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Turán numbers for odd wheels

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ABSTRACT

The Turán number $\mathrm{ex}(n, G)$ is the maximum number of edges in any n-vertex graph that does not contain a subgraph isomorphic to G. A *wheel* W_n is a graph on n vertices obtained from a C_{n-1} by adding one vertex w and making w adjacent to all vertices of the C_{n-1}. We obtain two exact values for small wheels:

$$\mathrm{ex}(n, W_5) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} \right\rfloor,$$

$$\mathrm{ex}(n, W_7) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} + 1 \right\rfloor.$$

Given that $\mathrm{ex}(n, W_6)$ is already known, this paper completes the spectrum for all wheels up to 7 vertices. In addition, we present the construction which gives us the lower bound $\mathrm{ex}(n, W_{2k+1}) > \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$ in general case.

1. Introduction

In this paper, all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. Let $d_G(v)$ be the degree of vertex v in G, $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of vertices of G, $\omega(G)$ be the clique number of a graph G and $\chi(G)$ be the chromatic number of graph G. Define $G[S]$ to be a subgraph of G induced by a set of vertices $S \subseteq V(G)$ and $G[S, R]$ to be a bipartite subgraph of G with the bipartition $\{S, R\}$. $G_1 \cup G_2$ denotes the graph which consists of two disconnected subgraphs G_1 and G_2. We will use $G_1 + G_2$ to denote the join of G_1 and G_2 defined as $G_1 \cup G_2$ together with all edges between G_1 and G_2. C_m denotes the cycle of length m. A *wheel* W_n is a graph on n vertices obtained from a C_{n-1} by adding one vertex w and making w adjacent to all vertices of the C_{n-1}.

The Turán number $\mathrm{ex}(n, G)$ is the maximum number of edges in any n-vertex graph that does not contain a subgraph isomorphic to G. A graph on n vertices is said to be *extremal with respect to* G if it does not contain a subgraph isomorphic to G and has exactly $\mathrm{ex}(n, G)$ edges. $\mathrm{EX}(n, G)$ is the set of all extremal graphs of order n with respect to G.

A main motivation for proving results for Turán numbers is that they are often useful in Ramsey Theory where the original extremal statements would not suffice (see [3] for example). Our goal is to determine the Turán numbers of wheels W_k for odd k. We describe families of extremal graphs for $k = 5, 7$ and present a very simple lower bound for all odd k.

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2. Known results

First, we recall the result which was proved by Mantel in 1907.

Theorem 1 (*Mantel,* [5]). *The maximum number of edges in an n-vertex triangle-free graph is* $\lfloor \frac{n^2}{4} \rfloor$.

By Theorem 1 and since $W_3 = C_3$, it is easy to have the property that for all integers $n, n \geq 3$, $\mathrm{ex}(n, W_3) = \lfloor \frac{n^2}{4} \rfloor$. The famous Turán's theorem may be stated as follows.

Theorem 2 (*Turán,* [8]). *Let G be any subgraph of* K_n *such that G is* K_{r+1}*-free. Then the number of edges in G is* $e(G) = \lfloor \frac{(r-1)n^2}{2r} \rfloor$. *In particular,* $ex(n, K_4) = \lfloor \frac{n^2}{3} \rfloor$.

As a special case, for $r = 2$, one obtains Mantel's theorem. Since $W_4 = K_4$, we obtain that for all integers $n, n \geq 3$, $\mathrm{ex}(n, W_4) = \lfloor \frac{n^2}{3} \rfloor$. In 1964 Erdős proved the following theorem.

Theorem 3 (*Erdős,* [4]). *Let G be any graph such that* $|E(G)| \geq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor + 1$. *Then G contains a* W_5.

By Theorem 3 we immediately obtain the upper bound for $\mathrm{ex}(n, W_5)$, namely $\mathrm{ex}(n, W_5) \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n+1}{4} \rfloor + 1$. The first author [2] proved that for all $k \geq 3$ and $n \geq 6k - 10$, if G is a graph that contains no subgraph isomorphic to W_{2k}, then $\mathrm{ex}(n, W_{2k}) = \lfloor \frac{n^2}{3} \rfloor$. In addition, he showed that $\mathrm{ex}(n, W_6) = \lfloor \frac{n^2}{3} \rfloor$.

If G is an arbitrary graph whose chromatic number is $r > 2$, then by Erdős–Stone–Simonovits theorem [7] we have that $\mathrm{ex}(n, G) = (\frac{r-2}{r-1} + o(1)) \binom{n}{2}$. This result determines the asymptotic behavior of $\mathrm{ex}(n, W_k)$.

It is interesting that exact values for $\mathrm{ex}(n, C_4)$ and $\mathrm{ex}(n, C_6)$, i.e. for rims of wheels W_5 and W_7 remain unknown in general. Even in the case of the C_4 cycle values are known only for $n \leq 32$ (the last result being $\mathrm{ex}(32, C_4) = 92$, obtained in 2009 by Shao, Xu and Xu), whereas for larger n only the upper or lower bounds are known.

3. Progress on $\mathrm{ex}(n, W_{2k+1})$

3.1. $ex(n, W_5)$

If G and H have maximum degree 1, then the join $G+H$ does not contain W_5. So define M_n by taking $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and adding a maximum matching within each partite set.

Lemma 4. *The graph* M_n *does not contain a* W_5 *as a subgraph.*

Proof. Every subgraph induced on 3 vertices of W_5 is connected. If i, j, k have the same parity then, by definition of M_n, graph $M_n[v_i, v_j, v_k]$ has at most one edge, so it is a disconnected graph. If we assume that M_n has a subgraph W_5, then at least 3 vertices of this subgraph W_5 are indexed by numbers which have the same parity (we denote the vertices of M_n as in the definition). A graph induced in W_5 by these three vertices is connected, but a graph induced in M_n by these vertices is not connected. This means that M_n does not contain a subgraph W_5. □

Theorem 5. *The graph* M_n *is an extremal graph with respect to* W_5.

Proof. We know that $M_1 = K_1, M_2 = K_2, M_3 = K_3$ and $M_4 = K_4$ are extremal. Assume that each M_n is extremal for $n < N$. We will show that M_N is also extremal. Let G be an extremal graph of order N. Let H be a 4-vertex subgraph of G with maximum possible number of edges. □

Lemma 6. *A graph G of order* 5 *contains* W_5 *as a subgraph if and only if* $\delta(G) \geq 3$.

Proof. If G contains W_5, it must be a spanning subgraph and so $\delta(G) \geq 3$. If $\delta(G) \geq 3$, then G contains a vertex of degree 4 and G contains a W_5. □

Consider the graph $G \setminus V(H)$. From Lemma 6 we know that each vertex from $G \setminus V(H)$ is adjacent to at most 2 vertices from H. If any $v \in G \setminus V(H)$ was adjacent to three vertices of H, then the graph $G[V(H) \cup \{v\}]$ would contain W_5 as a subgraph or a 4-vertex subgraph with a greater number of edges than H. From the above it follows that

$$\begin{aligned} e(G) &\leq e(H) + 2 \cdot |V(G \setminus V(H))| + e(G \setminus V(H)) \\ &\leq \binom{4}{2} + 2 \cdot (N-4) + \mathrm{ex}(N-4, W_5) = e(M_n). \end{aligned}$$

If G is extremal, then M_N does not contain W_5. In addition, $e(M_N) \geq e(G)$, so M_N is also extremal. □

Table 1
The values of $\mathrm{ex}(n, W_7)$ and $|\mathrm{EX}(n, W_7)|$ for all $7 \le n \le 26$.

n	7	8	9	10	11	12	13	14	15	16
$\mathrm{ex}(n, W_7)$	17	21	25	31	37	43	50	57	65	73
$\|\mathrm{EX}(n, W_7)\|$	2	1	5	1	1	2	1	2	1	2
n	17	18	19	20	21	22	23	24	25	26
$\mathrm{ex}(n, W_7)$	82	91	101	111	122	133	145	157	170	183
$\|\mathrm{EX}(n, W_7)\|$	1	2	1	2	1	3	2	3	1	2

Corollary 7.

$$ex(n, W_5) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} \right\rfloor.$$

Bataineh, Jaradat and Jaradat [1] presented a very extensive characterization of all extremal W_5-free graphs.

3.2. $ex(n, W_7)$

It is not hard to verify that if G has maximum degree 1 and H has maximum degree 2 and does not contain P_5, then the join $G + H$ does not contain W_7. So let G_m be the graph formed from m isolated vertices by adding a maximum matching. Further, let H_m be any 2-regular m-vertex graph formed by the disjoint union of copies of 3- or 4-cycles. (It can be checked that H_m exists for $m \ge 6$.) Then define the graph N_n as $G_{k-1} + H_{k+1}$ if $n = 2k$, and $G_k + H_{k+1}$ if $n = 2k + 1$. It can be checked that N_n has $k^2 + k + 1$ edges if $n = 2k$, and $k^2 + 2k + 2$ edges if $n = 2k + 1$.

From this construction we see that $\mathrm{ex}(2k, W_7) \ge k^2 + k + 1$ and $\mathrm{ex}(2k + 1, W_7) \ge k^2 + 2k + 2$.

Theorem 8. *For all $k \ge 5$, if $ex(2k, W_7) = k^2 + k + 1$, then $ex(2k + 1, W_7) \le k^2 + 2k + 2$.*

Proof. Let G be a graph of order $2k + 1$ which does not contain W_7 and assume that $e(G) = k^2 + 2k + 3$.

Observe that $\delta(G) \ge e(G) - \mathrm{ex}(2k, W_7) = k + 2$. Since $e(G) \ge \frac{(2k+1)(k+2)}{2} > k^2 + 2k + 3 = e(G)$ for all $k \ge 5$, we deduce the result. □

Theorem 9. *For all $k \ge 5$, $ex(2k, W_7) = k^2 + k + 1$.*

Proof. The cases $5 \le k \le 8$ were checked by computational calculations (see Table 1).

Suppose that $k > 8$ is the smallest number such that $\mathrm{ex}(2k, W_7) > k^2 + k + 1$, then for all $5 \le l < k$ we have $\mathrm{ex}(2l, W_7) = l^2 + l + 1$ and by Theorem 8 $\mathrm{ex}(2l + 1, W_7) = l^2 + 2l + 2$.

Let G be a graph of order $2k$ with $e(G) = k^2 + k + 2$ edges and G does not contain W_7 as a subgraph. We see that $\delta(G) \ge e(G) - \mathrm{ex}(2k - 1, W_7) = k + 1$. If $\delta(G) \ge k + 2$, then $e(G) \ge \frac{2k(k+2)}{2} > e(G)$ for all $k > 2$. So we have $\delta(G) = k + 1$.

The remaining part of the proof is divided into four cases according to the value of $\omega(G)$. Clearly $\omega(G) < 7$.

Case 1. $\omega(G) = 6$

Let K be a clique of order 6 in G and $W = V(G) \setminus V(K)$. To avoid W_7, every vertex in W is joined to K by at most two edges. We have

$$\binom{6}{2} + 2(2k - 6) + \mathrm{ex}(2k - 6, W_7) = k^2 - k + 10 < e(G),$$

a contradiction.

Case 2. $\omega(G) = 5$

Let $K = \{v_1, v_2, v_3, v_4, v_5\}$ be a maximum clique and $W = V(G) \setminus K$. Consider the edges of the bipartite graph $H = G[K, W]$. Let $W^4 = \{v \in W : d_H(v) = 4\}$, $W^3 = \{v \in W : d_H(v) = 3\}$ and $W^r = W - W^4 - W^3$, obviously if $v \in W^r$ then $d_H(v) < 3$.

One can easily verify that if $|W^4| \ge 2$, then we immediately have W_7. If $|W^4| = 1$, then to avoid W_7 in G we have that $|W^3| = 0$. Since $e(H) \le 4 + 2(2k - 6) < 5(k - 3) = 5(\delta(G) - 4) \le e(H)$ for $k > 7$, we obtain that in fact $W^4 = \emptyset$. Note that W^3 in G is an independent set and each edge in $G[K, W^3]$ is adjacent to the same three vertices of K, say $\{v_1, v_2, v_3\}$. From $\delta(G) = k + 1$, it follows that $|W^r| + 3 \ge \delta(G)$, so $|W^3| \le k - 3$. In fact $|W^3| = k - 3$ because of the inequality $e(G) \le 10 + 3|W^3| + 2|W^r| + \mathrm{ex}(2k - 5, W_7) = k^2 + 5 + |W^3|$.

Note that for every vertex v in W^3 we have that $d_G(v) = k + 1$. The bipartite graph $G[W^r, W^3]$ is complete, therefore $\Delta(G[W^r]) \le 2$. If not, then we have W_7 in $G[W]$. Hence, $e(G[W]) \le |W^3||W^r| + \frac{2|W^r|}{2} = k^2 - 4k + 4$ and $e(G) \le 10 + 3|W^3| + 2|W^r| + e(G[W]) \le k^2 + k + 1$, a contradiction.

We have $W^4 = W^3 = \emptyset$, $|W^r| = 2k - 5$ but $e(G[K, W]) \le 2(2k - 5) < 5(k - 3) = 5(\delta(G) - 4) < e(G[K, W])$ for $k > 5$, a contradiction.

Case 3. $\omega(G) = 4$

Let $K = \{v_1, v_2, v_3, v_4\}$ be a maximum clique and $W = V(G) \setminus K$.

Let U_i be the set of vertices from W such that they are adjacent to all vertices from $V(K) \setminus \{v_i\}$. This means that if $v \in U_i$ then $d_{G[K,W]}(v) = 3$. To avoid K_5 all U_i are independent. Let the remaining vertices of W be W^r.

First observe that if U_i, U_j, U_l are not empty for $i \neq j \neq l \in \{1, 2, 3, 4\}$, then we immediately have W_7. Without loss of generality, let us assume that U_3, U_4 are empty. Observe that if $|U_1 \cup U_2| > 2$, then the set $U_1 \cup U_2$ is independent.

Subcase 3.1 $U_1 = U_2 = \emptyset$

We have $e(G) \leq \text{ex}(2k-4, W_7) + 6 + 2(2k-4) = k^2 + k + 1 < e(G)$, a contradiction.

Subcase 3.2 $|U_1 \cup U_2| = 1$

Without loss of generality, let $w \in U_1$. To avoid a contradiction similar to the previous subcase, for all vertices $v \in W^r$ we have $d_{G[K,W]}(v) = 2$. This means that one vertex from K has degree $k+2$ and the remaining three vertices have degree $k+1$ in G, so at least one vertex from W has degree greater than or equal to $k+2$ in G.

Let X be all vertices from W^r adjacent to w and $Y = W^r \setminus X$. Obviously $|X| \geq k-2$. It is not hard to see that if $G[X]$ contains P_4 or K_3 as a subgraph, then $G[K \cup U_1 \cup X]$ contains W_7 as a subgraph. If $|X| \geq 4$, then there exist at least 3 vertices of degree 1 in $G[X]$. These vertices are adjacent to all vertices in Y, therefore $\Delta(G[Y]) \leq 2$, $|X| = k-2$, $|Y| = k-3$, subsequently $\delta(G[X]) = 1$, $\delta(G[Y]) \geq 1$ and $\Delta(G[Y]) \leq 2$, so each vertex from Y is adjacent to all or all except one vertex from X.

If there exists a vertex $p \in Y$ such that $d_G(p) > k+1$, then $d_{G[Y]}(p) = 2$ and p is adjacent to every vertex in X. Let p_1, p_2 be the vertices adjacent to p in Y. If there exists P_3 in $G[X]$, then one end-vertex of the path is adjacent to p_1 and the other to p_2, then the graph induced by the path, p_1, p_2, p and an additional vertex from X adjacent to p_1 and p_2 contains W_7 as a subgraph. Contrary, there exist two independent edges in $G[X]$ such that their vertices are adjacent to p_1 or p_2. These edges with p_1, p_2 and p induce a graph with W_7 as a subgraph.

If there exists a vertex $p \in X$ such that $d_G(p) > k+1$, then $d_{G[X]}(p) \geq 2$. If $d_{G[X]}(p) = 2$ then p is adjacent to every vertex in Y. Let p_1 and p_2 be the adjacent vertices to p in X. Note that p_1, p_2 have degree 1 in $G[X]$. There exist two independent edges in $G[Y]$. Since p, p_1 and p_2 are adjacent to vertices incident to these independent edges, then they both with w induce a graph with a subgraph W_7. If $d_{G[X]}(p) > 2$, then vertex w, three vertices adjacent to p in X and two vertices adjacent to p in Y induce a graph with a subgraph W_7.

From the above arguments, every vertex of W has degree $k+1$ in G, so $e(G[W]) < \text{ex}(2k-5, W_7)$, a contradiction.

Subcase 3.3 $|U_1 \cup U_2| = 2$

Let $w_1, w_2 \in U_1$. There exists a vertex $p \in W^r$ adjacent to w_1 and two vertices of K, v_1 and another vertex. A graph induced by $K \cup U_1$ and p contains W_7 as a subgraph.

Let $w_1 \in U_1$, $w_2 \in U_2$ and Q_1, Q_2 be the set of neighbors of w_1, w_2 in W^r, respectively. Every vertex of W^r is adjacent to at least one vertex of K.

Let $s_1 \in Q_1 \cap Q_2$ such that s_1 is adjacent to a vertex in K and $s_2 \in Q_1$ is adjacent to two vertices in K. The graph induced by $K \cup U_1 \cup U_2 \cup \{s_1, s_2\}$ contains a subgraph W_7.

If there are no vertices in $Q_1 \cap Q_2$ adjacent to one vertex in K then every vertex in Q_1 or Q_2 is adjacent to two vertices in K. Without loss of generality, let Q_1 be such a set. It is easy to see that the set Q_1 is independent. The maximal degree of w_1 in $G[K \cup U]$ is 4. From the assumption $\delta(G) \geq k+1$, we conclude $|Q_1| \geq k-3$. Let $X = W^r \setminus Q_1$. Since each vertex from Q_1 has degree at least $k+1$ in G and Q_1 is independent, we conclude $|X| \geq k-3$, $|Q_1| = |X| = k-3$ and w_1 is adjacent to w_2. If $Q_1 \neq Q_2$, then a vertex from $Q_2 \setminus Q_1$, any two vertices from Q_1, vertices w_1, w_2 and K induce a graph which contains W_7 as a subgraph. Since $k \geq 7$, we have that $\Delta(G[X]) \leq 2$. From all previous considerations we have $e(G) \leq 6 + 7 + 2(2k-6) + 2(k-3) + (k-3) + (k-3)(k-3) = k^2 + k + 1$, a contradiction.

Subcase 3.4 $|U_1 \cup U_2| > 2$

Let $W^2 = \{v \in W : d_{G[K,W]}(v) = 2\}$, $W^1 = \{v \in W : d_{G[K,W]}(v) \leq 1\}$ and $U = U_1 \cup U_2$. At least one of the sets U_1, U_2 has order greater than or equal to 2, say U_1 is such a set. Let $u_1, u_2 \in U_1$. If there exist vertices $w_1, w_2 \in W^2$ (not necessarily different) such that u_1 is adjacent to w_1 and u_2 is adjacent to w_2, then the graph $G[K \cup \{u_1, u_2, w_1, w_2\}]$ contains W_7. In the opposite case, one of the vertices u_1, u_2 is not adjacent to any vertex from W^2 and since U is an independent set, we have $W^1 \geq k-2$. By the inequalities $e(K) + 3|U| + 2|W^2| + |W^1| + \text{ex}(2k-4, W_7) \geq e(G)$ and $|U| + |W^2| + |W^1| = 2k-4$, we have $|U| \geq |W^1| + 1$, so $|U| + |W^1| \geq 2k-3$, a contradiction.

Case 4. $\omega(G) = 3$

Let $K = \{v_1, v_2, v_3\}$ be the clique in G and the remaining vertices are W. Let U_i be a set of all vertices from W such that they are adjacent to vertices $K - v_i$. This means that if $v \in U_i$ then $d_{G[K,W]}(v) = 2$. To avoid K_4 all U_i are independent. Let the remaining vertices of W be W^r and $U_1 \cup U_2 \cup U_3 = U$.

First observe that if there is a $K_2 \cup K_2$ between U_i and U_j where $i \neq j \in \{1, 2, 3\}$, then we immediately have W_7. Since $3(\delta(G) - 2) \leq e(G[K, W]) \leq (2k - 3 - |U|) + 2|U|$, we have $|U| \geq k$. There exists a vertex in U adjacent to at most two vertices in U. This vertex is adjacent to at least $k-3$ vertices in W^r. The equalities $|U| = k$ and $|W^r| = k-3$ are obtained by the above inequalities and the property $|W^r| + |U| = 2k-3$.

If there is a vertex of degree at most 1 in U, then we have a contradiction with $\delta(G) = k+1$. Since graphs $G[U_i \cup U_j]$ do not contain $K_2 \cup K_2$, the only graph with the property is $K_{k-2,1,1}$.

Note that all vertices of degree 2 in U are joined to every vertex of W^r but none of the vertices of degree $k-1$ in U are joined to any of vertices W^r. Moreover, to avoid W_7 we have $\Delta(G[W^r]) \leq 2$, so none of the vertices in G has degree greater than $k+1$, a contradiction. □

Corollary 10.

$$ex(n, W_7) = \left\lfloor \frac{n^2}{4} + \frac{n}{2} + 1 \right\rfloor.$$

At the end of this subsection we enumerate all of the extremal graphs for $7 \leq n \leq 26$. An important property to generate these graphs is that if they exist, then they can be selected from the sets of all W_7-free graphs with the number of edges greater than or equal to $\lceil \frac{n^2}{4} + \frac{n}{2} - 1 \rceil$. The sets were generated using the modified McKay's graph generation program geng [6].

For the cases when $n \in \{7, 8, 9\}$, the example of the extremal graph is $C_4 + (K_2 \cup (n-6)K_1)$. More precisely, the sets $\text{EX}(n, W_7)$ for these values of n are as follows:

- $\text{EX}(7, W_7)$={$C_4 + (K_2 \cup K_1)$, $K_2 + (K_4 \cup K_1)$ }
- $\text{EX}(8, W_7)$={$C_4 + (K_2 \cup 2K_1)$}
- $\text{EX}(9, W_7)$={$C_4 + (K_2 \cup 3K_1)$, $(K_3 \cup K_2) + (K_2 \cup 2K_1)$, $(C_4 \cup K_1) + (K_2 \cup 2K_1)$, $C_5 + 4K_1$, $2C_3 + (K_2 \cup K_1)$}.

3.3. $ex(n, W_{2k+1})$, where $n \geq 2k+1$ and $k \geq 4$

Let us recall that we denote by aG the graph consisting of a disconnected subgraphs G. It is not hard to see that the graph $(K_2 \cup aK_1) + bK_k$ does not contain W_{2k+1} as a subgraph for all $a, b \in \mathbb{N}$. We will try to maximize the number of its edges. We need to determine the number of disconnected copies of K_k. Consider the situation when $b = \lfloor \frac{n+k+1}{2k} \rfloor$. In this case, $a = n - 2 - k\lfloor \frac{n+k+1}{2k} \rfloor$ and $\text{ex}(n, W_{2k+1}) \geq e(K_k)b + kb(n-kb) + 1$.

Theorem 11. *Assume that $k \geq 4$ and $n \geq 2k+1$. Then*

$$ex(n, W_{2k+1}) \geq \left\lfloor \frac{n+k+1}{2k} \right\rfloor \left(\binom{k}{2} + kn - \left\lfloor \frac{n+k+1}{2k} \right\rfloor \right) + 1 > \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

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References

[1] M. Bataineh, M. Jaradat, A. Jaradat, Edge maximal graphs containing no specific wheels, Jordan J. Math. Statist. 8 (2) (2015) 107–120.
[2] T. Dzido, A note on Turán numbers for even wheels, Graphs Combin. 29 (5) (2013) 1305–1309.
[3] T. Dzido, M. Kubale, K. Piwakowski, On some Ramsey and Turán-type numbers for paths and cycles, Electron. J. Combin. 13 (2006) #R55.
[4] P. Erdős, Extremal problems in graph theory, Theory of graphs and its applications, in: Proc. Sympos. Smolenice, 1964, pp. 29–36.
[5] W. Mantel, Problem 28, soln. by H. Gouventak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W.A. Wythoff, Wiskundige Opgaven 10 (1907) 60–61.
[6] B.D. McKay, A. Piperno, Practical graph isomorphism, {II}, J. Symbolic Comput. 60 (2014).
[7] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Academic Press, New York, 1968, pp. 279–319.
[8] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452.