y) \leq \langle \nabla G(x), y \rangle \leq G(x + y) - G(x) \quad (5)$$

and

$$\langle x, \nabla G(x) \rangle = G(x) + G^*(\nabla G(x)).$$

Let $y = x$ in (5). Then $\langle \nabla G(x), x \rangle \leq G(2x) - G(x)$. Therefore, for all $x \in \mathbb{R}^n$

$$G^*(\nabla G(x)) \leq G(2x).$$

Directly from the above we get

Proposition 5.2. If $u \in \mathbf{L}^G$ then $\nabla G(u) \in \mathbf{L}^{G^*}$.

Lemma 5.3 (cf. [16, lemma A.5]). If $u_n \rightarrow u$ in \mathbf{L}^G then $R_{G^*}(\nabla G(u_n)) \rightarrow R_{G^*}(\nabla G(u))$.

Proof. There exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightarrow u$ a.e., $G(u_{n_k}) \rightarrow G(u)$ a.e. and $G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R})$. By continuity of ∇G and G^* we have $\nabla G(u_{n_k}) \rightarrow \nabla G(u)$ a.e. and

$$G^*(\nabla G(u_{n_k})) \rightarrow G^*(\nabla G(u)) \text{ a.e.}$$

Since $G^*(\nabla G(x)) \leq G(2x)$,

$$G^*(\nabla G(u_{n_k})) \leq G(2u_{n_k}) \leq C + K_1 G(u_{n_k}) \leq C + K_1 h.$$

By dominated convergence theorem $R_{G^*}(\nabla G(u_{n_k})) \rightarrow R_{G^*}(\nabla G(u))$. Since this holds for any subsequence of $\{u_n\}$ we have that

$$R_{G^*}(\nabla G(u_n)) \rightarrow R_{G^*}(\nabla G(u)).$$

□

As a direct consequence of the above lemma and Lemma 3.16 we obtain

Proposition 5.4.

$$\|u_n - u\|_{\mathbf{L}^G} \rightarrow 0 \implies \|\nabla G(u_n) - \nabla G(u)\|_{\mathbf{L}^{G^*}} \rightarrow 0.$$

5.1. Case I

We shall first examine a special case $F(t, x, v) = G(v)$, now functional (4) takes the form

$$\mathcal{I}(u) = \int_I G(\dot{u}) dt.$$

Theorem 5.5. $\mathcal{I} \in C^1(\mathbf{W}^1 \mathbf{L}^G, \mathbb{R})$. Moreover

$$\mathcal{I}'(u)\varphi = \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt. \quad (6)$$

Proof. The proof follows similar lines as [2, th. 3.2] (see also [1, thm 1.4]). First, note that $\dot{u} \in \mathbf{L}^G$ implies

$$0 \leq \mathcal{I}(u) < \infty.$$

It suffices to show that \mathcal{I} has at every point u directional derivative $\mathcal{I}'(u) \in (\mathbf{W}^1 \mathbf{L}^G)^*$ given by (6) and that the mapping $\mathcal{I}' : \mathbf{W}^1 \mathbf{L}^G \rightarrow (\mathbf{W}^1 \mathbf{L}^G)^*$ is continuous. Let $u \in \mathbf{W}^1 \mathbf{L}^G$, $\varphi \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$, $t \in I$, $s \in [-1, 1]$. Define

$$H(s, t) := G(\dot{u}(t) + s\dot{\varphi}(t)).$$

By (5) we obtain

$$\int_I |H_s(s, t)| dt = \int_I |\langle \nabla G(\dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt \leq \int_I G(\dot{u} + (s+1)\dot{\varphi}) + \int_I G(\dot{u} + s\dot{\varphi}) dt < \infty.$$

Consequently, \mathcal{I} has a directional derivative and

$$\mathcal{I}'(u)\varphi = \frac{d}{ds} \mathcal{I}(u + s\varphi) \Big|_{s=0} = \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt.$$

By Proposition 5.2 and the Hölder inequality

$$|\mathcal{I}'(u)\varphi| = \left| \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|\nabla G(\dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \leq C \|\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

To finish the proof it suffices to show that if $u_n \rightarrow u$ in $\mathbf{W}^1 \mathbf{L}^G$, then $\mathcal{I}'(u_n) \rightarrow \mathcal{I}'(u)$ in $(\mathbf{W}^1 \mathbf{L}^G)^*$. Using the Hölder inequality and Proposition 5.4 we obtain

$$\begin{aligned} |\mathcal{I}'(u_n)\varphi - \mathcal{I}'(u)\varphi| &= \left| \int_I \langle \nabla G(\dot{u}_n) - \nabla G(\dot{u}), \dot{\varphi} \rangle dt \right| \leq \\ &\leq 2 \|\nabla G(\dot{u}_n) - \nabla G(\dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \rightarrow 0. \end{aligned}$$

□

5.2. Case II

We turn to general case. Suppose that $F: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies

- (F₁) $F \in C^1$,
- (F₂) $|F(t, x, v)| \leq a(|x|)(b(t) + G(v))$,
- (F₃) $|F_x(t, x, v)| \leq a(|x|)(b(t) + G(v))$,
- (F₄) $G^*(F_v(t, x, v)) \leq a(|x|)(c(t) + G^*(\nabla G(v)))$.

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$.

If $G(v) = |v|^p$ then conditions (F₂), (F₃) and (F₄) take the standard form (Theorem 1.4 from [1]). In [2] there are similar conditions with $G(v) = \Phi(|v|)$, where Φ is an N-function. In this case, condition (F₄) takes the form $|F_v(t, x, v)| \leq \tilde{a}(|x|)(\tilde{c}(t) + \Phi'(|u|))$. In anisotropic case we need to use G^* , because vector valued G-function is not necessarily monotone with respect to $|\cdot|$.

Lemma 5.6. If $u \in \mathbf{W}^1 \mathbf{L}^G$, then $F_x(\cdot, u, \dot{u}) \in \mathbf{L}^1$ and $F_v(\cdot, u, \dot{u}) \in \mathbf{L}^{G^*}$.

Proof. Define non decreasing function

$$\alpha(s) = \sup_{\tau \in [0, s]} a(\tau).$$

Then, for $u \in \mathbf{W}^1 \mathbf{L}^G$ we have

$$a(|u(t)|) \leq \alpha(\|u\|_{\mathbf{L}^\infty}) \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}). \quad (7)$$

Let $u \in \mathbf{W}^1 \mathbf{L}^G$. By (7) and (F₃)

$$\begin{aligned} \int_I |F_x(t, u, \dot{u})| dt &\leq \int_I a(|u(t)|)(b(t) + G(\dot{u})) dt \leq \\ &\leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u})) dt < \infty. \end{aligned}$$

Moreover, by (F_4) and Proposition 5.2

$$\int_I G^*(F_v(t, u, \dot{u})) dt \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (c(t) + G^*(\nabla G(\dot{u}))) dt < \infty.$$

□

Theorem 5.7. $\mathcal{I} \in C^1(\mathbf{W}^1 \mathbf{L}^G, \mathbb{R})$. Moreover

$$\mathcal{I}'(u)\varphi = \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt + \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt. \quad (8)$$

Proof. By (F_2)

$$|\mathcal{I}(u)| \leq \int_I a(|u|)(b(t) + G(\dot{u})) dt \leq \alpha(\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u})) dt < \infty.$$

It suffices to show that directional derivative $\mathcal{I}'(u) \in (\mathbf{W}^1 \mathbf{L}^G)^*$ exists, is given by (8) and that the mapping $\mathcal{I}' : \mathbf{W}^1 \mathbf{L}^G \rightarrow (\mathbf{W}^1 \mathbf{L}^G)^*$ is continuous.

Let $u \in \mathbf{W}^1 \mathbf{L}^G$, $\varphi \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$, $t \in I$, $s \in [-1, 1]$. Define

$$H(s, t) := F(t, u + s\varphi, \dot{u} + s\dot{\varphi}).$$

By (F_3) , continuity of φ , (7) and the fact that $u + s\varphi \in \mathbf{W}^1 \mathbf{L}^G$ we obtain

$$\begin{aligned} \int_I |\langle F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle| dt &\leq \int_I |F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi})| |\varphi| dt \leq \\ &\leq \int_I a(|u + s\varphi|)(b(t) + G(\dot{u} + s\dot{\varphi})) |\varphi| dt \leq \\ &\leq \alpha(\|u + s\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u} + s\dot{\varphi})) |\varphi| dt < \infty. \end{aligned}$$

By the Fenchel inequality, (F_4) and Lemma 5.6 we obtain

$$\int_I |\langle F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt \leq \int_I [G^*(F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi})) + G(\dot{\varphi})] dt < \infty.$$

It follows that

$$\int_I |H_s(s, t)| dt = \int_I |\langle F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle + \langle F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt < \infty.$$

Consequently, \mathcal{I} has a directional derivative and

$$\mathcal{I}'(u)\varphi = \left. \frac{d}{ds} \mathcal{I}(u + s\varphi) \right|_{s=0} = \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt + \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$



By Lemma 5.6, the Hölder inequality and (3) we get

$$|\mathcal{I}'(u)\varphi| \leq \|F_x(\cdot, u, \dot{u})\|_{\mathbf{L}^1} \|\varphi\|_{\mathbf{L}^\infty} + \|F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \leq C \|\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

To finish the proof it suffices to show that \mathcal{I}' is continuous. Since $u_n \rightarrow u$ in $\mathbf{W}^1 \mathbf{L}^G$, it follows that $u_n \rightarrow u$ in \mathbf{L}^G , $\dot{u}_n \rightarrow \dot{u}$ in \mathbf{L}^G and there exists $M > 0$ such that $\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G} < M$.

By Lemma 3.15 we have $G(\dot{u}_n) \rightarrow G(\dot{u})$ in $\mathbf{L}^1(I, \mathbb{R})$. Hence there exists a subsequence $\{u_{n_k}\}$ and $h \in \mathbf{L}^1(I, \mathbb{R})$ such that

$$G(\dot{u}_{n_k}) \rightarrow G(\dot{u}) \text{ a.e and } G(\dot{u}_{n_k}) \leq h.$$

By (F₃) and since $\{u_{n_k}\}$ is bounded, we obtain

$$|F_x(t, u_{n_k}, \dot{u}_{n_k})| \leq \alpha(\|u_{n_k}\|_{\mathbf{W}^1 \mathbf{L}^G})(b(t) + G(\dot{u}_{n_k})) \leq \alpha(M)(b(t) + h(t)).$$

By (F₁) we have

$$F_x(t, u_{n_k}(t), \dot{u}_{n_k}(t)) \rightarrow F_x(t, u(t), \dot{u}(t))$$

for a.e $t \in I$. Applying dominated convergence theorem we obtain

$$\int_I \langle F_x(t, u_{n_k}, \dot{u}_{n_k}), \varphi \rangle dt \rightarrow \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt.$$

Since this holds for any subsequence of $\{u_n\}$ we have that

$$\int_I \langle F_x(t, u_n, \dot{u}_n), \varphi \rangle dt \rightarrow \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt.$$

By (F₄) and Lemma 5.6

$$G^*(F_v(t, u_{n_k}(t), \dot{u}_{n_k}(t))) \leq \alpha(M)(c(t) + G^*(\nabla G(\dot{u}_{n_k}(t)))).$$

In the same way as in the proof of Lemma 5.3 we obtain

$$G^*(F_v(t, u_{n_k}(t), \dot{u}_{n_k}(t))) \leq \alpha(M)(c(t) + C + K_1 h(t)).$$

By continuity of F_v we obtain

$$G^*(F_v(t, u_{n_k}(t), \dot{u}_{n_k}(t))) \rightarrow G^*(F_v(t, u(t), \dot{u}(t)))$$

for a.e $t \in I$ and consequently

$$\int_I G^*(F_v(t, u_{n_k}, \dot{u}_{n_k})) dt \rightarrow \int_I G^*(F_v(t, u, \dot{u})) dt.$$

It follows that

$$\int_I G^*(F_v(t, u_n, \dot{u}_n)) dt \rightarrow \int_I G^*(F_v(t, u, \dot{u})) dt.$$

Application of Lemma 3.16 to R_{G^*} yields $\|F_v(\cdot, u_n, \dot{u}_n) - F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \rightarrow 0$. By the Hölder inequality

$$\left| \int_I \langle F_v(t, u_n, \dot{u}_n) - F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|F_v(\cdot, u_n, \dot{u}_n) - F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \rightarrow 0.$$

Finally,

$$\int_I \langle F_v(t, u_n, \dot{u}_n), \dot{\varphi} \rangle dt \rightarrow \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$

□

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