Anisotropic Orlicz-Sobolev spaces of vector valued functions and Lagrange equations

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Abstract

In this paper we study some properties of anisotropic Orlicz and Orlicz-Sobolev spaces of vector valued functions for a special class of G-functions. We introduce a variational setting for a class of Lagrangian Systems. We give conditions which ensure that the principal part of variational functional is finitely defined and continuously differentiable on Orlicz-Sobolev space.

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1. Introduction

In this paper we make some preliminary steps for variational analysis in anisotropic Orlicz-Sobolev spaces of vector valued functions. We consider the Euler-Lagrange equation

$$\frac{d}{dt}L_v(t,u(t),\dot{u}(t)) = L_x(t,u(t),\dot{u}(t)), \quad t \in (a,b) \tag{1}$$

where Lagrangian is of the form $L(t,x,v) = F(t,x,v) + V(t,x)$.

If $F(v) = \frac{1}{2}|v|^2$ then the equation (1) reduces to $\ddot{u}(t) + \nabla V(t,u(t)) = 0$. One can consider more general case $F(v) = \phi(|v|)$, where ϕ is convex and nonnegative. In the above cases F does not depend on v directly but rather on its norm $|v|$ and the growth of F is the same in all directions, i.e. F has isotropic growth. Equation (1) with Lagrangian $L(t,x,v) = \frac{1}{p}|v|^p + V(t,x)$ has been studied by

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many authors under different conditions. The classical reference is [1]. The isotropic Orlicz-Sobolev space setting was considered in [2].

We are interested in anisotropic case. This means that F depends on all components of v not only on $|v|$ and has different growth in different directions. A simple example of such function is $F(v) = \sum_{i=1}^{N} |v_i|^{p_i}$ or $F(v) = \sum_{i=1}^{N} \phi_i(|v_i|)$, where ϕ_i are N-functions. We wish to consider more general situation. We assume that $F\colon [a,b] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies

(F_1) $F \in C^1$,

(F_2) $|F(t,x,v)| \le a(|x|)(b(t) + G(v))$,

(F_3) $|F_x(t,x,v)| \le a(|x|)(b(t) + G(v))$,

(F_4) $G^*(F_v(t,x,v)) \le a(|x|)(c(t) + G^*(\nabla G(v)))$,

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$ and $G\colon \mathbb{R}^N \to \mathbb{R}$ is a G-function. Conditions (F_1)–(F_4) are direct generalization of standard growth conditions from [1] (see also [2]). We show (see Theorem 5.7) that under these conditions the functional $\mathcal{I}\colon \mathbf{W}^1\mathbf{L}^G \to \mathbb{R}$ given by

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u})\, dt$$

is continuously differentiable.

We restrict our considerations to a special class of G-functions. Here $G\colon \mathbb{R}^n \to [0,\infty)$ is convex, $G(-x) = G(x)$, supercoercive, $G(0) = 0$ and satisfies Δ_2 and ∇_2 conditions. We define the anisotropic Orlicz space to be

$$\mathbf{L}^G(I, \mathbb{R}^N) = \{u\colon I \to \mathbb{R}^N\colon \int_I G(u)\, dt < \infty\}.$$

The Orlicz space $\mathbf{L}^G$ equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf\left\{\alpha > 0\colon \int_I G\left(\frac{u}{\alpha}\right)\, dt \le 1\right\}$$

is a reflexive Banach space. An important example of Orlicz space is classical Lebesgue $\mathbf{L}^p$ space, defined by $G(x) = \frac{1}{p}|x|^p$. In this case, the Luxemburg norm and the standard $\mathbf{L}^p$ norm are equivalent. Therefore, Orlicz spaces can be viewed as a straightforward generalization of $\mathbf{L}^p$ spaces.

Properties of N-functions and Orlicz spaces of real-valued functions has been studied in great details in monographs [3, 4, 5] and [6]. The standard references for vector-valued case are [7, 8, 9] and [10, 11] for Banach-space valued functions. In [7, 8] author considers a class of G-functions together with a uniformity conditions which, for example, excludes the function $G(x) = \sum |x_i|^{p_i}$ unless $1 < p_1 = \cdots = p_N < \infty$. Moreover G is not necessarily assumed to be an even function. As was pointed out in [11], if G is not even then $\mathbf{L}^G$ is no longer a vector space (see also [10, Example 2.1]).

Our strong conditions on G allow us to work in Orlicz spaces without worry about some technical difficulties arising in general case. For example, it is well known that the set $\mathbf{L}^G(I, \mathbb{R}^N)$ is a vector space if and only if G satisfies Δ_2 condition. Otherwise $\mathbf{L}^G$ is only a convex set. Another difficulty is the convergence notion. In Lebesgue spaces $\|u_n - u\|_{\mathbf{L}^p} \to 0$ means simply $\int |u_n - u|^p \to 0$. For arbitrary G-function G, convergence in Luxemburg norm is not equivalent to $\int G(u_n - u)\, dt \to 0$ unless G satisfies Δ_2. The Δ_2 condition is also crucial for separability and reflexivity of $\mathbf{L}^G$.

The main consequence of anisotropic nature of G is the lack of monotonicity of the norm. It is no longer true that $|u| \leq |v|$ implies $\|u\|_{\mathbf{L}^G} \leq \|v\|_{\mathbf{L}^G}$. In anisotropic case, standard dominance condition $|u_n| \leq f$ does not implies convergence in $\mathbf{L}^G$ norm and must be replaced by $G(u_n) \leq f$ (see Theorem 3.17).

For every G there exist $p, q \in (1, \infty)$ such that $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$. If $G(x) = \sum |x_i|^{p_i}$ then $\mathbf{L}^G$ can be identified with the product of $\mathbf{L}^{p_i}$ but in many cases an anisotropic Orlicz Space is not equal to the space $\mathbf{L}^{p_1} \times \mathbf{L}^{p_2} \times ... \times \mathbf{L}^{p_N}$ (see Example 3.7).

To give a proper variational setting for equation (1) we introduce a notion of an anisotropic Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G$ of vector-valued functions. It is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^N) = \{u \in \mathbf{L}^G(I, \mathbb{R}^N) \colon \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^N)\}$$

with the norm

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

To the authors best knowledge there is no reference for the case of anisotropic norm and vector-valued functions of one variable. The references for other cases are [2, 9, 12, 13, 14, 15, 16, 17, 18, 19].

In [9] and [18] the space $H^0(G, \Omega)$, $\Omega \subset \mathbb{R}^n$ is defined as a completion of $C_0^1(\Omega, \mathbb{R}^n)$ under norm $\|u\|_{H^0(G,\Omega)} = \|Du\|_{G,\Omega}$. It is classical result due to Trudinger $H^0(G, \Omega) \hookrightarrow L_A(\Omega)$, where A is some N-function (see also Cianchi [14]).

In [17] and [19] the anisotropic Orlicz-Sobolev space $W^1 L_G$ is defined for G-function $G : \mathbb{R}^{n+1} \to [0, \infty]$ as a space of weakly differentiable functions $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$ such that $(u, D_1 u, D_2 u, ..., D_n u)$ belongs to the Orlicz space generated by G. A norm for $W^1 L_G$ is given by

$$\|u\|_{1,G,\Omega} = \|(u, Du)\|_{G,\Omega}.$$

In [12] we can find definition of isotropic Orlicz-Sobolev space of real valued functions

$$W^1_A(\Omega) = \{u \in \Omega \to \mathbb{R} \text{ measurable } : u, |\nabla u| \in L_A\},$$

where L_A is Orlicz Space and A is an N-function.

In [2] the isotropic Orlicz-Sobolev space of vector-valued functions is defined to be a space of absolutely continuous functions $u : [0, T] \to \mathbb{R}^d$ such that u and $\dot{u}$ belongs to Orlicz space generated by an N-function. Similar treatment can be found in [20].

2. G-functions

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^N$ and $| \cdot |$ is the induced norm. We assume that $G\colon \mathbb{R}^N \to [0, \infty)$ satisfies the following conditions:

(G_1) $G(0) = 0$,

(G_2) G is convex,

(G_3) G is even,

(G_4) G is supercoercive:

$$\lim_{|x|\to\infty} \frac{G(x)}{|x|} = \infty,$$

(G_5) G satisfies the Δ_2 condition:

$$\exists_{K_1\geq 2}\, \exists_{M_1>0}\, \forall_{|x|\geq M_1}\; G(2x) \leq K_1 G(x), \tag{Δ_2}$$

(G_6) G satisfies the ∇_2 condition:

$$\exists_{K_2\geq 1}\, \exists_{M_2>0}\, \forall_{|x|\geq M_2}\; G(x) \leq \frac{1}{2K_2} G(K_2 x). \tag{∇_2}$$

A function G is a G-function in the sense of Trudinger [9]. In general, G-function can be unbounded on bounded sets and need not satisfy conditions (G_4)–(G_6) but only $\lim_{x\to\infty} G(x) = \infty$. A G-function of one variable is called N-function. Some typical examples of G are:

1. $G_p(x) = \frac{1}{p}|x|^p$, $1 < p < \infty$,
2. $G(x) = \sum_{i=1}^N G_{p_i}(x_i)$, $1 < p_i < \infty$,
3. $G(x_1, x_2) = (x_1 - x_2)^2 + x_2^4$.

A function G can be equal to zero in some neighborhood of 0. So that a function

$$G(x) = \begin{cases} 0 & |x| \leq 1 \\ |x|^2 - 1 & |x| > 1 \end{cases}$$

is also admissible. Condition Δ_2 implies that G is of polynomial growth (see Lemma 2.4 below and [3]). A function $f : \mathbb{R}^2 \to \mathbb{R}$ $f(x) = e^{|x|} - |x| - 1$ does not satisfy Δ_2.

Since G is convex and finite on $\mathbb{R}^n$, G is locally Lipschitz and therefore continuous. Note that for every $x \in \mathbb{R}^N$

$$G(\alpha x) \leq \alpha G(x), \text{ if } 0 \leq \alpha \leq 1,$$
$$\alpha G(x) \leq G(\alpha x), \text{ if } 1 \leq \alpha.$$

We get immediately that G is non-decreasing along any half-line through the origin i.e. for every $x \in \mathbb{R}^N$

$$0 < \alpha \leq \beta \implies G(\alpha x) \leq G(\beta x). \tag{2}$$

Our assumptions on G imply that for every $x_0 \in \mathbb{R}^N$ there exists $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$

$$\langle a, x_0 \rangle + b = G(x_0) \text{ and } \langle a, x \rangle + b \leq G(x).$$

From this, we can easily obtain the Jensen integral inequality. Let $I \subset \mathbb{R}$ be a finite interval and let $u \in \mathbf{L}^1(I, \mathbb{R}^N)$. Then

$$G\left(\frac{1}{\mu(I)} \int_I u \, dt\right) \leq \frac{1}{\mu(I)} \int_I G(u) \, dt.$$

We will often make use of the following simple observation.

Proposition 2.1. For all $\alpha \in \mathbb{R}$ there exists $K_1(\alpha) > 0$ such that

$$G(\alpha x) \leq K_1(\alpha) G(x)$$

for all $|x| \geq M_1$.

In fact, the above proposition provides a characterization of Δ_2 (see [7, 11]). It follows that for every $\alpha \in \mathbb{R}$ there exists $C_\alpha > 0$ such that for $x \in \mathbb{R}^N$

$$G(\alpha x) \leq C_\alpha + K_1(\alpha) G(x).$$

We recall a notion of Fenchel conjugate. Define $G^* : \mathbb{R}^N \to [0, \infty)$ by

$$G^*(y) := \sup_{x \in \mathbb{R}^N} \{\langle x, y \rangle - G(x)\}.$$

A function G^* is called Fenchel conjugate of G. As an immediate consequence of definition we have the so called Fenchel inequality:

$$\forall_{x,y\in\mathbb{R}^N}\ \langle x, y\rangle \leq G(x) + G^*(y).$$

Consider arbitrary $f\colon \mathbb{R}^N \to [0,\infty)$. It is obvious that the conjugate function f^* is always convex. But in general f^* need not be continuous, finite or coercive, even if f is. From the other hand, it is well known that if f is convex and l.s.c. then $f^* \not\equiv \infty$ and $(f^*)^* = f$.

Example 2.2.

1. If
$$g(x) = \begin{cases} 0 & |x| \leq 1 \\ \infty & |x| > 1 \end{cases}$$
then $g^*(x) = |x|$. Note that g and g^* are G-functions but do not satisfy our assumptions.
2. If $G_p(x) = \frac{1}{p}|x|^p$, then $G_p^*(x) = \frac{1}{q}|x|^q$, $\frac{1}{p} + \frac{1}{q} = 1$.
3. If $G(x) = \sum_{i=1}^N G_{p_i}(x_i)$, then $G^*(x) = \sum_{i=1}^N G_{p_i}^*(x_i)$,
4. If $G(x, y) = (x - y)^2 + y^4$, then
$$G^*(x, y) = \frac{1}{4}x^2 + \frac{3}{4}(x + y)\left(\frac{x + y}{4}\right)^{\frac{1}{3}}.$$

More information on general theory of conjugate functions can be found in standard books on convex analysis, see for instance [21, 22].

If a function $G\colon \mathbb{R}^n \to [0,\infty)$ satisfies conditions (G_1)–(G_6) then the same is true for its conjugate G^*. This is main reason we want to restrict class of considered functions.

Theorem 2.3. If G satisfies conditions (G_1)–(G_6) then G^* also satisfies (G_1)–(G_6) and $(G^*)^* = G$.

Proof. It is evident that G^* satisfies (G_1), (G_2) and (G_3). It is well known that, under our conditions, G^* is finite (proposition 1.3.8, [21]), G^* is supercoercive (proposition 1.3.9, [21]) and G^* satisfies (G_5) and (G_6) (remark 2.3, [10]). Corrollary [21, cor. 1.3.6] gives $(G^*)^* = G$. □

In order to compare growth rate of G-functions we define two relations. Let G_1 and G_2 be G-functions. Define

$$G_1 \prec G_2 \iff \exists_{M\geq 0}\ \exists_{K>0}\ \forall_{|x|\geq M}\ G_1(x) \leq G_2(K\,x)$$

and

$$G_1 \prec\prec G_2 \iff \forall_{\alpha>0} \lim_{|x|\to\infty} \frac{G_2(\alpha x)}{G_1(x)} = \infty.$$

For conjugate functions we have (see [3, thm. 3.1])

$$G_1 \prec G_2 \Rightarrow G_2^* \prec G_1^*.$$

Obviously $G_1 \prec\prec G_2$ implies $G_1 \prec G_2$. Assumption (G_4) implies $|x| \prec\prec G$. It is true that $|x| \prec G$ holds under weaker assumption: $G(x) \to \infty$. Note that, if $p > 1$ then $|x| \prec\prec |x|^p$. Hence, if $|x|^p \prec G$ then $|x| \prec\prec G$. Since G satisfies (G_5) and (G_6) we have the following bounds for the growth of G.

Lemma 2.4 ([10, Lemma 2.4])**.** There exist $p, q \in (1, \infty)$ such that

$$|x|^p \prec G \prec |x|^q.$$

The exponents p and q depend on the constants in the ∇_2 and Δ_2 conditions respectively. Immediately from the above we get $|x|^{\frac{q}{q-1}} \prec G^* \prec |x|^{\frac{p}{p-1}}$.

3. Orlicz spaces

Let $I \subset \mathbb{R}$ be a finite interval. The Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^n)$ is defined to be

$$\mathbf{L}^G(I, \mathbb{R}^n) = \left\{ u\colon I \to \mathbb{R}^n \colon u - \text{measurable}, \int_I G(u)\, dt < \infty \right\}.$$

As usual, we identify functions equal a.e. For $u \in \mathbf{L}^G$ define:

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0 \colon \int_I G\left(\frac{u}{\alpha}\right) dt \le 1 \right\}.$$

The function $\|\cdot\|_{\mathbf{L}^G}$ is called the Luxemburg norm. It is easy to see that

$$\int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1,$$

since G satisfies Δ_2. Moreover

$$\int_I G\left(\frac{u}{k}\right) dt \le 1 \iff \|u\|_{\mathbf{L}^G} \le k.$$

Using Fenchel's inequality we obtain the Hölder inequality

$$\int_I \langle u, v \rangle\, dt \le 2\|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^*}}, \ u \in \mathbf{L}^G \text{ and } v \in \mathbf{L}^{G^*}$$

Similarly to [3] and [8] one can show that $\mathbf{L}^G$ is a linear ([3, thm. 8.2]) and normed space ([8, thm. 2.3]). Completeness and separability of $\mathbf{L}^G$ can be obtained in the same way as in [11, thm. 6.1, thm. 6.3, cor. 6.1]. Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^p \hookrightarrow\hookrightarrow \mathbf{L}^{p_0} \hookrightarrow \mathbf{L}^1$ (see propositions 3.3 and 3.4 below) and $1 < p_0 < p$, it follows that $\mathbf{L}^G$ is reflexive space. The proof, in more general case, can be found in [11].

According to above remarks, we have the following theorem.

Theorem 3.1. If $G\colon \mathbb{R}^n \to [0,\infty)$ satisfies (G_1)–(G_6), then $(\mathbf{L}^G(I,\mathbb{R}^n), \|\cdot\|_{\mathbf{L}^G})$ is a separable, reflexive Banach space.

Remark 3.2.

1. All properties of $\mathbf{L}^G$ remains true for $\mathbf{L}^{G^*}$, since G and G^* belongs to the same class of functions.

2. For an arbitrary G-function $f\colon \mathbb{R}^n \to [0,\infty)$ which does not satisfies Δ_2 the set $\mathbf{L}^f$ is not a linear space but only a convex set. In fact, it is well known that the set $\mathbf{L}^f$ is linear space if and only if a G-function f satisfies Δ_2 condition.

3. It was pointed out by Schappacher [11, example 3.1] that if f is not bounded on bounded sets (i.e. we allow $f(x) = +\infty$ for some $x \in \mathbb{R}^n$) then $\mathbf{L}^f$ need not be a linear space, even if f satisfies Δ_2 condition (see [3, 11]).

4. It is well known that if G-function does not satisfies Δ_2 condition then $\mathbf{L}^G$ is not separable. One can define a subspace E^G as the closure of bounded functions under Luxemburg norm. In this case, the space E^G is a proper subset of $\mathbf{L}^G$ and is always separable (see [3, 11]).

5. For every $F \in (\mathbf{L}^G)^*$ there exists unique $v \in \mathbf{L}^{G^*}$ such that for every $u \in \mathbf{L}^G$
$$Fu = \int_I \langle u, v\rangle \, dt.$$
As a consequence we obtain that $\mathbf{L}^{G^*} \simeq (\mathbf{L}^G)^*$. Since $G^{**} = G$, we also get $\mathbf{L}^G \simeq (\mathbf{L}^{G^*})^*$ (see [3, 8, 11]).

6. If G-function does not satisfies Δ_2 and ∇_2 conditions, then $\mathbf{L}^G$ is not reflexive and $(\mathbf{L}^G)^*$ is not isomorphic to $\mathbf{L}^{G^*}$ (see [3, 11]).

An important example of Orlicz space is a classical Lebesgue space $(\mathbf{L}^p, \|\cdot\|_{\mathbf{L}^p})$, $p \in (1,\infty)$ defined by $G(x) = \frac{1}{p}|x|^p$. It is easy to check that in this case $\mathbf{L}^G = \mathbf{L}^p$ and the Luxemburg norm and standard $\mathbf{L}^p$ norm are equivalent. Two important examples of Lebesgue spaces are not covered in our setting, namely $\mathbf{L}^1$ and $\mathbf{L}^\infty$.

The space $\mathbf{L}^1$ is generated by $f(x) = |x|$ and the space $\mathbf{L}^\infty$ generated by f^*. We exclude these two spaces because we want to have only reflexive spaces in the class of Orlicz spaces we consider.

We will use the symbols $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$ for, respectively, continuous and compact embeddings. Using the same methods as in [6, th. 8.12, 8.24] we obtain basic embedding theorems for anisotropic Orlicz spaces.

Proposition 3.3. Assume that $F \prec G$. Then $L^G \hookrightarrow L^F$ and

$$\|u\|_{\mathbf{L}^F} \leq K(C\mu(I) + 1)\|u\|_{\mathbf{L}^G}$$

for some $C > 0$.

Proposition 3.4. If $F \prec\prec G$ then $\mathbf{L}^G \hookrightarrow\hookrightarrow \mathbf{L}^F$.

Directly from Lemma 2.4 we obtain that Orlicz spaces can be viewed as a spaces between two Lebesgue spaces determined by constants in Δ_2 and ∇_2 conditions.

Proposition 3.5. For every G there exist $p, q \in (1, \infty)$ such that

$$\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p\,.$$

In particular $\mathbf{L}^\infty \hookrightarrow \mathbf{L}^G \hookrightarrow\hookrightarrow \mathbf{L}^1$.

In some cases $\mathbf{L}^G$ is simply a product of $\mathbf{L}^{p_i}(I, \mathbb{R})$, but there exist Orlicz spaces which are not in the form $\mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R})$ (cf. [9, pp. 18-20]).

Example 3.6. Consider the Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$ generated, by $G(x) = |x_1|^{p_1} + |x_2|^{p_2}$, $p_1, p_2 > 0$. If $u = (u_1, u_2) \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$, then

$$\int_I G(u)\, dt = \int_I |u_1|^{p_1}\, dt + \int_I |u_2|^{p_2}\, dt < \infty.$$

Conversely, if $u = (u_1, u_2) \in \mathbf{L}^G$ then

$$\int_I |u_1|^{p_1}\, dt \leq \int_I G(u)\, dt < \infty \text{ and } \int_I |u_2|^{p_2}\, dt \leq \int_I G(u)\, dt < \infty.$$

Hence $u \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$.

Example 3.7. Consider the Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$ generated, by $G(x) = (x_1 - x_2)^4 + x_2^2$. From Lemma 2.4 and Proposition 3.5 we obtain that $\mathbf{L}^4(I, \mathbb{R}^2) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}^2)$. Let u_1 be a function in $\mathbf{L}^2(I, \mathbb{R})$ such that $u_1 \notin \mathbf{L}^p(I, \mathbb{R})$, for $p > 2$. Set $u = (u_1, u_1)$, then

$$\int_I G(u)\, dt = \int_I |u_1|^2\, dt < \infty$$

but

$$\int_I |u|^p\,dt = \infty.$$

Therefore for every $p > 2$ there exists $u \in \mathbf{L}^G$ such that $u \notin \mathbf{L}^p(I, \mathbb{R}^2)$. Moreover, $u \notin \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$ for any $p > 2$. From the other hand if $u = (u_1, u_2) \in \mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R})$ then $u \in \mathbf{L}^G$. Therefore

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$$

but $\mathbf{L}^G$ cannot be identified with any

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R}) \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R}).$$

3.1. Convergence

Now we investigate relations between Luxemburg norm and the integral

$$R_G(u) := \int_I G(u)\,dt.$$

A functional R_G is called modular. Theory of modulars is well known and is developed in more general setting than ours. More information can be found in [23, 5].

For Lebesgue spaces a notions of modular and norm are indistinguishable because modular $\int_I |u|^p\,dt$ is equal to $\|u\|^p_{\mathbf{L}^p}$. But in Orlicz spaces relation between R_G and $\|\cdot\|_{\mathbf{L}^G}$ is more complex.

There is remarkable difference between isotropic and anisotropic spaces. It is clear that if $u, v \in \mathbf{L}^p$ (or more generally in isotropic Orlicz space) then $|u(t)| \leq |v(t)|$ a.e. implies $\|u\|_{\mathbf{L}^p} \leq \|v\|_{\mathbf{L}^p}$. In anisotropic case it is no longer true, even if $G(u(t)) < G(v(t))$. Next two examples illustrates this point.

Example 3.8. Let $G(x, y) = (x - y)^2 + y^4$, $I = [0, 1]$, $u(t) = (2, 0)$ and $v(t) = (2, 3/2)$. Then $|u(t)| < |v(t)|$, $G(u(t)) < G(v(t))$ and $R_G(u) \leq R_G(v)$, but $2 = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.6$.

Example 3.9. Let $G(x, y) = x^2 + y^4$, $u(t) = (1, 0)$ and $v(t) = \frac{11}{10}(\cos t, \sqrt{\sin t})$. In $\mathbf{L}^G([0, \pi], \mathbb{R}^2)$ we have $\sqrt{\pi} = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.7$, but $|u(t)| < |v(t)|$, $G(u(t)) < G(v(t))$ for all $t \in [0, \pi]$ and $R_G(u) < R_G(v)$.

Definition 3.10. We say that a subset $K \subset \mathbf{L}^G$ is modular bounded if there exists $C > 0$ such that

$$R_G(u) \leq C, \text{ for all } u \in K.$$

Modular boundedness is sometimes called mean boundedness. It is evident that $R_G(u) \leq \|u\|_{\mathbf{L}^G}$ if $\|u\|_{\mathbf{L}^G} \leq 1$ and $R_G(u) > \|u\|_{\mathbf{L}^G}$ if $\|u\|_{\mathbf{L}^G} > 1$.

Lemma 3.11. Let $u \in \mathbf{L}^G$.

1. If $R_G(u) \leq C$ then $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$.
2. If $\|u\|_{\mathbf{L}^G} \leq C$ then $R_G(u) \leq \mu(I)\widetilde{C} + K_1(C)$ for some $\widetilde{C} > 0$.

Moreover, a set $K \subset \mathbf{L}^G$ is modular bounded if and only if is norm bounded.

Proof. Assume that $R_G(u) \leq C$. If $C \leq 1$ then $\|u\|_{\mathbf{L}^G} \leq 1$. If $C > 1$ then

$$\int_I G\left(\frac{u}{C}\right)\, dt \leq \frac{1}{C}\int_I G(u)\, dt \leq 1.$$

This implies $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$. For the second statement, assume $\|u\|_{\mathbf{L}^G} \leq C$. Then

$$R_G(u) = \int_{I_1} G\left(u\right)\, dt + \int_{I\setminus I_1} G\left(C\,\frac{u}{C}\right)\, dt \leq \mu(I_1)\,\widetilde{C} + K_1(C)\int_I G\left(\frac{u}{C}\right)\, dt,$$

where $I_1 = \{t \in I \colon |u(t)| \leq M_1 C\}$ and $\widetilde{C} > 0$. To finish the proof observe that

$$\int_I G\left(\frac{u}{C}\right)\, dt \leq \int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right)\, dt = 1.$$

$\square$

Definition 3.12. We say that a sequence of functions $u_k \in \mathbf{L}^G$ is modular convergent to $u \in \mathbf{L}^G$ if $R_G(u_k - u) \to 0$ as $k \to \infty$.

Modular convergence is sometimes called mean convergence. Norm convergence always implies modular convergence. Let $\|u_k\|_{\mathbf{L}^G} \to 0$ as $k \to \infty$. We can assume that $\forall_k\ \|u_k\|_{\mathbf{L}^G} \leq 1$, then

$$\frac{1}{\|u_k\|_{\mathbf{L}^G}} R_G(u_k) \leq R_G\Big(\frac{u_k}{\|u_k\|_{\mathbf{L}^G}}\Big) = 1.$$

Hence $0 \leq R_G(u_k) \leq \|u_k\|_{\mathbf{L}^G}$. In general, converse is not true unless G satisfies Δ_2 condition (see [3, 11]).

Theorem 3.13. Norm convergence is equivalent to modular convergence.

Proof. We need only to prove that modular convergence implies norm convergence. Fix $\varepsilon > 0$ and assume that $\{u_k\}$ is modular convergent to 0. Define

$$I_{1,k} = \{t \in I \colon |u_k(t)| \leq M_1\}.$$

Since G satisfies Δ_2, for all $k > 0$ we have

$$\int_I G(u_k/\varepsilon)\,dt \le \mu(I_{1,k})\,C_{1/\varepsilon} + K_1(1/\varepsilon)\int_{I\setminus I_{1,k}} G(u_k)\,dt \le$$
$$\le \mu(I)\,C_{1/\varepsilon} + K_1(1/\varepsilon)\int_I G(u_k)\,dt.$$

For sufficiently large k we have

$$\int_I G(u_k)\,dt \le \frac{1}{K_1(1/\varepsilon)}$$

and

$$\int_I G(u_k/\varepsilon)\,dt \le \mu(I)\,C_{1/\varepsilon} + 1 = C.$$

Finally, Lemma 3.11 shows that $\|u_k\|_{\mathbf{L}^G} \le C\varepsilon$ and hence $\|u_k\|_{\mathbf{L}^G} \to 0$. $\square$

It is standard result due to Riesz that for $f_n,\ f \in \mathbf{L}^p$

$$f_n \to f \text{ a.e.} \implies (\|f_n\|_{\mathbf{L}^p} \to \|f\|_{\mathbf{L}^p} \iff \|f_n - f\|_{\mathbf{L}^p} \to 0)\,.$$

Following lemmas establish Orlicz space version of this fact.

Lemma 3.14. For every $k > 1$ and $0 < \varepsilon < \frac{1}{k}$ and $x, y \in \mathbb{R}^n$

$$|G(x+y) - G(x)| \le \varepsilon|G(kx) - kG(x)| + 2G(C_\varepsilon y)$$

where $C_\varepsilon = \frac{1}{\varepsilon(k-1)}$

The proof can be found in [24] (see also [25]).

Lemma 3.15. If $u_n \to u$ in $\mathbf{L}^G$ then $R_G(u_n) \to R_G(u)$.

Proof. In Lemma 3.14 set $x + y = u_n$, $x = u$, $k = 2$. Then $\varepsilon < 1/2$, $C_\varepsilon = \frac{1}{\varepsilon}$ and

$$|G(u_n) - G(u)| \le \varepsilon|G(2u) - 2G(u)| + 2G\left(\frac{u_n - u}{\varepsilon}\right).$$

Since $u_n \to u$ in $\mathbf{L}^G$, there exists n_0 such that for $n > n_0$ we have $\|u_n - u\|_{\mathbf{L}^G} < \varepsilon^2 \le \varepsilon < 1$. Thus

$$\int_I G\left(\frac{u_n - u}{\varepsilon}\right)\,dt \le \frac{1}{\varepsilon}\|u_n - u\|_{\mathbf{L}^G} < \varepsilon.$$

From this and inequality above we obtain

$$|R_G(u_n) - R_G(u)| \le \varepsilon\int_I |G(2u) - 2G(u)|\,dt + 2\varepsilon.$$

Letting $\varepsilon \to 0$ we have $R_G(u_n) \to R_G(u)$. $\square$

According to the above lemma, if $u_n \to u$ in $\mathbf{L}^G$ then:

1. Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^1$ (see Lemma 3.5 below), we can extract a subsequence u_{n_k} such that
$$u_{n_k} \to u \text{ a.e and } |u_{n_k}| \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

2. Since $R_G(u_n - u) \to 0$, $G(u_n - u) \to 0$ in $\mathbf{L}^1$. Thus we can extract a subsequence $\{u_{n_k}\}$ such that
$$G(u_{n_k} - u) \to 0 \text{ a.e and } G(u_{n_k} - u) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

3. Since $R_G(u_n) \to R_G(u)$, $G(u_n) \to G(u)$ in $\mathbf{L}^1$. Hence there exists a subsequence $\{u_{n_k}\}$ such that
$$G(u_{n_k}) \to G(u) \text{ a.e and } G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

Lemma 3.16. Let $\{u_n\} \subset \mathbf{L}^G$ and $u \in \mathbf{L}^G$. Suppose that

1. $u_n \to u$ a.e.,
2. $R_G(u_n) \to R_G(u)$.

Then $u_n \to u$ in $\mathbf{L}^G$.

Proof. This lemma was proved in [4, p. 83] for N-functions. Since G is convex, we get $\frac{1}{2}\left(G(u_n(t)) + G(u(t))\right) - G\left(\frac{u_n(t)-u(t)}{2}\right) \geq 0$. Continuity of G and $u_n \to u$ a.e. implies
$$\frac{1}{2}\left(G(u_n(t)) + G(u(t))\right) - G\left(\frac{u_n(t) - u(t)}{2}\right) \to G(u) \text{ a.e.}$$

So that by the Fatou Lemma, we have
$$\begin{aligned}\int_I G(u)\,dt \leq \liminf_{n\to\infty} \int_I \frac{1}{2}(G(u_n) + G(u))\,dt - G\left(\frac{u_n - u}{2}\right)\,dt \leq \\ \leq \lim_{n\to\infty} \int_I \frac{1}{2}(G(u_n) + G(u))\,dt - \limsup_{n\to\infty} \int_I G\left(\frac{u_n - u}{2}\right)\,dt = \\ = \int_I G(u)\,dt - \limsup_{n\to\infty} \int_I G\left(\frac{u_n - u}{2}\right)\,dt.\end{aligned}$$

This implies that
$$\int_I G\left(\frac{u_k(t) - u(t)}{2}\right)\,dt \to 0$$
and $\|u_k - u\|_{\mathbf{L}^G} \to 0$ by Theorem 3.13. $\square$

As a consequence we obtain dominated convergence theorem for anisotropic Orlicz spaces:

Theorem 3.17. Suppose that $\{u_n\} \subset \mathbf{L}^G$ and

1. $u_n \to u$ a.e.

2. there exists $h \in \mathbf{L}^1(I, \mathbb{R})$ such that $G(u_n) \leq h$ a.e.

Then $u \in \mathbf{L}^G$ and $u_n \to u$ in $\mathbf{L}^G$.

Proof. Since G is continuous and $u_n \to u$ a.e., $G(u_n) \to G(u)$ a.e. It follows that $G(u) \leq h$ a.e. Thus $G(u) \in \mathbf{L}^1(I, \mathbb{R})$ and hence $u \in \mathbf{L}^G$. In a standard way we get $R_G(u_n) \to R_G(u)$. Hence $u_n \to u$ in $\mathbf{L}^G$, by the Lemma 3.16. □

In the above theorem, assumption $G(u_n) \leq h$ can be replaced by $G(u_n) \leq G(h)$, $h \in \mathbf{L}^G$. Consider a sequence $\{u_n\} \subset \mathbf{L}^G$ convergent pointwise to measurable function u. Under standard dominance condition (i.e. $|u_n| \leq |g|$, $g \in \mathbf{L}^G$) it is not true in general that $u_n \to u \in \mathbf{L}^G$.

Example 3.18. Let $G(x, y) = x^2 + y^4$, $I = (0, 1)$, $u(t) = (0, t^{-1/4})$ and $h(t) = (t^{-3/8}, 0)$. Define

$$u_n(t) = \begin{cases} u(t) & |u(t)| \leq n \\ 0 & |u(t)| > n \end{cases}$$

Then $u_n \to u$ a.e., $u_n, h \in \mathbf{L}^G$ and $|u_n| \leq |h|$ for every t. But $G(u(t)) = t^{-1} \notin \mathbf{L}^1(I, \mathbb{R})$. Hence $u \notin \mathbf{L}^G$.

Remark 3.19. Modular R_G is called monotone modular if $|x| \leq |y|$ implies $R_G(x) \leq R_G(y)$. If R_G is monotone modular then $u_k \to u$ a.e and $|u_k| \leq |g|$, $g \in \mathbf{L}^G$ implies $u \in \mathbf{L}^G$ and $\|u_k - u\|_{\mathbf{L}^G} \to 0$. We refer the reader to [25] for more details.

4. Orlicz-Sobolev spaces

The Orlicz-Sobolev space $\mathbf{W}^1\,\mathbf{L}^G = \mathbf{W}^1\,\mathbf{L}^G(I, \mathbb{R}^n)$ is defined to be

$$\mathbf{W}^1\,\mathbf{L}^G(I, \mathbb{R}^n) := \{u \in \mathbf{L}^G(I, \mathbb{R}^n) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^n)\}.$$

For $u \in \mathbf{W}^1\,\mathbf{L}^G$ we define

$$\|u\|_{\mathbf{W}^1\,\mathbf{L}^G} := \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

Define $\mathbf{W}_0^1\,\mathbf{L}^G = \mathbf{W}_0^1\,\mathbf{L}^G(I, \mathbb{R}^n)$ as the closure of $C_0^1(I, \mathbb{R}^n)$ in $\mathbf{W}^1\,\mathbf{L}^G$ with respect to the $\|\cdot\|_{\mathbf{W}^1\,\mathbf{L}^G}$.

Theorem 4.1. The space $(\mathbf{W}^1\,\mathbf{L}^G, \|\cdot\|_{\mathbf{W}^1\,\mathbf{L}^G})$ is a separable reflexive Banach space.

Proof is standard and will be omitted, see for instance [26]. If $G(x) = \frac{1}{p}|x|^p$, then the Orlicz-Sobolev space $\mathbf{W}^1\,\mathbf{L}^G$ coincides with the Sobolev space $\mathbf{W}^{1,p}(I,\mathbb{R}^n)$. Observe that $u_n \to u$ in $\mathbf{W}^1\,\mathbf{L}^G$ is equivalent to $R_G(u_n - u) \to 0$ and $R_G(\dot{u}_n - \dot{u}) \to 0$.

Since there exist $p, q \in (1,\infty)$ such that $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$, the following continuous embeddings exist

$$\mathbf{W}^{1,q} \hookrightarrow \mathbf{W}^1\,\mathbf{L}^G \hookrightarrow \mathbf{W}^{1,p}$$

Using standard results from the theory of Sobolev spaces we get

1. $\mathbf{W}^1\,\mathbf{L}^G(I,\mathbb{R}^n) \hookrightarrow\hookrightarrow \mathbf{W}^{1,1}$,
2. $\mathbf{W}^1\,\mathbf{L}^G(I,\mathbb{R}^n) \hookrightarrow\hookrightarrow \mathbf{L}^q$, for all $1 \le q \le \infty$,
3. $\mathbf{W}^1\,\mathbf{L}^G(I,\mathbb{R}^n) \hookrightarrow\hookrightarrow C(\bar{I})$.

As a consequence we have

Theorem 4.2. A function $u \in \mathbf{W}^1\,\mathbf{L}^G$ is absolutely continuous. Precisely, there exists absolutely continuous representative of u such that for all $a, b \in I$

$$u(b) - u(a) = \int_a^b \dot{u}(t)dt.$$

Directly from definition of $\mathbf{W}^1_0\,\mathbf{L}^G$ we obtain important property of functions in $\mathbf{W}^1_0\,\mathbf{L}^G$.

Theorem 4.3. If $u \in \mathbf{W}^1_0\,\mathbf{L}^G$, then $u = 0$ on ∂I.

Using embeddings mentioned above we have for every $u \in \mathbf{W}^1\,\mathbf{L}^G$

$$\|u\|_{\mathbf{L}^\infty} \le C\|u\|_{\mathbf{W}^1\,\mathbf{L}^G} \tag{3}$$

Theorem 4.4 (Sobolev inequality)**.** For every function $u \in \mathbf{W}^1\,\mathbf{L}^G$

$$\|u - u_I\|_{\mathbf{L}^G} \le \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$$

where $u_I = \frac{1}{\mu(I)}\int_I u$.

Proof. Since u is absolutely continuous, there exists $t_0 \in I$ such that $u(t_0) = \frac{1}{\mu(I)} \int_I u$ and for every $t \in I$ we have

$$u(t) - u(t_0) = \int_{t_0}^{t} \dot{u}\, dt.$$

By Jensen's inequality,

$$G\left(\frac{u(t) - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) = G\left(\frac{1}{|t - t_0|} \int_{t_0}^{t} \frac{|t - t_0|}{\mu(I)} \frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\, dt\right) \leq$$
$$\leq \frac{1}{|t - t_0|} \int_{t_0}^{t} G\left(\frac{|t - t_0|}{\mu(I)} \frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)} \int_I G\left(\frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)}.$$

Integrating both sides over I we get

$$\int_I G\left(\frac{u - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq 1.$$

Thus $\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$. □

In similar way we get

Theorem 4.5 (Poincaré inequality)**.** For every $u \in \mathbf{W}_0^1\,\mathbf{L}^G$

$$\|u\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$$

It follows that one can introduce equivalent norm in $\mathbf{W}_0^1\,\mathbf{L}^G$:

$$\|u\|_{\mathbf{W}_0^1\,\mathbf{L}^G} = \|\dot{u}\|_{\mathbf{L}^G}.$$

Every linear functional F on $\mathbf{W}_0^1\,\mathbf{L}^G$ can be represented in the form

$$F(u) = \int_I \langle u, v_0\rangle + \langle \dot{u}, v_1\rangle\, dt,$$

where $v_0, v_1 \in \mathbf{L}^{G^*}$. Moreover, $\|F\| = \max\{\|v_0\|_{\mathbf{L}^{G^*}}, \|v_1\|_{\mathbf{L}^{G^*}}\}$. In the case of Sobolev space $\mathbf{W}^{1,p}$ the proof is given in [26, proposition 8.14], but it remains the same for Orlicz-Sobolev spaces. As was pointed out in [26], the first assertion of the above proposition holds for every linear functional on $\mathbf{W}^1\,\mathbf{L}^G$.

5. Variational setting

In this section we examine the principal part

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u})\, dt \tag{4}$$

of the variational functional associated with Euler-Lagrange equation

$$\frac{d}{dt} F_v(t, u, \dot{u}) = F_x(t, u, \dot{u}) + \nabla V(t, u), \quad t \in I$$

where $u\colon I \to \mathbb{R}^N$ and the Lagrangian $L\colon I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is given by $L(t, x, v) = F(t, x, v) + V(t, x)$.

In definition of the Orlicz space we need not to assume that G is differentiable, but when we consider the functional $\mathcal{I}$ we need it to show that $\mathcal{I} \in C^1$. Throughout this section we will assume, in addition to (G_1)–(G_6), that G satisfies

(G_7) G is of a class C^1.

Remark 5.1. Differentiability of f is not sufficient to differentiability of f^*. But if f is finite, strictly convex, 1-coercive and differentiable then so is f^*. This result is in close relation with Legendre duality (see [21, p. 239] and [1] for more details).

It is well known that if G is continuously differentiable then for all $x, y \in \mathbb{R}^n$

$$G(x) - G(x - y) \leq \langle \nabla G(x), y \rangle \leq G(x + y) - G(x) \tag{5}$$

and

$$\langle x, \nabla G(x) \rangle = G(x) + G^*(\nabla G(x)).$$

Let $y = x$ in (5). Then $\langle \nabla G(x), x \rangle \leq G(2x) - G(x)$. Therefore, for all $x \in \mathbb{R}^N$

$$G^*(\nabla G(x)) \leq G(2x).$$

Directly from the above we get

Proposition 5.2. If $u \in \mathbf{L}^G$ then $\nabla G(u) \in \mathbf{L}^{G^*}$.

Lemma 5.3 (cf. [16, lemma A.5])**.** If $u_n \to u$ in $\mathbf{L}^G$ then $R_{G^*}(\nabla G(u_n)) \to R_{G^*}(\nabla G(u))$.

Proof. There exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ a.e., $G(u_{n_k}) \to G(u)$ a.e. and $G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R})$. By continuity of ∇G and G^* we have $\nabla G(u_{n_k}) \to \nabla G(u)$ a.e. and

$$G^*(\nabla G(u_{n_k})) \to G^*(\nabla G(u)) \text{ a.e.}$$

Since $G^*(\nabla G(x)) \leq G(2x)$,

$$G^*(\nabla G(u_{n_k})) \leq G(2u_{n_k}) \leq C + K_1 G(u_{n_k}) \leq C + K_1 h.$$

By dominated convergence theorem $R_{G^*}(\nabla G(u_{n_k})) \to R_{G^*}(\nabla G(u))$. Since this holds for any subsequence of $\{u_n\}$ we have that

$$R_{G^*}(\nabla G(u_n)) \to R_{G^*}(\nabla G(u)).$$

□

As a direct consequence of the above lemma and Lemma 3.16 we obtain

Proposition 5.4.

$$\|u_n - u\|_{\mathbf{L}^G} \to 0 \implies \|\nabla G(u_n) - \nabla G(u)\|_{\mathbf{L}^{G^*}} \to 0.$$

5.1. Case I

We shall first examine a special case $F(t,x,v) = G(v)$, now functional (4) takes the form

$$\mathcal{I}(u) = \int_I G(\dot{u})\, dt.$$

Theorem 5.5. $\mathcal{I} \in C^1(\mathbf{W}^1\mathbf{L}^G, \mathbb{R})$. Moreover

$$\mathcal{I}'(u)\varphi = \int_I \langle \nabla G(\dot{u}), \dot{\varphi}\rangle dt. \tag{6}$$

Proof. The proof follows similar lines as [2, th. 3.2] (see also [1, thm 1.4]). First, note that $\dot{u} \in \mathbf{L}^G$ implies

$$0 \leq \mathcal{I}(u) < \infty.$$

It suffices to show that $\mathcal{I}$ has at every point u directional derivative $\mathcal{I}'(u) \in (\mathbf{W}^1\mathbf{L}^G)^*$ given by (6) and that the mapping $\mathcal{I}' : \mathbf{W}^1\mathbf{L}^G \to (\mathbf{W}^1\mathbf{L}^G)^*$ is continuous. Let $u \in \mathbf{W}^1\mathbf{L}^G$, $\varphi \in \mathbf{W}^1\mathbf{L}^G \setminus\{0\}$, $t \in I$, $s \in [-1,1]$. Define

$$H(s,t) := G(\dot{u}(t) + s\dot{\varphi}(t)).$$

By (5) we obtain

$$\int_I |H_s(s,t)|\, dt = \int_I |\langle \nabla G(\dot{u}+s\dot{\varphi}), \dot{\varphi}\rangle|\, dt \leq \int_I G(\dot{u}+(s+1)\dot{\varphi}) + \int_I G(\dot{u}+s\dot{\varphi})\, dt < \infty.$$

Consequently, $\mathcal{I}$ has a directional derivative and

$$\mathcal{I}'(u)\varphi = \frac{d}{ds}\mathcal{I}(u+s\varphi)\Big|_{s=0} = \int_I \langle \nabla G(\dot{u}), \dot{\varphi}\rangle dt.$$

By Proposition 5.2 and the Hölder inequality

$$|\mathcal{I}'(u)\varphi| = \Big|\int_I \langle \nabla G(\dot u), \dot\varphi\rangle dt\Big| \le 2\|\nabla G(\dot u)\|_{\mathbf{L}^{G^*}} \|\dot\varphi\|_{\mathbf{L}^G} \le C\|\varphi\|_{\mathbf{W}^1\mathbf{L}^G}.$$

To finish the proof it suffices to show that if $u_n \to u$ in $\mathbf{W}^1\mathbf{L}^G$, then $\mathcal{I}'(u_n) \to \mathcal{I}'(u)$ in $(W^1L^G)^*$. Using the Hölder inequality and Proposition 5.4 we obtain

$$|\mathcal{I}'(u_n)\varphi - \mathcal{I}'(u)\varphi| = \left|\int_I \langle \nabla G(\dot u_n) - \nabla G(\dot u), \dot\varphi\rangle\, dt\right| \le$$
$$\le 2\|\nabla G(\dot u_n) - \nabla G(\dot u)\|_{\mathbf{L}^{G^*}} \|\dot\varphi\|_{\mathbf{L}^G} \to 0.$$

□

5.2. Case II

We turn to general case. Suppose that $F\colon I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies

(F_1) $F \in C^1$,

(F_2) $|F(t,x,v)| \le a(|x|)(b(t) + G(v))$,

(F_3) $|F_x(t,x,v)| \le a(|x|)(b(t) + G(v))$,

(F_4) $G^*(F_v(t,x,v)) \le a(|x|)(c(t) + G^*(\nabla G(v)))$.

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$.

If $G(v) = |v|^p$ then conditions (F_2), (F_3) and (F_4) take the standard form (Theorem 1.4 from [1]). In [2] there are similar conditions with $G(v) = \Phi(|v|)$, where Φ is an N-function. In this case, condition (F_4) takes the form $|F_v(t,x,v)| \le \tilde a(|x|)(\tilde c(t) + \Phi'(|u|))$. In anisotropic case we need to use G^*, because vector valued G-function is not necessarily monotone with respect to $|\cdot|$.

Lemma 5.6. If $u \in \mathbf{W}^1\mathbf{L}^G$, then $F_x(\cdot, u, \dot u) \in \mathbf{L}^1$ and $F_v(\cdot, u, \dot u) \in \mathbf{L}^{G^*}$.

Proof. Define non decreasing function

$$\alpha(s) = \sup_{\tau \in [0,s]} a(\tau).$$

Then, for $u \in \mathbf{W}^1\mathbf{L}^G$ we have

$$a(|u(t)|) \le \alpha(\|u\|_{\mathbf{L}^\infty}) \le \alpha(C\|u\|_{\mathbf{W}^1\mathbf{L}^G}). \tag{7}$$

Let $u \in \mathbf{W}^1\mathbf{L}^G$. By (7) and (F_3)

$$\int_I |F_x(t,u,\dot u)|\, dt \le \int_I a(|u(t)|)(b(t) + G(\dot u))\, dt \le$$
$$\le \alpha(C\|u\|_{\mathbf{W}^1\mathbf{L}^G}) \int_I (b(t) + G(\dot u))\, dt < \infty.$$

Moreover, by (F_4) and Proposition 5.2

$$\int_I G^*(F_v(t,u,\dot{u}))\,dt \ \le \alpha(C\|u\|_{\mathbf{W}^1\mathbf{L}^G})\int_I (c(t)+G^*(\nabla G(\dot{u})))\,dt < \infty.$$

□

Theorem 5.7. $\mathcal{I}\in C^1(\mathbf{W}^1\mathbf{L}^G,\mathbb{R})$. Moreover

$$\mathcal{I}'(u)\varphi = \int_I \langle F_x(t,u,\dot{u}),\varphi\rangle dt + \int_I \langle F_v(t,u,\dot{u}),\dot{\varphi}\rangle dt. \tag{8}$$

Proof. By (F_2)

$$|\mathcal{I}(u)| \le \int_I a(|u|)(b(t)+G(\dot{u}))dt \le \alpha(\|u\|_{\mathbf{W}^1\mathbf{L}^G})\int_I (b(t)+G(\dot{u}))dt < \infty.$$

It suffices to show that directional derivative $\mathcal{I}'(u)\in(\mathbf{W}^1\mathbf{L}^G)^*$ exists, is given by (8) and that the mapping $\mathcal{I}':\mathbf{W}^1\mathbf{L}^G\to(\mathbf{W}^1\mathbf{L}^G)^*$ is continuous.

Let $u\in\mathbf{W}^1\mathbf{L}^G$, $\varphi\in\mathbf{W}^1\mathbf{L}^G\setminus\{0\}$, $t\in I$, $s\in[-1,1]$. Define

$$H(s,t) := F(t,u+s\varphi,\dot{u}+s\dot{\varphi}).$$

By (F_3), continuity of φ, (7) and the fact that $u+s\varphi\in\mathbf{W}^1\mathbf{L}^G$ we obtain

$$\begin{aligned}\int_I |\langle F_x(t,u+s\varphi,\dot{u}+s\dot{\varphi}),\varphi\rangle|\,dt &\le \int_I |F_x(t,u+s\varphi,\dot{u}+s\dot{\varphi})||\varphi|\,dt \le \\ &\le \int_I a(|u+sv|)(b(t)+G(\dot{u}+s\dot{\varphi}))|\varphi|\,dt \le \\ &\le \alpha(\|u+s\varphi\|_{\mathbf{W}^1\mathbf{L}^G})\int_I (b(t)+G(\dot{u}+s\dot{\varphi}))|\varphi|\,dt < \infty.\end{aligned}$$

By the Fenchel inequality, (F_4) and Lemma 5.6 we obtain

$$\int_I |\langle F_v(t,u+s\varphi,\dot{u}+s\dot{\varphi}),\dot{\varphi}\rangle|dt \le \int_I [G^*(F_v(t,u+s\varphi,\dot{u}+s\dot{\varphi}))+G(\dot{\varphi})]dt < \infty.$$

It follows that

$$\int_I |H_s(s,t)|dt = \int_I |\langle F_x(t,u+s\varphi,\dot{u}+s\dot{\varphi}),\varphi\rangle + \langle F_v(t,u+s\varphi,\dot{u}+s\dot{\varphi}),\dot{\varphi}\rangle|dt < \infty.$$

Consequently, $\mathcal{I}$ has a directional derivative and

$$\mathcal{I}'(u)\varphi = \frac{d}{ds}\mathcal{I}(u+s\varphi)\Big|_{s=0} = \int_I \langle F_x(t,u,\dot{u}),\varphi\rangle dt + \int_I \langle F_v(t,u,\dot{u}),\dot{\varphi}\rangle dt.$$

By Lemma 5.6, the Hölder inequality and (3) we get

$$|\mathcal{I}'(u)\varphi| \le \|F_x(\cdot,u,\dot u)\|_{\mathbf{L}^1}\|\varphi\|_{\mathbf{L}^\infty} + \|F_v(\cdot,u,\dot u)\|_{\mathbf{L}^{G^*}}\|\dot\varphi\|_{\mathbf{L}^G} \le C\|\varphi\|_{\mathbf{W}^1\mathbf{L}^G}.$$

To finish the proof it suffices to show that $\mathcal{I}'$ is continuous. Since $u_n \to u$ in $\mathbf{W}^1\mathbf{L}^G$, it follows that $u_n \to u$ in $\mathbf{L}^G$, $\dot u_n \to \dot u$ in $\mathbf{L}^G$ and there exists $M > 0$ such that $\|u_n\|_{\mathbf{W}^1\mathbf{L}^G} < M$.

By Lemma 3.15 we have $G(\dot u_n) \to G(\dot u)$ in $\mathbf{L}^1(I,\mathbb{R})$. Hence there exists a subsequence $\{u_{n_k}\}$ and $h \in \mathbf{L}^1(I,\mathbb{R})$ such that

$$G(\dot u_{n_k}) \to G(\dot u) \text{ a.e and } G(\dot u_{n_k}) \le h.$$

By (F_3) and since $\{u_{n_k}\}$ is bounded, we obtain

$$|F_x(t,u_{n_k},\dot u_{n_k})| \le \alpha(\|u_{n_k}\|_{\mathbf{W}^1\mathbf{L}^G})(b(t)+G(\dot u_{n_k}))dt \le \alpha(M)(b(t)+h(t)).$$

By (F_1) we have

$$F_x(t,u_{n_k}(t),\dot u_{n_k}(t)) \to F_x(t,u(t),\dot u(t))$$

for a.e $t \in I$. Applying dominated convergence theorem we obtain

$$\int_I \langle F_x(t,u_{n_k},\dot u_{n_k}),\varphi\rangle dt \to \int_I \langle F_x(t,u,\dot u),\varphi\rangle dt.$$

Since this holds for any subsequence of $\{u_n\}$ we have that

$$\int_I \langle F_x(t,u_n,\dot u_n),\varphi\rangle dt \to \int_I \langle F_x(t,u,\dot u),\varphi\rangle dt.$$

By (F_4) and Lemma 5.6

$$G^*(F_v(t,u_{n_k}(t),\dot u_{n_k}(t))) \le \alpha(M)(c(t) + G^*(\nabla G(\dot u_{n_k}(t)))).$$

In the same way as in the proof of Lemma 5.3 we obtain

$$G^*(F_v(t,u_{n_k}(t),\dot u_{n_k}(t))) \le \alpha(M)(c(t) + C + K_1 h(t)).$$

By continuity of F_v we obtain

$$G^*(F_v(t,u_{n_k}(t),\dot u_{n_k}(t))) \to G^*(F_v(t,u(t),\dot u(t)))$$

for a.e $t \in I$ and consequently

$$\int_I G^*(F_v(t,u_{n_k},\dot u_{n_k}))dt \to \int_I G^*(F_v(t,u,\dot u))dt.$$

It follows that

$$\int_I G^*(F_v(t,u_n,\dot u_n))dt \to \int_I G^*(F_v(t,u,\dot u))dt.$$

Application of Lemma 3.16 to R_{G^*} yields $\|F_v(\cdot,u_n,\dot u_n)-F_v(\cdot,u,\dot u)\|_{\mathbf{L}^{G^*}} \to 0$. By the Hölder inequality

$$\left|\int_I \langle F_v(t,u_n,\dot u_n)-F_v(t,u,\dot u),\dot\varphi\rangle\, dt\right| \le 2\|F_v(\cdot,u_n,\dot u_n)-F_v(\cdot,u,\dot u)\|_{\mathbf{L}^{G^*}}\|\dot\varphi\|_{\mathbf{L}^G} \to 0.$$

Finally,

$$\int_I \langle F_v(t,u_n,\dot u_n),\dot\varphi\rangle dt \to \int_I \langle F_v(t,u,\dot u),\dot\varphi\rangle dt.$$

□

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