CURVES OF THERMODYNAMIC STATES IN SOME FLUIDS WITH DISPERSION

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Abstract: Variations in the thermodynamic state of a dispersive medium, caused by sound, are studied. A bubbly liquid and a Maxwell fluid are considered as examples. Curves in the plane of thermodynamic states are plotted. They are in fact pictorial images of linear relations of excess pressure and excess density in the acoustic wave which reflect irreversible attenuation of the sound energy. The curves account for the nonlinear generation of the entropy mode in the field of sound. In the case of Maxwell fluids, loops may form under some conditions. Curves and loops for some kinds of stationary waveforms and impulse sound are discussed and compared.

Keywords: Equation of state, Nonlinear acoustics, Hysteresis

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1. Introduction

As a rule, a connection between acoustic pressure and density is non-local and reflects the memory effects. In general, it is integral with some kernel which is conditioned by the molecular properties of a fluid. In Newtonian fluids, the relation between acoustic pressure and excess acoustic density includes a term proportional to the partial derivative of excess acoustic density with respect to time which differs from zero in the thermoconducting fluids [1]. As usual, a dispersive medium is damping. There is irreversible loss of acoustic energy into the energy of the thermal mode in viscous nonlinear media. It makes the temperature of a medium of sound propagation to increase. The process is isobaric and is followed by a corresponding decrease in the medium density [1–3]. There are many reasons for an irreversible loss of sound energy in fluids. Among them, thermal conductivity, molecular absorption, scattering and relaxation processes of different origin, may be listed [1, 3–6]. The irreversible loss depends on the kind of attenuation in a fluid, on the intensity of the wave, but also on gradients of acoustic perturbations. The thermodynamic state of a fluid depends on a prehistory of sound perturbations. Thereby, some kind of acoustic hysteresis may take place.

2. Dynamics of total excess density and pressure

A relation between the total excess pressure and the total excess density follows from the continuity, momentum and energy equations describing a nonlinear flow of a relaxing medium. The entropy mode is isobaric, hence the total excess pressure equals the acoustic pressure, and the total excess density is a sum of parts belonging to acoustic and entropy modes. A thermodynamic state of a fluid is described by the total excess quantities, so that the hysteresis images should be plotted in terms of the total excess density, $\rho' = \rho_a + \rho_e$ and pressure, $p' = p_a$ (indexes a and e relate to acoustic and entropy quantities, respectively). The analysis below concerns stationary planar waveforms which may propagate over a relaxing medium, and some impulses. We will consider two examples of relaxing media: Maxwell fluids and bubbly liquids.

2.1. The bubbly liquids

We consider a bubbly liquid which consists of an incompressible liquid of density ρ_0 involving identical spherical bubbles of an ideal gas. All bubbles are of the same radii at equilibrium, L_0, there is no heat and mass transfer between liquid and gas. Bubbles are well separated, and they pulsate in the lowest, radially symmetric mode. The characteristic scale of perturbation in a bubbly liquid is much larger than a bubble radius, so that a bubbly liquid as a whole may be treated as the homogeneous continuum. Pressure in a bubbly liquid coincides with pressure in the liquid phase [7, 8]. Quantities relating to gas, liquid or to a bubbly liquid as a whole, are marked by indices g, l and 0, respectively. The leading-order relation between total excess pressure and density in the case of a bubbly liquid, takes the form [9]

$$\begin{aligned} P = \frac{p_a}{M\rho_0 c_0^2} = \frac{\rho_a}{M\rho_0} + \frac{\varepsilon - 1}{M\rho_0^2}\rho_a^2 + \frac{D}{M\rho_0^2}\frac{\partial^2 \rho_a}{\partial t^2} = \\ R + M(\varepsilon-1)R^2 + D\frac{\partial^2 R}{\partial t^2} - R_e = R + M(\varepsilon-1)R^2 + D\frac{\partial^2 R}{\partial t^2} - \int^t Q_a dt \end{aligned} \tag{1}$$

where $P = \frac{p_a}{M\rho_0 c_0^2}$ denotes the dimensionless total pressure, and $R = \frac{\rho_a + \rho_e}{M\rho_0} = R_a + R_e$ is the dimensionless total density, which is a sum of specific terms belonging to sound and the entropy mode, M is the Mach number, c_0 is a linear sound speed in a bubbly liquid, ε is the parameter of nonlinearity of a bubbly liquid, and

$$D = \frac{L_0^2 \rho_l}{3\gamma_g p_g} \tag{2}$$

is the parameter responsible for dispersion (p_g is the initial pressure of a gas inside a bubble, and γ_g is the ratio of specific heats in a gas). Its dimension is square seconds; for dispersion to be small it should be much smaller than the characteristic inverse squared sound frequency, ω^{-2}. The sound speed in a bubbly liquid depends strongly on the initial volume concentration of bubbles, α:

$$c_0 = \sqrt{\frac{\gamma_g p_g}{\alpha(1-\alpha)\rho_l}} \tag{3}$$

The parameter of nonlinearity in a bubbly liquid may vary in orders of magnitude due to variability of α. In the case of an incompressible liquid including bubbles, it equals [4]

$$\varepsilon = \frac{\gamma_g + 1}{\alpha} \tag{4}$$

The acoustic source of the thermal mode, Q_a, takes the leading-order form [9]:

$$Q_a = D(\varepsilon - 2)R\frac{\partial^3 R}{\partial t^3} \tag{5}$$

The lower limit of integration of Q_a should be chosen in accordance with the beginning of sound transmission. An acoustic pressure in a beam progressive in the positive direction of axis OX is described in the leading order by the equation

$$\frac{\partial p_a}{\partial x} - \frac{\varepsilon}{c_0^3 \rho_0} p_a \frac{\partial p_a}{\partial \theta} - \frac{D}{2c_0}\frac{\partial^3 p_a}{\partial \theta^3} = 0 \tag{6}$$

with $\theta = t - x/c_0$ denoting the retarded time. The dispersion in its pure form is considered in this example. Damping due to mechanical viscosity and thermal conductivity is ignored. A dimensionless excess density which specifies the entropy mode, may be readily evaluated:

$$R_e = \int^t Q_a dt = D(\varepsilon - 2)\left(R\frac{\partial^2 R}{\partial t^2} - 0.5\left(\frac{\partial R}{\partial t}\right)^2\right) \tag{7}$$

The conclusion is that the integral over period of the periodic sound is zero. It is zero always for impulses, at the times, when the impulse has gone away; that is due to zero boundary conditions for R at both infinities. Hence, pure dispersion does no lead to a noticeable trace after passing of the impulse sound, but may vary the thermodynamic state of domains over which sound propagates.

2.2. Maxwell fluids

In Maxwell relaxing fluids which do not conduct heat, the link between dimensionless total pressure and density takes the form:

$$P = R + (\varepsilon - 1)R^2 - \int^t Q_a dt \tag{8}$$

where ε is the parameter of nonlinearity of a Maxwell fluid. The acoustic source was derived by the author in [10]

$$Q_a = -m(\gamma - 1)\frac{\partial R}{\partial t}\int_{-\infty}^{t} \frac{\partial R}{\partial t'} e^{-(t-t')/\tau_R} dt' \tag{9}$$

where τ_R denotes the characteristic time of thermodynamic relaxation, m is the dimemsionless dispersion parameter,

$$m=\frac{c_\infty^2-c_0^2}{c_0^2} \tag{10}$$

and c_∞ denotes the linear speed of sound at very high frequencies (the "frozen" sound speed). The acoustic source may be approximately evaluated in the two limiting cases, the low-frequency or the high-frequency sound. When $\omega\tau_R \ll 1$, $e^{-(t-t')/\tau_R}$ varies much more quickly than $\frac{\partial R}{\partial t'}$, and the fluid behaves as a Newtonian with the corresponding acoustic source of heating:

$$Q_{a,low}=-m(\gamma-1)\tau_R\left(\frac{\partial R}{\partial t}\right)^2 \tag{11}$$

The low-frequency acoustic pressure is governed by the Burgers equation [1]:

$$\frac{\partial p_a}{\partial x}-\frac{\varepsilon}{c_0^3\rho_0}p_a\frac{\partial p_a}{\partial \theta}-\frac{m\tau_R}{2c_0}\frac{\partial^2 p_a}{\partial \theta^2}=0 \tag{12}$$

In the other limiting case, $\omega\tau_R \gg 1$, the acoustic source takes the leading-order form

$$Q_{a,high}=-m(\gamma-1)R\frac{\partial R}{\partial t} \tag{13}$$

In this case, the acoustic pressure is described in the leading order by equation

$$\frac{\partial p_a}{\partial x}-\frac{\varepsilon}{c_0^3\rho_0}p_a\frac{\partial p_a}{\partial \Theta}+\frac{m}{2c_0\tau_R}p_a=0 \tag{14}$$

with $\Theta=t-x/c_\infty$. The form of Equation (11), corresponding to a Newtonian fluid, leads to a negative integral of the acoustic source independently on the limits of integration. The acoustic source Equation (13) yields readily

$$\int Q_{a,high}dt=-0.5m(\gamma-1)R^2 \tag{15}$$

The integral over period is zero for nearly periodic sound. In the case of impulses, similarly to the case of a bubbly liquid with pure dispersion, the integral over the domains where a pulse has already passed away, approximately equals zero. This reflects the very low attenuation of the short-scale sound impulses. Actually, generation of the entropy mode occurs over the length of an impulse. The case when $\omega\tau_R$ equals unity is of the most interest in view of the fact that the attenuation of sound is the largest and therefore, the contribution of the entropy mode in the total excess density achieves a maximum. The main difficulty in evaluation of the total excess density is to establish a solution of the nonlinear equation which describes the acoustic pressure. It takes the general form [4]:

$$\frac{\partial p_a}{\partial x}-\frac{\varepsilon}{c_0^3\rho_0}p_a\frac{\partial p_a}{\partial \theta}-\frac{m}{2c_0}\frac{\partial}{\partial \theta}\int_{-\infty}^{\theta}\frac{\partial p_a}{\partial \theta}e^{-(\theta-\theta')/\tau_R}d\theta'=0 \tag{16}$$

which has no general analytical solutions.

3. Curves for some kinds of sound

The idea to plot curves $P \Leftrightarrow R$ which reflect the memory effects in fluids with attenuation and relaxation, comes from the papers of Rudenko, Hedberg. In the study [11], only acoustic perturbations were considered. The authors pointed out the existence of the hysteresis acoustic loops; the nonlinear distortions of sound itself were also accounted, including formation of the shock waves. In the plane of total perturbations, the nonlinear generation of the entropy mode should be considered. This was not done in the study [11]. Generation of the entropy mode is, in some sense, also "the memory effect" which makes density of a medium to decrease in dependence on a kind of attenuation and sound excitation. As for the intersections of the curve itself, which form loops, they are determined by the linear dispersive term in the link between acoustic pressure and excess acoustic density. They are hardly expected in the case of symmetric pulses in a bubbly liquid: for their formation, there should be different temporal behavior of acoustic pressure in the domains when it enlarges and decreases. The linear dispersive term is totally absent in Maxwell fluids without thermal conduction. Thermal conduction of fluids may lead to loops in the plane of thermodynamic states in the temporal domains when pressure decreases in time and thermal conduction is enough large. The main difficulty in the plotting of graphs is the establishment of acoustic pressure which satisfies the corresponding nonlinear equation. Among all solutions, the stationary waveforms are especially noteworthy.

3.1. Some exact sound waveforms

3.1.1. Bubbly liquid, stationary waveform

One of stationary solutions for acoustic pressure in the solitary form propagates faster than sound, with the speed $c_s = (1 + D\omega^2/2)c_0$ [7]:

$$P = \frac{3D\omega^2}{\varepsilon M} \left(1 + \cosh(t - x/c_s)\right)^{-1} \tag{17}$$

The acoustic source of the thermal mode may be evaluated, expressing the upper limit of integration, $\tau = t - x/c_s$ in terms of P. The curves in the $R(P)$ plane with account for Q_a in Equation (1) and without it are plotted in Figure 1. Loops for a symmetric stationary impulse are absent. The relative decrease in the total density peaks at the top of the soliton.

The following set of parameters are used: $L_0 = 2\,\mathrm{mm}$, $\alpha = 10^{-4}$, $\rho_l = 10^3\,\mathrm{kg/m^3}$, $\gamma_g = 1.4$, $p_g = 10^5\,\mathrm{Pa}$. This corresponds to $c_0 = 1183\,\mathrm{m/s}$, $\varepsilon = 1.2 \cdot 10^4$ and $D = 10^{-8}\,\mathrm{s^2}$. Hence, an equality $D\omega^2 = 0.5$ determines ω about $20\,\mathrm{kHz}$. The Mach number M equals 10^{-2}. The soliton's maximum dimensionless pressure equals $6 \cdot 10^{-3}$. The total density decreases at the length of an impulse, but after the impulse has passed away, there is no variation in the total density which might form some kind of a trace with increased temperature as it happens to Newtonian fluids.

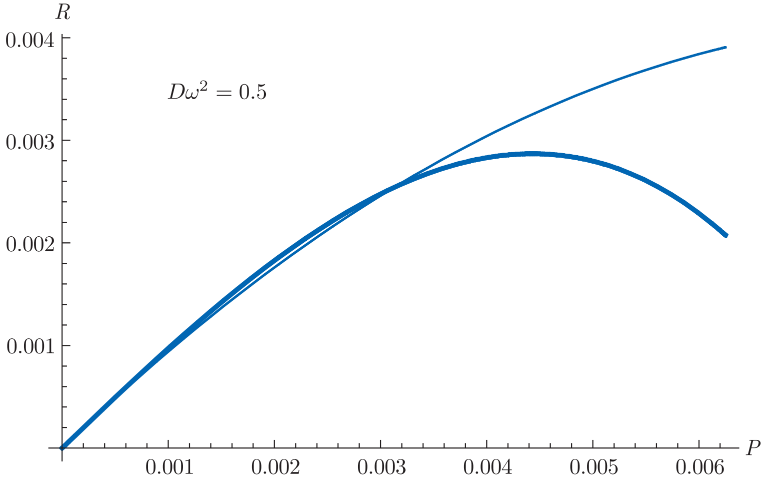

Figure 1. $R \Leftrightarrow P$ diagrams with account for entropy mode generated in acoustic field (bold) and without it (thin)

3.1.2. Maxwell fluids, stationary waveform

The stationary acoustic pressure in the case of weak nonlinearity, when $m \gg 2\varepsilon M$, takes the form

$$P = \tanh(\theta\varepsilon M/m) \tag{18}$$

Evaluations in the case of the high-frequency sound are plotted in Figure 2 for the following set of parameters: $\gamma_g = 1.4$, $\varepsilon = 1.2$ $M = 10^{-2}$, $m = 0.1$, $\omega\tau_R = 10^3$. The total excess density which is represented by the bold line, gets smaller with account for the part belonging to the entropy mode. The difference is more noticeable for large acoustic pressures, that is, the density jump in the shock wave gets smaller.

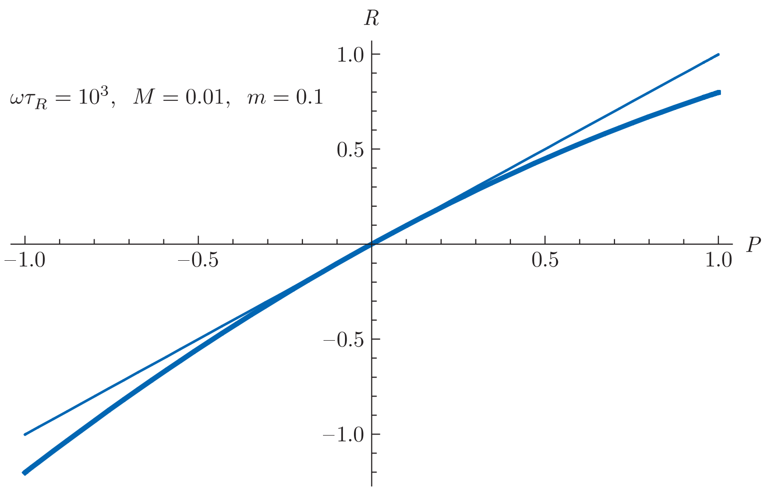

Figure 2. $R \Leftrightarrow P$ diagrams with account for entropy mode generated in acoustic field (bold) and without it (thin)

This corresponds to the reduced density of the background of the shock wave propagation.

3.1.3. The low-frequency saw-tooth sound in a Maxwell fluid

The next example is a periodic solution of the Burgers equation, Equation (12), which has the shock front, one period of which takes the form [1]:

$$P = -\frac{\omega\theta}{\pi} + \tanh\left(\frac{\theta}{T}\right), \quad -\pi < \omega\theta < \pi \tag{19}$$

where $T = \frac{m\tau_R}{M\varepsilon}$ is the characteristic width of the shock front. Assuming that the beginning of transmission of sound starts at $\theta = -\pi/\omega$, the integral of the acoustic source may be evaluated in terms of P, taking in mind, that at the straight parts of the n^{th} period of the shock wave an acoustic pressure is described by the leading-order equalities $(n = 1, 2, \ldots)$,

$$P = \begin{cases} -\dfrac{\omega\theta}{\pi} - 1 + 2n, & -\pi + 2\pi n < \omega\theta < 2\pi n \\ -\dfrac{\omega\theta}{\pi} + 1 + 2n, & 2\pi n < \omega\theta < \pi + 2\pi n \end{cases} \tag{20}$$

and in the vicinity of shocks $(\omega\theta \approx 2\pi n)$ by the leading-order relation

$$P = \tanh\left(\frac{\omega\theta - 2\pi n}{\omega T}\right) \tag{21}$$

5/4 periods of the sawtooth wave are plotted in Figure 3. The total density gets constantly smaller over each period of the saw-tooth wave.

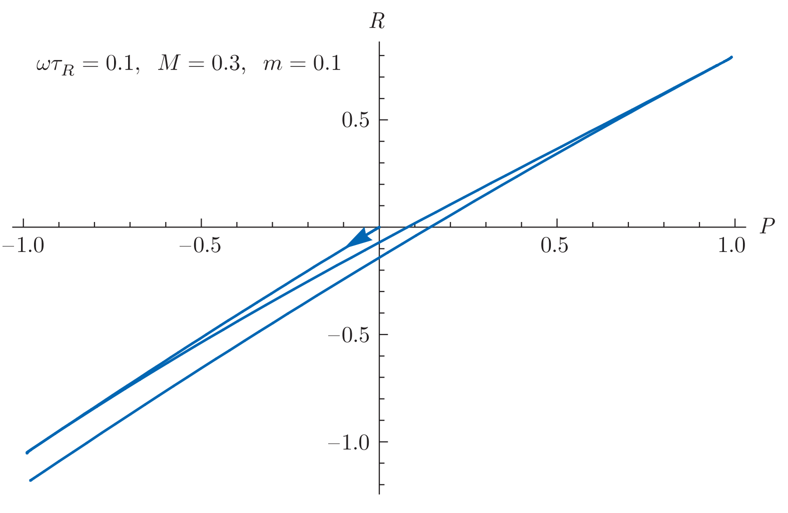

Figure 3. Dependence of the total excess density on the total excess pressure in low-frequency perturbations in a Maxwell fluid. Case of the saw-tooth wave

3.1.4. The high-frequency saw-tooth sound in a Maxwell fluid

The Equation (14) may be transformed in a pure nonlinear equation which has, inter alia, a solution in the form of a saw-tooth wave one period of which takes the form:

$$P = \frac{\exp(-mx/(2c_0\tau_R)}{1+\frac{2\varepsilon M\omega\tau_R}{m\pi}(\exp(-mx/(2c_0\tau_R))-1)}\begin{cases}-\dfrac{\omega\theta}{\pi}-1, & -\pi<\omega\theta<0\\ -\dfrac{\omega\theta}{\pi}+1, & 0<\omega\theta<\pi\end{cases} \tag{22}$$

This readily allows expressing θ in terms of P at different domains of an impulse. At any distance from the transducer, the form of the wave is triangular, but its maximum depends on a distance from the transducer and varies from 1 till 0 far from the transducer.

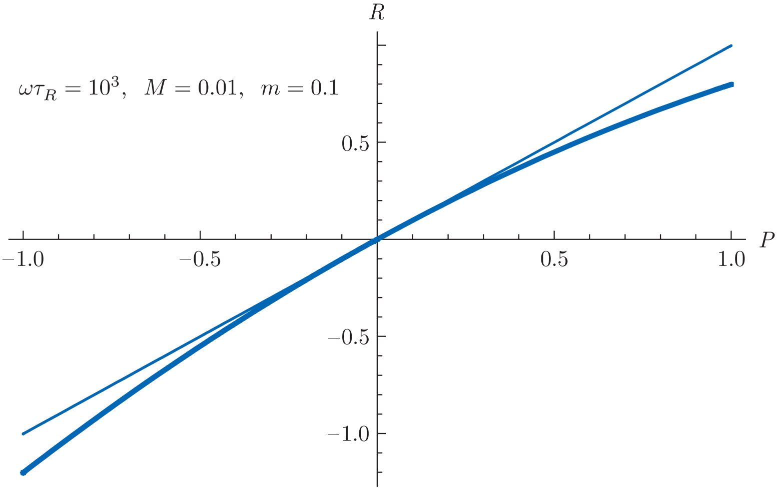

Figure 4. Pictorial images of perturbations in plane of thermodynamic states at transducer with account for entropy mode (bold line) and without it (thin line)

Figure 4 is very close to the Figure 2 which concerns the stationary waveform. The conclusion is that the density jump in the saw-tooth wave decreases, but the effect does not accumulate with the number of periods of the saw-tooth wave, in contrast to the low-frequency perturbations in a Maxwell fluid.

4. Some impulses

The curves in the plane of thermodynamic states may be plotted approximately, assuming that the sound propagates without any change in the wave form. This allows in many cases evaluating the acoustic source in the most interesting domain of sound frequencies $\omega\tau_R \approx 1$ propagating in a Maxwell fluid.

4.1. A Maxwell fluid

As an example, one period of sinusoidal wave is considered:

$$P = \sin(\omega\theta) \tag{23}$$

when $-\pi \le \omega\theta \le \pi$. Equation (23) determines the expression of θ in terms of P at different domains,

$$\theta = \begin{cases} \dfrac{\arcsin P}{\omega}, & \text{if } P \text{ enlarges} \\ \dfrac{-\pi - \arcsin P}{\omega}, & \text{if } P \text{ is negative and decreases} \\ \dfrac{\pi - \arcsin P}{\omega}, & \text{if } P \text{ is positive and decreases} \end{cases} \tag{24}$$

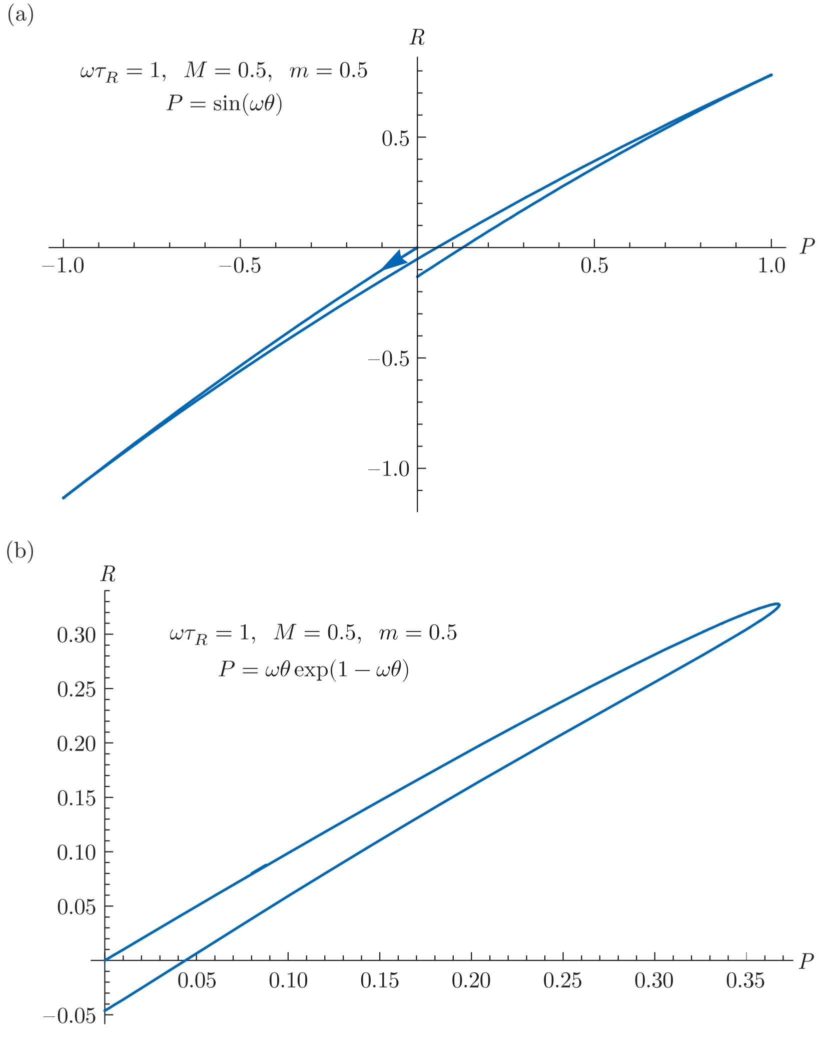

Figure 5. $R \Leftrightarrow P$ diagrams for some impulse sound perturbations in Maxwell fluid

The curve in the pale $R(P)$ is shown in Figure 5(a) in the case of a Maxwell fluid for the data as follows: $M=0.5$, $m=0.5$, $\omega\tau_R=1$. This is the case where a dispersive flow reveals the greatest attenuation. The second example relates to the impulse determined for positive arguments θ,

$$P=\omega\theta\exp(1-\omega\theta), \quad \theta>0 \tag{25}$$

In the both cases, an impulse is followed by a trail of increased temperature, and, relatively decreased density.

4.2. A bubbly liquid

As exemplary impulses, the impulse in the form of Equation (25), and the asymmetric impulse which is determined for any θ,

$$P=2\omega\theta(1+(\omega\theta)^2)^{-1} \tag{26}$$

are considered.

Plots in Figure 6 correspond to the set of parameters listed in Section 3.1.1. The second asymmetric impulse yields loops in a curve. The last example also relates to the asymmetric pulse which is defined for any θ,

$$P=2(\omega\theta)^3(1+(\omega\theta)^6)^{-1} \tag{27}$$

Equation (27) determines θ in terms of P at different domains,

$$\omega\theta=\begin{cases} -\left(-\dfrac{\sqrt{(1-P^2)}+1}{P}\right)^{1/3}, & \text{if } P \text{ is negative and decreases} \\ -\left(\dfrac{\sqrt{(1-P^2)}-1}{P}\right)^{1/3}, & \text{if } P \text{ is negative and increases} \\ \left(\dfrac{1-\sqrt{(1-P^2)}}{P}\right)^{1/3}, & \text{if } P \text{ is positive and increases} \\ \left(\dfrac{1+\sqrt{(1-P^2)}}{P}\right)^{1/3}, & \text{if } P \text{ is positive and decreases} \end{cases} \tag{28}$$

5. Conclusions

The nonlinear propagation of sound in Newtonian fluids is always followed by an irreversible loss in the acoustic energy: the macroscopic wave energy transfers into the thermal energy of chaotic motion of molecules. The corresponding enlargement of medium temperature and a decrease in its density are isobaric. The total density in the plane of the thermodynamic states (ρ', p') gets smaller by the nature of the case. In dispersive flows, a similar nonlinear generation of the entropy mode occurs. In a pure dispersive media like a bubbly liquid without account for attenuation of liquid and gaseous phases, thermal conduction and radiation, this nonlinear excitation is ineffective as compared with a Newtonian fluid. There is no

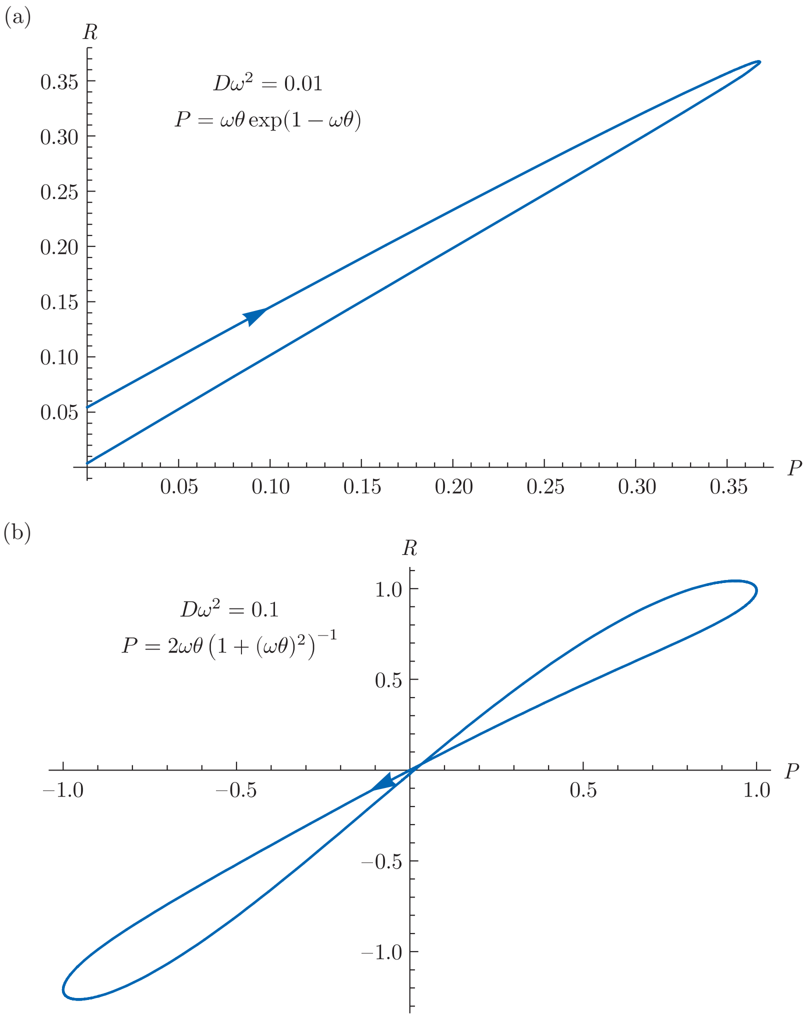

Figure 6. $R \Leftrightarrow P$ diagrams for some impulse perturbations in bubbly liquid

accumulation of variations in the entropy density; the thermodynamic state alters only at the duration of an impulse. Dispersion in Maxwell fluids is always followed by attenuation, which gets maximum at the characteristic frequency of sound equal to the inverse time of relaxation. The main difficulty in evaluations of an excess density associating with the entropy mode, is in establishing the acoustic pressure which satisfies a relative nonlinear dynamic equation, and in evaluation of the acoustic source of the entropy mode and its integral. Acoustic heating may be readily evaluated analytically for some stationary waveforms, which are exact solutions of the nonlinear dynamic equations describing acoustic pressure. The approximate evaluations are possible for traveling without distortion waves.

As for the low-frequency Maxwell fluid, it corresponds to a Newtonian fluid. The high-frequency Maxwell fluid has very low attenuation, and generation of

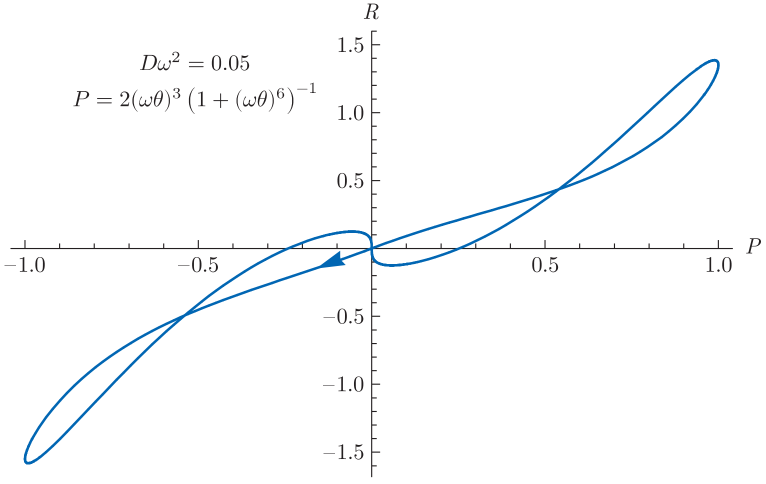

Figure 7. The $R \Leftrightarrow P$ diagram for an asymmetric acoustic impulse in a bubbly liquid

the entropy mode is ineffective as compared with a Newtonian fluid. The pure dispersion of a bubbly liquid yields an acoustic source different from that in a Maxwell fluid.

In this study, thermal conductivity is not accounted for. In the Newtonian fluids, there is an additional linear term in the link between acoustic pressure and excess acoustic density, which is proportional to the thermal conduction and the first partial derivative of density with respect to time. This may result in a loop in a curve of thermodynamic states, because the link is different at domains where pressure increases or decreases. Maxwell fluids with pure relaxation do not have this term. In a bubbly liquid, there is a linear term proportional to dispersion and the second derivative of density with respect to time. Therefore, loops are absent for symmetric impulses propagating in a bubbly liquid. The nonlinear phenomena occur unusually in acoustically active fluids. The entropy excess density enlarges, and the direction of the curves changes oppositely [12].

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